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EXISTENCE OF HALF-TRAJECTORIES IN PRESCRIBED REGIONS AND ASYMPTOTIC ORBITAL STABILITY

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A theorem is proved concerning the existence of a half-trajectory in the neighborhood of a semi-invariant set of a general dynamical system. A corollary of this theorem strengthens a result of P. Mendelson. The theorem is further used to obtain a necessary and sufficient condition for a compact positively invariant set to be positively asymptotically orbitally stable, and the condition is compared with another one due to S. Lefschetz.

Mendelson [3] applied a topological method due to Wazewski to obtain a sufficient condition for a neighborhood of a rest point of an autonomous system of differential equations to contain a half trajectory other than the rest point. The condition is that all points of egress be points of strict egress. Actually, as we show in § 2, this condition is redundant. Any neighborhood of a rest point of a dynamical system defined on an open set in E^n contains a half trajectory other than the rest point (possibly another rest point).

The purpose of this paper is twofold. First, in § 2 we prove a theorem on the existence of a half trajectory in the neighborhood of a semi-invariant set of a general dynamical system and from it deduce three corollaries. Corollary 2 is a generalization of Mendelson's theorem ([3], p. 221). Corollary 3 gives a sufficient condition for a neighborhood of a rest point to contain a half trajectory which is not a rest point, and it is shown by example that, in a sense, the condition is the best possible. Second, in § 3 Theorem 1 is used to obtain a necessary and sufficient condition for a compact positively invariant set to be positively asymptotically orbitally stable. This condition is compared with another one due to Lefschetz [2].

An autonomous system of differential equations fails to generate a dynamical system in the sense of Nemytskii-Stepanov [5, part II] if the solutions do not exist for all values of the independent variable. However, although the present discussion pertains to a dynamical system, with minor modifications the results in § 2 can be extended to autonomous systems of differential equations whose right hand sides

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are continuous and whose solutions are uniquely determined by their initial values.

2. Existence of half trajectories in a neighborhood of a semi-invariant set. Let R be a metric space with distance function ρ, and denote the closure, complement, and boundary of a set $A \subset R$ by $\bar{A}$, A', and ∂A respectively. The open ε-sphere about either a point $A \in R$ or a set $A \subset R$ will be written $S(A, \varepsilon) = \{p \in R: \rho(p, A) < \varepsilon\}$. We denote the real intervals $(-\infty, \infty)$, $(-\infty, 0]$ and $[0, \infty)$ by I, I^-, and I^+ respectively. Let f be a dynamical system on R, i.e., f is a continuous transformation on $R \times I$ onto R such that $f(p, 0) = p$ and $f(f(p, t'), t) = f(p, t' + t)$ for all $p \in R, t, t' \in I$. The sets $f(p, I^+)$ and $f(p, I^-)$, for any $p \in R$, are called (positive and negative) half trajectories. The $\alpha-$ and $\omega-$ limit sets of a motion $f(p, t)$ will be denoted by A_p and Ω_p respectively. A nonempty set $A \subset R$ is said to be *positively (negatively) invariant* if $f(A, t) \subset A$ for all $t \geqq 0$ $(t \leqq 0)$, *invariant* if $f(A, I) = A$.

THEOREM 1. *Let F be a closed positively (negatively) invariant set and let G be an open set which contains F and has a compact boundary. If there exist $p_n \in G, p_0 \in F$, and $t_n \in I$, $n = 1, 2, \cdots$, such that $p_n \to p_0$, $t_n > 0$ $(t_n < 0)$, and $f(p_n, t_n) \in G'$, then $\bar{G} \cap F'$ contains a negative (positive) half trajectory.*

Proof.[1] We give the proof for the case in which F is positively invariant. It follows from the hypothesis and well known properties of dynamical systems that there exist $T_n > 0$ such that $f(p_n, t) \in G$ if $0 \leqq t < T_n$, $f(p_n, T_n) \in \partial G$ and $T_n \to \infty$. Let $q_n = f(p_n, T_n)$. Since ∂G is compact, $\{q_n\}$ contains a convergent subsequence, and we can assume $q_n \to q_0 \in \partial G$. The inclusion $F \subset G$ implies $q_0 \notin F$, and it follows from the positive invariance of F that $f(q_0, I^-) \cap F = \emptyset$. Given any $\tau < 0$, $f(p_n, T_n + \tau) = f(q_n, \tau) \to f(q_0, \tau)$. Since $T_n \to \infty$, $0 \leqq T_n + \tau < T_n$ whence $f(p_n, T_n + \tau) \in G$ for n sufficiently large. Therefore $f(q_0, \tau) \in \bar{G}$ for all $\tau \leqq 0$.

