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MINIMAL GERSCHGORIN SETS II

B. W. LEVINGER AND R. S. VARGA

The Gerschgorin Circle Theorem, which yields n disks whose union contains all the eigenvalues of a given $n \times n$ matrix $A = (a_{i,j})$, applies equally well to any matrix $B = (b_{i,j})$ of the set Ω_A of $n \times n$ matrices with $b_{i,i} = a_{i,i}$ and $|b_{i,j}| = |a_{i,j}|$, $1 \leqq i, j \leqq n$. This union of n disks thus bounds the entire spectrum $S(\Omega_A)$ of the matrices in Ω_A. The main result of this paper is a precise characterization of $S(\Omega_A)$, which can be determined by extensions of the Gerschgorin Circle Theorem based only on the use of positive diagonal similarity transformations, permutation matrices, and their intersections.

Given any $n \times n$ complex matrix $A = (a_{i,j})$, it is well known that the simplest of Gerschgorin arguments, which depends upon row sums of the moduli of off-diagonal entries of the matrix $X^{-1}AX$, X a positive diagonal matrix, yields the union of n disks which contains all the eigenvalues of A. It is clear that this union of n disks necessarily contains all the eigenvalues of any $n \times n$ matrix in the set Ω_A defined as follows: $B = (b_{i,j}) \in \Omega_A$ if $b_{i,i} = a_{i,i}$, $1 \leqq i \leqq n$, and $|b_{i,j}| = |a_{i,j}|$ for all $1 \leqq i, j \leqq n$, $i \neq j$. Hence, this union of n Gerschgorin disks can be viewed as giving bounds for the entire spectrum $S(\Omega_A) = \{z \mid \det(zI - B) = 0 \text{ for some } B \in \Omega_A\}$ of the set Ω_A.

It is logical to ask to what extent the spectrum $S(\Omega_A)$ can be more precisely determined by extensions of Gerschgorin's original argument [3]. In the previous paper [6], it was shown that

$$\partial G(\Omega_A) \subset S(\Omega_A) \subset G(\Omega_A) , \tag{1.1}$$

where $G(\Omega_A)$ is the *minimal Gerschgorin set* deduced from A and $\partial G(\Omega_A)$ is its boundary. The first inclusion of (1.1) states that every point of the boundary $\partial G(\Omega_A)$ of the minimal Gerschgorin set is then an eigenvalue of some $B \in \Omega_A$. We now extend the results of [6] by making use of results of Schneider [4], and Camion and Hoffman [1]. In so doing, we shall *precisely* determine $S(\Omega_A)$.

To begin, let $P_\phi = (\delta_{i,\phi(j)})$ be an $n \times n$ permutation matrix, where ϕ is a permutation of the integers $1 \leqq i \leqq n$ and $\delta_{i,j}$ is the Kronecker delta function, and let $X = \operatorname{diag}(x_1, x_2, \cdots, x_n)$, where $\boldsymbol{x} > \boldsymbol{0}$. Given $B \in \Omega_A$, we define the $n \times n$ matrix $M^\phi(\boldsymbol{x})$ by

$$M^\phi(\boldsymbol{x}) = (X^{-1}BX - \lambda I)P_\phi = (m_{i,j}) , \tag{1.2}$$

so that

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$$(1.3)\qquad m_{i,j} = b_{i,\phi(j)}x_{\phi(j)}/x_i - \lambda\delta_{i,\phi(j)}, \quad 1 \leqq i, j \leqq n .$$

Following Schneider [4], if λ is an eigenvalue of B, then $M^{\phi}(\boldsymbol{x})$ is surely singular and thus not strictly diagonally dominant. Hence,

$$(1.4)\qquad |m_{i,i}| \leqq \sum_{j\neq i} |m_{i,j}|$$

must be true for at least one i, $1 \leqq i \leqq n$. Defining first

$$(1.5)\qquad \Lambda_i(\boldsymbol{x}) \equiv \left(\sum_{j\neq i} |a_{i,j}|\, x_j\right)\Big/x_i , \qquad 1\leqq i\leqq n ,$$

then (1.4) implies that either

$$(1.6)\qquad |\lambda - a_{i,i}| \leqq \Lambda_i(\boldsymbol{x}) \quad \text{if} \quad \phi(i) = i ,$$

or

$$(1.6')\qquad 2x_{\phi(i)}\,|a_{i,\phi(i)}|/x_i \leqq |\lambda - a_{i,i}| + \Lambda_i(\boldsymbol{x}) \quad \text{if} \quad \phi(i) \neq i .$$

