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INVARIANT SUBSPACES OF POLYNOMIALLY COMPACT OPERATORS

P. R. HALMOS

This paper is a comment on the solution of an invariant subspace problem by A. R. Bernstein and A. Robinson [2]. The theorem they prove can be stated as follows: if A is an operator on a Hilbert space H of dimension greater than 1, and if p is a nonzero polynomial such that $p(A)$ is compact, then there exists a nontrivial subspace of H invariant under A. ("Operator" means bounded linear transformation; "Hilbert space" means complete complex inner product space; "compact" means completely continuous; "subspace" means closed linear manifold; "nontrivial", for subspaces, means distinct from $\{0\}$ and from H.) The Bernstein-Robinson proof has two aspects: it is an ingenious adaptation of the proof by N. Aronszajn and K. T. Smith of the corresponding theorem for compact operators [1], and it makes strong use of metamathematical concepts such as nonstandard models of higher order predicate languages. The purpose of this paper is to show that by appropriate small modifications the Bernstein-Robinson proof can be converted (and shortened) into one that is expressible in the standard framework of classical analysis.

A quick glance at the problem is sufficient to show that there is no loss of generality in assuming the existence of a unit vector e such that the vectors $e, Ae, A^2e, \cdots$ are linearly independent and have H for their (closed linear) span. (This comment appears in both [1] and [2].) The Gram-Schmidt orthogonalization process applied to the sequence $\{e, Ae, A^2e, \cdots\}$ yields an orthonormal basis $\{e_1, e_2, e_3, \cdots\}$ with the property that the span of $\{e, \cdots, A^{n-1}e\}$ is the same as the span of $\{e_1, \cdots, e_n\}$ for each positive integer n. It follows that if $a_{mn} = (Ae_n, e_m)$, then $a_{mn} = 0$ unless $m \leqq n+1$; in other words, in the matrix of A all entries more than one step below the main diagonal must vanish. The matrix entries of the kth power of A are given by $a_{mn}^{(k)} = (A^k e_n, e_m)$. A straightforward induction argument, based on matrix multiplication, yields the result that $a_{mn}^{(k)} = 0$ unless $m \leqq n+k$, and

$$a_{n+k,n}^{(k)} = \Pi_{1 \leqq j \leqq k} a_{n+j,\, n+j-1}.$$

(With the usual understanding about an empty product having the value 1, the result is true for $k = 0$ also.) This result for powers has an implication for polynomials. If the degree of p (the only polynomial

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needed) is k ($\geqq 1$), and if the matrix entries of $p(A)$ are given by $a_{mn}^{(p)} = (p(A)e_n, e_m)$, then $a_{n+k,n}^{(p)}$ is a constant multiple (by the leading coefficient of p) of $a_{n+k,n}^{(k)}$. Since $\| p(A)e_n \| \to 0$ as $n \to \infty$ (because of the compactness of $p(A)$), there exists an increasing sequence $\{k(n)\}$ of positive integers (in fact a sequence with no gaps of length greater than the degree of p) such that the corresponding subdiagonal terms $a_{k(n)+1,k(n)}$ tend to 0 as n tends to ∞. (This very useful conclusion is one of the analytic tools used in [2], where it is described in terms of "infinite positive integers".)

If H_n is the span of $\{e_1, \cdots, e_{k(n)}\}$, then $\{H_n\}$ is an increasing sequence of finite-dimensional subspaces of H whose span is H. If P_n is the projection with range H_n, then $P_n \to 1$ (the identity operator) strongly. Since, for each n, the operator P_nAP_n leaves H_n invariant, it follows that, for each n, there exists a chain of subspaces invariant under P_nAP_n,

$$\{0\} = H_n^{(0)} \subset H_n^{(1)} \subset \cdots \subset H_n^{(k(n))} = H_n ,$$

with $\dim H_n^{(i)} = i$, $i = 0, 1, \cdots, k(n)$. (The consideration of such chains is essential in both [1] and [2].)

If $\{f_n\}$ and $\{g_n\}$ are sequences of vectors in H, it is convenient to write $f_n \sim g_n$ to mean that $\| f_n - g_n \| \to 0$ as $n \to \infty$. Assertion: if $\{f_n\}$ is a bounded sequence of vectors in H, then

$$AP_nf_n \sim P_nAP_nf_n . \tag{1}$$

(Intuitively: H_n is approximately invariant under A.) The proof is a straightforward computation, based on the fact that $P_nf = \sum_{j=1}^{k(n)} (f, e_j)e_j$ whenever $f \in H$. Since $AP_nf_n - P_nAP_nf_n = \sum_{j=1}^{k(n)} (f_n, e_j) \sum_{i=k(n)+1}^{\infty} a_{ij}e_i$, since the largest j here is $k(n)$ and the smallest i is $k(n) + 1$, and since $a_{ij} = 0$ unless $i \leqq j + 1$, it follows that $\| AP_nf_n - P_nAP_nf_n \| \leqq \| f_n \| \cdot | a_{k(n)+1,k(n)} |$.