COROLLARY 1. *Given a closed positively or negatively invariant set F, let $H = f(F, I)$. If $(\partial H) \cap F \neq \emptyset$, and if G is an open set which contains F and has a compact boundary, then $\bar{G} \cap F'$ contains a half trajectory.*

Proof. Let F be positively invariant and suppose $\bar{G} \cap F'$ contains no positive half trajectory. Choose $p_0 \in F \cap (\partial H)$. Then there exist $p_n \in G \cap H'$ such that $p_n \to p_0$, and it follows from the invariance of

[1] I am indebted to the referee for pointing out a simplification of this proof.

H that $f(p_n, I^+) \cap F = \varnothing$. Hence, by assumption, there exist $t_n > 0$ such that $f(p_n, t_n) \in G'$, and it follows from Theorem 1 that $\bar{G} \cap F'$ contains a negative half trajectory.

A simple example shows that the hypothesis $(\partial H) \cap F \neq \varnothing$ is essential to Corollary 1. Let R be the closed unit interval, and consider a dynamical system which has rest points at 0 and 1 and for which the motion through any point in the open interval $(0, 1)$ has $\{0\}$ as its α-limit set and $\{1\}$ as its ω-limit set. If $F = [1/2, 1]$ and $G = (1/4, 1]$, then $H = (0, 1]$ and $(\partial H) \cap F = \{0\} \cap [1/2, 1] = \varnothing$. The rest of the hypothesis of Corollary 1 is fulfilled but the conclusion fails to hold.

COROLLARY 2. *If R is locally compact, if F is a compact positively or negatively invariant set such that $(\partial H) \cap F \neq \varnothing$ where $H = f(F, I)$, and if G is an open set which contains F, then $G \cap F'$ contains a half trajectory whose closure is compact.*

Proof. Since R is locally compact and F is compact, there exists an open set U such that $F \subset U$, $\bar{U} \subset G$, and $\bar{U}$ is compact. By corollary 1, $F' \cap \bar{U}$ contains a half trajectory; its closure is contained in $\bar{U}$ and is therefore compact.

COROLLARY 3. *If R is connected, if G is an open set which contains a rest point, and if ∂G is compact, nonempty and contains no rest point, then $\bar{G}$ contains a half trajectory which is not a rest point.*

Proof. Let F be the set of all rest points in G; obviously F is invariant. Since the set of all rest points in R is closed and disjoint from ∂G, F is closed. Since R is connected and $\partial G \neq \varnothing$, $F \cap (\partial F) \neq \varnothing$, and the result follows from corollary 1.

Corollaries 1 and 2 are valid if $\partial G = \varnothing$. This follows from the fact that any finite arc $f(p, [\tau_1, \tau_2])$ is connected and so if $\partial G = \varnothing$, $f(p, I) \subset G$ for all $p \in G$. Thus if $(\partial H) \cap F \neq \varnothing$, there exists $p \in G \cap F'$ and it follows from the one-side invariance of F that either $f(p, I^+) \subset F'$ or $f(p, I^-) \subset F'$. However, the hypothesis $\partial G \neq \varnothing$ is essential to Corollary 3. For example, let every point of R be a rest point and take $G = R$. An example which shows that Corollary 3 is not valid if ∂G is permitted to have even one rest point is the dynamical system in the plane determined by the autonomous system of differential equations

$$\dot{x} = y$$
$$\dot{y} = \begin{cases} -x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x \geqq 0 . \end{cases}$$

The semi-axis $y = 0$, $x \geqq 0$ consists of rest points. All other motions $x(t)$, $y(t)$ traverse semi-circles above the origin in the left half plane and $x(t) \to \infty$ as $t \to \infty$ and as $t \to -\infty$. Thus the boundary of any sphere about the origin contains precisely one rest point, but no such sphere contains a half trajectory which is not a rest point.