For any complex number σ, we consequently define

$$(1.7)\qquad r_i^{\phi}(\sigma; \boldsymbol{x}) \equiv \Lambda_i(\boldsymbol{x}) - |\sigma - a_{i,i}| \quad \text{if} \quad \phi(i) = i ,$$

and let

$$(1.7')\qquad r_i^{\phi}(\sigma; \boldsymbol{x}) \equiv |\sigma - a_{i,i}| + \Lambda_i(\boldsymbol{x}) - 2\,|a_{i,\phi(i)}|\, x_{\phi(i)}/x_i \quad \text{if} \quad \phi(i) \neq i .$$

With this, we next define the set $G_i^{\phi}(\boldsymbol{x})$ as

$$(1.8)\qquad G_i^{\phi}(\boldsymbol{x}) \equiv \{\sigma \mid r_i^{\phi}(\sigma; \boldsymbol{x}) \geqq 0\} , \qquad 1 \leqq i \leqq n .$$

If $\phi(i) = i$, then $G_i^{\phi}(\boldsymbol{x})$ reduces to the familiar Gerschgorin disk $|z - a_{i,i}| \leqq \Lambda_i(\boldsymbol{x})$. If $\phi(i) \neq i$, we observe from (1.7′) that $G_i^{\phi}(\boldsymbol{x})$ is the closed exterior of a disk, and is thus an *unbounded* set.

Defining $G^{\phi}(\boldsymbol{x})$ to be the union of the sets $G_i^{\phi}(\boldsymbol{x})$:

$$(1.9)\qquad G^{\phi}(\boldsymbol{x}) \equiv \bigcup_{i=1}^{n} G_i^{\phi}(\boldsymbol{x}) ,$$

the inequalities of (1.6) and (1.6′) show that if $\lambda \in S(\Omega_A)$, then $\lambda \in G_i^{\phi}(\boldsymbol{x})$ for some i, and hence $\lambda \in G^{\phi}(\boldsymbol{x})$. Thus, $S(\Omega_A) \subset G^{\phi}(\boldsymbol{x})$ for *every* $\boldsymbol{x} > \boldsymbol{0}$, and we then have that

$$(1.10)\qquad G^{\phi}(\Omega_A) \equiv \bigcap_{\boldsymbol{x}>0} G^{\phi}(\boldsymbol{x}) ,$$

called the *minimal Gerschgorin set relative to the permutation* ϕ, is such that

$$(1.11)\qquad S(\Omega_A) \subset G^{\phi}(\Omega_A)$$

for *every* permutation ϕ. It is clear that $G^{\phi}(\Omega_A)$ is a *closed* set for any permutation ϕ. Since $G_i^\phi(\boldsymbol{x})$ is a bounded set only when $\phi(i) = i$, it follows that $G^\phi(\Omega_A)$ is a bounded set only when ϕ is the identity permutation. We remark that the results of [6] are for the special case when ϕ is the identity permutation.

Since (1.11) is valid for any permutation ϕ, it then follows that

$$S(\Omega_A) \subset H(\Omega_A) \, , \tag{1.12}$$

where

$$H(\Omega_A) \equiv \bigcap_\phi G^\phi(\Omega_A) \, . \tag{1.13}$$

In § 2, we first characterize (Theorem 1) the minimal Gerschgorin sets $G^\phi(\Omega_A)$, and then show (Theorem 2) that their boundaries $\partial G^\phi(\Omega_A)$ are subsets of $S(\Omega_A)$. Finally, using a result of Camion and Hoffman [1], we prove (Theorem 3) in § 3 our main result that

$$S(\Omega_A) = H(\Omega_A) \, . \tag{1.14}$$

Summarizing, the now elementary Gerschgorin Circle Theorem [3], applied to a particular matrix A, actually gives eigenvalue bounds for a set Ω_A of related matrices. Our main result is that the *exact* spectrum $S(\Omega_A)$ of Ω_A can be determined from extensions of the Gerschgorin Circle Theorem based only on positive diagonal similarity transformations, permutation matrices, and intersections.