The conclusion (1) can be generalized to higher exponents:

$$A^kP_nf_n \sim (P_nAP_n)^kf_n , \qquad k = 1, 2, 3, \cdots ; \tag{2}$$

the proof is by induction on k and is omitted. For $k = 0$, (2) says that $\| P_nf_n - f_n \| \to 0$, which is a stringent condition on the bounded sequence $\{f_n\}$; if that condition is satisfied, then (2) implies that

$$p(A)P_nf_n \sim p(P_nAP_n)f_n . \tag{3}$$

Return now to the unit vector e. Since $P_ne = e$ for each n, it follows that $p(P_nAP_n)e \sim p(A)e$. Since $p(A)e \neq 0$ (because the vectors $e, Ae, A^2e, \cdots$ are linearly independent), it follows that

$$\varepsilon = \lim_n \| p(P_nAP_n)e \| = \| p(A)e \| > 0 .$$

Consider, for each n, the numbers

$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(0)}e \| ,$$
$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(1)}e \| ,$$
$$\cdots$$
$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(k(n))}e \| ,$$

where $P_n^{(i)}$ is the projection with range $H_n^{(i)}$. Since $P_n^{(0)}$ is the zero projection, the first of these numbers tends to ε. Since, on the other hand, $P_n^{(k(n))} = P_n$, the last of these numbers is always 0. In view of these facts it is possible to choose for each n (with possibly a finite number of exceptions) a positive integer $i(n)$, $1 \leqq i(n) \leqq k(n)$, such that

$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(i(n)-1)}e \| \geqq \frac{\varepsilon}{2} , \tag{4}$$

and

$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(i(n))}e \| < \frac{\varepsilon}{2} ; \tag{5}$$

the simplest way to do it is to let $i(n)$ be the smallest positive integer for which these inequalities are true. (The construction of this particular "infinite positive integer" i is the second major analytic insight in [2].)

Since both $\{P_n^{(i(n)-1)}\}$ and $\{P_n^{(i(n))}\}$ are bounded sequences of operators, there exists an increasing sequence $\{n_j\}$ of positive integers such that both $\{P_{n_j}^{(i(n_j)-1)}\}$ and $\{P_{n_j}^{(i(n_j))}\}$ are weakly convergent. Write, for typographical convenience, $Q_j^- = P_{n_j}^{(i(n_j)-1)}$ and $Q_j^+ = P_{n_j}^{(i(n_j))}$. Let M^- be the set of all those vectors f in H for which $Q_j^- f \to f$ (strongly), and, similarly, let M^+ be the set of those vectors f for which $Q_j^+ f \to f$ (strongly). The purpose of what follows is to prove that both M^- and M^+ are subspaces of H, that both are invariant under A, and that at least one of them is nontrivial.

Since linear combinations are continuous, it follows that M^- is a linear manifold. To prove that M^- is closed, suppose that g is in the closure of M^-; it is to be proved that $g \in M^-$, i.e., that $Q_j^- g \to g$. Given a positive number δ, find f in M^- so that $\| f - g \| < \delta/3$, and then find j_0 so that $\| Q_j f - f \| < \delta/3$ whenever $j \geqq j_0$. It follows that if $j \geqq j_0$, then $\| Q_j^- g - g \| \leqq \| Q_j^- g - Q_j^- f \| + \| Q_j^- f - f \| + \| f - g \| < \delta$. This proves that M^- is closed; the proof for M^+ is the same.

To prove that M^- is invariant under A, suppose that $f \in M^-$, so that $Q_j^- f \to f$, and infer, first, that $AQ_j^- f \to Af$, just because A is bounded, and, second, that $Q_j^- AQ_j^- f \sim Q_j^- Af$, because Q_j^- is uniformly bounded. Then reason as follows: $Q_j^- Af \sim Q_j^- AQ_j^- f = Q_j^- P_{n_j} AP_{n_j} Q_j^- f$ (because $Q_j^- \leqq P_{n_j}$) $= P_{n_j} AP_{n_j} Q_j^- f$ (because the range of Q_j^- is invariant under $P_{n_j}AP_{n_j}) \sim AP_{n_j}Q_j^- f$ (by (1)) $= AQ_j^- f \to Af$. This proves that M^- is invariant; the proof for M^+ is the same.