3. Asymptotic orbital stability. A compact positively invariant set F is said to be *positively orbitally stable* is given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $q \in S(F, \delta)$, $f(q, I^+) \subset S(F, \varepsilon)$. Because of the compactness of F, this is equivalent to the requirement that given $\varepsilon > 0$ and $p \in F$, there exist $\delta > 0$ such that for all $q \in S(p, \delta)$, $f(q, I^+) \subset S(F, \varepsilon)$. A compact positively invariant set F is said to be *positively asymptotically orbitally stable* if it is positively orbitally stable and if there exists $\delta' > 0$ such that for all $q \in S(F, \delta')$, $\Omega_q \subset F$. This implies that $f(q, t)$ is positively Lagrange stable for all $q \in S(F, \delta')$, and it is a consequence of [5, Theorem 3.07, p. 341] that the condition $\Omega_q \subset F$ is equivalent to the more usual condition $\lim_{t\to\infty} \rho(f(q, t), F) = 0$.[1] These definitions of stability agree with the terminology of Auslander-Seibert [1] and Lefschetz [2] except that they omit the prefix "orbital." The qualifier will be retained here in order to avoid confusion with other types of stability and because it conforms with the terminology usually applied to limit cycles. A necessary condition[2] for F to be positively orbitally stable is that $A_q \cap F = \varnothing$ for all $q \in F'$. For if $q \in F'$ and $p \in A_q \cap F$, then there exist $t_n < 0$ such that $t_n \to -\infty$ and $q_n = f(q, t_n) \to p$. Since F is closed, $\rho(q, F) = \varepsilon > 0$. Hence given any $\delta > 0$, there exists n for which $q_n \in S(p, \delta)$ and $\rho(f(q_n, -t_n), F) = \rho(q, F) = \varepsilon$. Therefore F is not positively orbitally stable. That F can fail to be positively orbitally stable in precisely this way is indicated in [4], where there is sketched (Figure 1) a dynamical system in the plane for which the origin 0 is a rest point and $A_q = \Omega_q = \{0\}$ for all $q \in R$. In the next theorem we show that the requirement $A_q \cap F = \varnothing$ for $q \in F'$ together with a weaker requirement than $\Omega_q \subset F$ for q in some neighborhood of F constitute a necessary and sufficient condition for positive asymptotic orbital stability. Because of the compactness of F, any open set containing F contains $S(F, \varepsilon)$ for some $\varepsilon > 0$, and therefore it is only a matter of notational convenience to

[1] This is also observed [**1**], p. 459.

[2] This condition is explicitly stated in [**2**, (15.2)] for the case where F is a rest point of an autonomous system of differential equations.

make use of arbitrary open sets containing F rather than sets of the specific form $S(F, \varepsilon)$.

THEOREM 2. *Let F be a compact positively invariant set which is contained in an open set G with compact closure. A necessary and sufficient condition for F to be positively asymptotically orbitally stable is that there exist an open set G_1 containing F such that $A_q \cap F = \varnothing$ and $\Omega_q \cap F \neq \varnothing$ for all $q \in \bar{G}_1 \cap F'$.*

Proof. In view of the remark in the preceding paragraph, the necessity is obvious. To prove the sufficiency we first show that F is positively orbitally stable. Assuming the contrary we obtain $p \in F$, $p_n \in G$, $t_n > 0$ and $\varepsilon > 0$ such that $p_n \to p$ and $\rho(f(p_n, t_n), F) \geqq \varepsilon$. Thus, letting $G_2 = S(F, \varepsilon) \cap G_1 \cap G$, $f(p_n, t_n) \in G_2'$ and it follows from Theorem 1 (note that $p_n \in G_2$ for n sufficiently large) that there exists $q \in \bar{G}_2$ such that $f(q, I^-) \subset \bar{G}_2 \cap F'$. Since $\bar{G}_2$ is compact, $A_q \neq \varnothing$, say $q_1 \in A_q$, and $A_q \subset \bar{G}_2$. Since A_q is closed and invariant, $\Omega_{q_1} \subset A_q$; by hypothesis, $\Omega_{q_1} \cap F \neq \varnothing$ and so $A_q \cap F \neq \varnothing$. But this contradicts the other part of the hypothesis and therefore F is positively orbitally stable.

Now assume F is not positively asymptotically orbitally stable. Then there exists $q \in G_1$ such that $\Omega_q \cap F' \neq \varnothing$ and, by hypothesis, $\Omega_q \cap F \neq \varnothing$. Choose $p_0 \in \Omega_q \cap F$, $q_0 \in \Omega_q \cap F'$, and let $\rho(q_0, F) = \varepsilon$. Then $\varepsilon > 0$ and, given any $\delta > 0$, there exist $t > 0$ and $t' > t$ such that $f(q, t) \in S(p_0, \delta)$ and $f(q, t') \in S(q_0, \varepsilon/2)$. Hence

$$f(q, t') = f(f(q, t), t' - t) \notin S\left(F, \frac{\varepsilon}{2}\right)$$

and $t' - t > 0$. This contradicts the positive orbital stability of F and the proof is complete.

Another set of necessary and sufficient condition for positive asymptotic orbital stability of an invariant set F is given in [2, (14.3)]. These conditions comprise a necessary and sufficient condition for positive orbital stability together with the requirement that there exist a neighborhood of F which contains no entire trajectory other than those in F. On the other hand, no part of the hypothesis of Theorem 2 is both a necessary and sufficient condition for positive orbital stability alone, and in this respect the two results are essentially different.

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