In § 4, we include an extension of a result of [6] concerning the number of eigenvalues of any $B \in \Omega_A$ in a bounded component of $G^\phi(\Omega_A)$. Finally, in § 5 we include several examples to show how $S(\Omega_A)$ can be determined.

2. The Function $\nu_\phi(\sigma)$.

In order to determine $G^\phi(\Omega_A)$, let σ be any complex number, and consider the real $n \times n$ matrix $Q^\phi(\sigma) = (q_{i,j})$ whose entries are defined by

$$q_{i,j} = (-1)^{\delta_{i,j}} \, | a_{i,\phi(j)} - \sigma \delta_{i,\phi(j)} | \, , \qquad 1 \leqq i, j \leqq n \, . \tag{2.1}$$

Since the off-diagonal entries of $Q^\phi(\sigma)$ are nonnegative, then $Q^\phi(\sigma)$ is *essentially nonnegative* [2; 5, p. 260], and hence we can associate with the matrix $Q^\phi(\sigma)$ the real number $\nu_\phi(\sigma)$, where $\nu_\phi(\sigma)$ is the (possibly multiple) eigenvalue of $Q^\phi(\sigma)$ with largest real part. From the Perron-Frobenius theory of nonnegative matrices [5, pp. 46–47], $\nu_\phi(\sigma)$ corresponds to a nonnegative eigenvector $\boldsymbol{y} \geqq \boldsymbol{0}$, i.e., $Q^\phi(\sigma)\boldsymbol{y} = \nu_\phi(\sigma)\boldsymbol{y}$, and it is further known that

$$\nu_\phi(\sigma) = \inf_{\boldsymbol{u}>0} \max_{1 \leqq i \leqq n} \left\{ \frac{(Q^\phi(\sigma)\boldsymbol{u})_i}{u_i} \right\} \, . \tag{2.2}$$

We remark that $\nu_\phi(\sigma)$ is a continuous function of σ.

THEOREM 1. *Let $A = (a_{i,j})$ be an $n \times n$ complex matrix, let ϕ be any permutation, and let σ be a complex number. Then, $\sigma \in G^\phi(\Omega_A)$ if and only if $\nu_\phi(\sigma) \geqq 0$.*

Proof. From the definitions of $Q^\phi(\sigma)$ in (2.1) and $r_i^\phi(\sigma; \boldsymbol{x})$ in (1.7)–(1.7′), it follows that

$$r_i^\phi(\sigma; \boldsymbol{x}) = \left(\frac{x_{\phi(i)}}{x_i}\right)\left[\frac{(Q^\phi(\sigma)\boldsymbol{z})_i}{z_i}\right], \text{ where } z_i \equiv x_{\phi(i)} . \tag{2.3}$$

Now, if $\sigma \in G^\phi(\Omega_A)$, then $\sigma \in G^\phi(\boldsymbol{x})$ for every $\boldsymbol{x} > \boldsymbol{0}$. But for every $\boldsymbol{x} > \boldsymbol{0}$, there is an i such that $\sigma \in G_i^\phi(\boldsymbol{x})$, so that $r_i^\phi(\sigma; \boldsymbol{x}) \geqq 0$. Since $\boldsymbol{x} > \boldsymbol{0}$, then $(x_{\phi(i)}/x_i)$ is positive for all $1 \leqq i \leqq n$, and it therefore follows from (2.2) that

$$\max_{1 \leqq i \leqq n} [(Q^\phi(\sigma)\boldsymbol{z})_i / z_i] \geqq 0 \quad \text{for every} \quad \boldsymbol{x} > \boldsymbol{0} .$$