The next step is to prove that $M^- \neq H$; this is done by proving that e does not belong to M^-. For this purpose observe first that the operators $p(P_nAP_n)$ are uniformly bounded. (Observe that

$$\| (P_nAP_n)^k \| \leqq \| P_nAP_n \|^k \leqq \| A \|^k$$

and use the polynomial whose coefficients are the absolute values of the coefficients of p.) Now use (4):

$$\frac{\varepsilon}{2} \leqq \| p(P_{n_j}AP_{n_j}) \| \cdot \| e - Q_j^- e \| .$$

Since $\| p(P_{n_j}AP_{n_j}) \|$ is bounded from above, its reciprocal is bounded away from zero, and, consequently, $\| e - Q_j^- e \|$ is bounded away from zero, which makes the convergence $Q_j^- e \to e$ impossible.

The corresponding step for M^+ says that $M^+ \neq \{0\}$; the proof is quite different. The choice of the sequence $\{n_j\}$ implies that the sequence $\{Q_j^+ e\}$ is weakly convergent; the compactness of $p(A)$ implies, therefore, that the sequence $\{p(A)Q_j^+ e\}$ is strongly convergent to, say, f. The proof that follows consists of two parts: (i) $f \neq 0$, (ii) $f \in M^+$. Part (i): $p(A)Q_j^+ e \sim p(P_{n_j}AP_{n_j})Q_j^+ e$ (by (3)), which is within $\varepsilon/2$ of $p(P_{n_j}AP_{n_j})e$ (by (5)), whose norm tends to ε; it follows that $\| p(A)Q_j^+ e \|$ cannot tend to 0, and hence that $f \neq 0$. Part (ii): $Q_j^+ f \sim Q_j^+ p(A)Q_j^+ e$ (since Q_j^+ is uniformly bounded) $\sim Q_j^+ p(P_{n_j}AP_{n_j})Q_j^+ e$ (by (3), and, again, uniform boundedness) $= p(P_{n_j}AP_{n_j})Q_j^+ e$ (because the range of Q_j^+ is invariant under $p(P_{n_j}AP_{n_j})$) $\sim p(A)Q_j^+ e$ (by (3)) $\to f$ (by definition).

If $M^+ \neq H$, all is well; it remains to be proved that if $M^+ = H$, then $M^- \neq \{0\}$. If $M^+ = H$, then $Q_j^+ f \to f$ for all f, and, a fortiori, $Q_j^+ f \to f$ weakly. At the same time the sequence $\{Q_j^-\}$ is known to be weakly convergent to, say, Q^-. The operators Q_j^- and Q_j^+ are projections such that $Q_j^- \leqq Q_j^+$ and such that $Q_j^+ - Q_j^-$ has rank 1. It follows that, for each j, there exists a unit vector f_j such that $(Q_j^+ - Q_j^-)f = (f, f_j)f_j$ for all f. Observe now that $Q_j^- e$ cannot tend weakly to e, for, if it did, then it would tend strongly to e (an elementary property of projections), and that was proved to be not so. This implies that $Q^- e \neq e$, or, equivalently, that $(1 - Q^-)e \neq 0$. Can the numbers $|(e, f_j)|$ be arbitrarily small? Since $|((Q_j^+ - Q_j^-)e, g)| \leqq |(e, f_j)| \cdot \| g \|$ for all g, an affirmative answer would imply that $((1 - Q^-)e, g) = 0$ for all g, so that $(1 - Q^-)e = 0$—a contradiction. The fact so obtained (that the numbers $|(e, f_j)|$ are bounded away from zero) makes it possible to prove that $M^- \neq \{0\}$; it turns out that if $g \perp (1 - Q^-)e$, then $g \in M^-$. Indeed, since $(e, f_j)(f_j, g) \to ((1 - Q^-)e, g) = 0$, it follows that $(f_j, g) \to 0$, and hence that $(f, f_j)(f_j, g) \to 0$ for all

f. This implies that $((1 - Q^-)f, g) = 0$ for all f, and hence that $(1 - Q^-)g = 0$. In other words, $Q_j^- g \rightarrow g$ weakly, and therefore strongly (the same property of projections that was alluded to above); from this it follows, finally, that $g \in M^-$.

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UNIVERSITY OF MICHIGAN