Clearly, as $\boldsymbol{x} > \boldsymbol{0}$ runs over all positive vectors, so does the corresponding vector $\boldsymbol{z} > \boldsymbol{0}$. Hence, $\nu_\phi(\sigma) \geqq 0$ from (2.2). Conversely, assume that $\nu_\phi(\sigma) \geqq 0$. From (2.2) and (2.3), it follows that $r_i^\phi(\sigma; \boldsymbol{x}) \geqq 0$ for some i for every $\boldsymbol{x} > 0$. Hence, $\sigma \in G^\phi(\boldsymbol{x})$ for every $\boldsymbol{x} > \boldsymbol{0}$, and thus $\sigma \in G^\phi(\Omega_A)$, which completes the proof.

Our interest turns now to the boundary $\partial G^\phi(\Omega_A)$ of the minimal Gerschgorin set $G^\phi(\Omega_A)$. As usual, it is defined by

$$\partial G^\phi(\Omega_A) = \overline{G^\phi(\Omega_A)} \cap \overline{G^\phi(\Omega_A)'} , \tag{2.4}$$

where $\overline{G^\phi(\Omega_A)'}$ is the closure of the complement $G^\phi(\Omega_A)'$ of $G^\phi(\Omega_A)$. It follows from Theorem 1 that $G^\phi(\Omega_A)'$ is the set of all σ which satisfy $\nu_\phi(\sigma) < 0$. Similarly, the boundary $\partial G^\phi(\Omega_A)$ of the minimal Gerschgorin set is the set of all σ for which $\nu_\phi(\sigma) = 0$, and to which there exists a sequence of complex numbers $\{z_j\}_{j=1}^\infty$ with $\lim_{j\to\infty} z_j = \sigma$ such that $\nu_\phi(z_j) < 0$.

As in [6], we now show that *every* point of the boundary $\partial G^\phi(\Omega_A)$ is an eigenvalue of some matrix $B \in \Omega_A$.

THEOREM 2. *Let $A = (a_{i,j})$ be an $n \times n$ complex matrix, and let ϕ be any permutation. If $\nu_\phi(\sigma) = 0$, then σ is an eigenvalue of some matrix $B \in \Omega_A$, and thus $\sigma \in S(\Omega_A)$.*

Proof. If $\nu_\phi(\sigma) = 0$, then there exists a vector $\boldsymbol{y} \geqq \boldsymbol{0}$ with $\boldsymbol{y} \neq \boldsymbol{0}$ such that $Q^\phi(\sigma)\boldsymbol{y} = \boldsymbol{0}$. Writing $(\sigma - a_{k,k}) = |\sigma - a_{k,k}| \exp(i\psi_k)$, $1 \leqq k \leqq n$, let the $n \times n$ matrix $B = (b_{k,j})$ be defined by

(2.5) $\quad b_{k,k} = a_{k,k};\ b_{k,j} = |a_{k,j}| \exp i\,\{\psi_k + \pi[-1 + \delta_{k,\phi(k)} + \delta_{j,\phi(k)}]\},\ k \neq j\ .$

It is evident that $B \in \Omega_A$, and if $y_j = z_{\phi(j)}$, it can be verified (upon considering separately the cases when $\phi(i) = i$ and $\phi(i) \neq i$) that $Q^{\phi}(\sigma)\boldsymbol{y} = \boldsymbol{0}$ is equivalent to

$$\sum_{j=1}^{n} b_{k,j} z_j = \sigma z_k\ , \qquad 1 \leqq k \leqq n\ . \tag{2.6}$$

Since $\boldsymbol{y} \neq \boldsymbol{0}$, then $\boldsymbol{z} \neq \boldsymbol{0}$, and we conclude from (2.6) that σ is an eigenvalue of B, which completes the proof.

In order to prove a somewhat stronger result, let $\sigma \in \partial G^{\phi}(\Omega_A)$. Then, $\nu_{\phi}(\sigma) = 0$ and $\sigma \in S(\Omega_A)$. But as $S(\Omega_A) \subset G^{\phi}(\Omega_A)$ from (1.11), we have the

COROLLARY 1. *Let A be an $n \times n$ complex matrix. Then, for any permutation ϕ,*

$$\partial G^{\phi}(\Omega_A) \subset \partial S(\Omega_A)\ . \tag{2.7}$$

In [6], an interesting geometrical property of the boundary $\partial G^{\phi}(\Omega_A)$ was given when ϕ was the identity permutation, and A was assumed to be irreducible. In that case, each boundary point of $G^{\phi}(\Omega_A)$ was shown to be the intersection of n Gerschgorin circles. An analogous result is true for an arbitrary permutation ϕ, under slightly stronger hypotheses.

COROLLARY 2. *Let A be an $n \times n$ complex matrix, let ϕ be any permutation, and let $\sigma \in \partial G^{\phi}(\Omega_A)$. If $Q^{\phi}(\sigma)$ is irreducible, then there exists a vector $\boldsymbol{x} > \boldsymbol{0}$ such that $\sigma \in \partial G_i^{\phi}(\boldsymbol{x})$ for all $1 \leqq i \leqq n$.*

Proof. If $Q^{\phi}(\sigma)$ is irreducible, then $Q^{\phi}(\sigma)$ is essentially positive [5, p. 257]. Thus, there exists a vector $\boldsymbol{z} > \boldsymbol{0}$ such that $Q^{\phi}(\sigma)\boldsymbol{z} = \nu_{\phi}(\sigma)\boldsymbol{z}$. But, if $\sigma \in \partial G^{\phi}(\Omega_A)$, then $\nu_{\phi}(\sigma) = 0$, and $Q^{\phi}(\sigma)\boldsymbol{z} = \boldsymbol{0}$. Letting $\boldsymbol{x} > \boldsymbol{0}$ be defined component-wise by $z_i = x_{\phi(i)}$, it then follows from (2.3) that $r_i^{\phi}(\sigma; \boldsymbol{x}) = 0$ for all $1 \leqq i \leqq n$. Now, $r_i^{\phi}(\sigma; \boldsymbol{x})$ is obviously a continuous function of σ from (1.7)–(1.7′), and from (1.8) we deduce that $\partial G_i^{\phi}(\boldsymbol{x}) = \{\mu \mid r_i^{\phi}(\mu; \boldsymbol{x}) = 0\}$. Hence, $\sigma \in \partial G_i^{\phi}(\boldsymbol{x})$ for all $1 \leqq i \leqq n$, which completes the proof.

We remark that is ϕ if the identity permutation, then $Q^{\phi}(\sigma)$ is irreducible for any σ if and only if A is irreducible. For general ϕ, it is not difficult to show that A irreducible implies that $Q^{\phi}(\sigma)$ is irreducible when $\sigma \neq a_{i,i}$ for any i.

3. Main Result. We shall now show that $S(\Omega_A) = H(\Omega_A) \equiv \bigcap_{\phi} G^{\phi}(\Omega_A)$. Since $S(\Omega_A) \subset H(\Omega_A)$ by (1.12), it suffices to prove that

$S(\Omega_A)' \subset H(\Omega_A)'$, where $S(\Omega_A)'$ denotes the complement of $S(\Omega_A)$. This last inclusion will follow quite easily from the following theorem of Camion and Hoffman [1]:

Given an arbitrary $n \times n$ complex matrix $B = (b_{i,j})$, let $\mathring{\Omega}_B$ be the set of all matrices $C = (c_{i,j})$ with $|c_{i,j}| = |b_{i,j}|$ for *all* $1 \leqq i, j \leqq n$. Then, if all matrices $C \in \mathring{\Omega}_B$ are nonsingular, there exists a positive diagonal matrix $X = \operatorname{diag}(x_1, \cdots, x_n)$, $x_i > 0$, and a permutation matrix $P_\phi = (\delta_{i,\phi(j)})$ such that the matrix $M \equiv BXP_\phi = (m_{i,j})$ is strictly diagonally dominant, i.e.,

$$|m_{i,i}| > \sum_{j \neq i} |m_{i,j}| \quad \text{for all} \quad 1 \leqq i \leqq n\,. \tag{3.1}$$

We first prove

LEMMA 1. *$\sigma \in S(\Omega_A)'$ if and only if each $R \in \mathring{\Omega}_{A-\sigma I}$ is nonsingular.*

Proof. It is clear that each $R \in \mathring{\Omega}_{A-\sigma I}$ can be uniquely expressed as $R = D(B - \sigma I)$, where $D = \operatorname{diag}(e^{i\psi_1}, \cdots, e^{i\psi_n})$, ψ_j is real, and $B \in \Omega_A$. Then, $\sigma \in S(\Omega_A)'$ implies that $\det(B - \sigma I) \neq 0$ for any $B \in \Omega_A$. But as $|\det D| = 1$, then $\det R = \det D \cdot \det(B - \sigma I) \neq 0$ for any $R \in \mathring{\Omega}_A$. The converse follows similarly.

Now, suppose $\sigma \in S(\Omega_A)'$. From Lemma 1 and the result of Camion and Hoffman applied to $B = A - \sigma I$, there exists a positive diagonal matrix $X = \operatorname{diag}(x_1, \cdots, x_n)$ and a permutation matrix $P_\phi = (\delta_{i,\phi(j)})$ such that the matrix $M \equiv (A - \sigma I)XP_\phi \equiv (m_{i,j})$ is strictly diagonally dominant, where

$$m_{i,j} = (a_{i,\phi(j)} - \sigma\delta_{i,\phi(j)})x_{\phi(j)}\,. \tag{3.2}$$

Comparing (3.2) with the definition of $Q^\phi(\sigma)$ in (2.1) and setting $z_j \equiv x_{\phi(j)}$, $1 \leqq j \leqq n$, (3.1) can be equivalently expressed as

$$0 > \sum_{j \neq i} |m_{i,j}| - |m_{i,i}| = (Q^\phi(\sigma)z)_i\,, \qquad 1 \leqq i \leqq n\,. \tag{3.3}$$

Since $\boldsymbol{z} > \boldsymbol{0}$, it follows from (2.2) that $\nu_\phi(\sigma) < 0$, and hence from Theorem 1 we deduce that $\sigma \notin G^\phi(\Omega_A)$. Consequently, $\sigma \notin S(\Omega_A)$ implies that $\sigma \notin G^\phi(\Omega_A)$, which in turn implies that $\sigma \notin H(\Omega_A)$, or

$$S(\Omega_A)' \subset H(\Omega_A)'\,. \tag{3.4}$$

This, coupled with the result that $S(\Omega_A) \subset H(\Omega_A)$, gives us

THEOREM 3. *Let $A = (a_{i,j})$ be any $n \times n$ complex matrix. Then*

$S(\Omega_A) = H(\Omega_A)$.

4. Disconnected minimal gerschgorin sets. A familiar result of Gerschgorin [3] states that if k disks of the Gerschgorin set $G^I(\boldsymbol{x})$ (where I is the identity permutation) are disjoint from the remaining $n-k$ disks, then these k disks contain exactly k eigenvalues of any matrix $B \in \Omega_A$. In this section, we give a generalization of this result (cf. Theorem 5 of [6]). For a given $n \times n$ matrix $A = (a_{i,j})$ and an arbitrary permutation ϕ, let $G_j^\phi(\Omega_A)$ denote the nonempty disjoint closed connected components of the minimal Gerschgorin set $G^\phi(\Omega_A)$:

$$G^\phi(\Omega_A) = \bigcup_{j=1}^{m} G_j^\phi(\Omega_A)\,, \qquad 1 \leqq m \leqq n\,. \tag{4.1}$$

For each *bounded* component $G_j^\phi(\Omega_A)$, let the *order* s_j^ϕ be defined as the number of diagonal elements $a_{i,i}$ of A contained in $G_j^\phi(\Omega_A)$ for which $\phi(i) = i$. We shall show that each matrix $B \in \Omega_A$ contains exactly s_j^ϕ eigenvalues in each bounded component $G_j^\phi(\Omega_A)$ of the minimal Gerschgorin set $G^\phi(\Omega_A)$.

To begin, we enlarge the set Ω_A. An $n \times n$ matrix $B = (b_{i,j})$ is defined to be an element of the extended set Ω_A^ϕ if

$$\begin{cases} b_{i,i} = a_{i,i},\ 1 \leqq i \leqq n\ ;\ |b_{i,\phi(i)}| \geqq |a_{i,\phi(i)}|\ ,\ \phi(i) \neq i\ , \\ |b_{i,j}| \leqq |a_{i,j}|,\ 1 \leqq i,\ j \leqq n, \quad \text{for which} \quad j \neq i \quad \text{and} \quad j \neq \phi(i)\ . \end{cases} \tag{4.2}$$

Clearly, $\Omega_A \subset \Omega_A^\phi$.

LEMMA 2. *Given $B \in \Omega_A^\phi$, then $G^\phi(\Omega_B) \subset G^\phi(\Omega_A)$.*

Proof. For any vector $\boldsymbol{u} > \boldsymbol{0}$ and any complex number σ, consider the vector $Q_B^\phi(\sigma)\boldsymbol{u}$, where we are using an obvious subscript notation. With $B \in \Omega_A^\phi$, one verifies from (4.2) and (2.1) that $Q_B^\phi(\sigma)\boldsymbol{u} \leqq Q_A^\phi(\sigma)\boldsymbol{u}$ for any $\boldsymbol{u} > \boldsymbol{0}$ and any σ, from which it follows that

$$\max_{1 \leqq i \leqq n} \left\{ \frac{(Q_B^\phi(\sigma)\boldsymbol{u})_i}{u_i} \right\} \leqq \max_{1 \leqq i \leqq n} \left\{ \frac{(Q_A^\phi(\sigma)\boldsymbol{u})_i}{u_i} \right\}. \tag{4.3}$$

Thus, from (2.2), $\nu_{\phi,B}(\sigma) \leqq \nu_{\phi,A}(\sigma)$. Hence, by Theorem 1, $\sigma \in G^\phi(\Omega_B)$ implies that $\sigma \in G^\phi(\Omega_A)$, which completes the proof.

For this extended set Ω_A^ϕ, we remark that it can be further shown that $S(\Omega_A^\phi) = G^\phi(\Omega_A)$ for any permutation ϕ. This generalizes another result (Theorem 6) of [6].

In the spirit of Gerschgorin's original continuity argument [3], we prove

THEOREM 4. *Let $A = (a_{i,j})$ be any $n \times n$ complex matrix, and let ϕ be any permutation. If $G^{\phi}(\Omega_A)$ has a bounded component $G_j^{\phi}(\Omega_A)$ of order s_j^{ϕ}, then, for any matrix $B \in \Omega_A$, B contains exactly s_j^{ϕ} eigenvalues in $G_j^{\phi}(\Omega_A)$.*

Proof. For any $B = (b_{i,j}) \in \Omega_A$, consider the family of matrices $B_m(\alpha) = (b_{i,j}(\alpha))$ defined by

$$(4.4)\quad \begin{cases} b_{i,i}(\alpha) = b_{i,i},\ 1 \leqq i \leqq n; \\ b_{i,\phi(i)}(\alpha) = b_{i,\phi(i)}[m(1-\alpha)+\alpha] \text{ when } \phi(i) \neq i\,; \\ b_{i,j}(\alpha) = \alpha b_{i,j} \text{ for any } 1 \leqq i, j \leqq n \text{ for which } j \neq i \text{ and } j \neq \phi(i)\,. \end{cases}$$

By definition, $B_m(\alpha) \in \Omega_A^{\phi}$ for all $0 \leqq \alpha \leqq 1$ and all $m \geqq 1$, and $B_m(1) = B$. Moreover, $B_m(\alpha) \in \Omega^{\phi}_{B_m(\alpha')}$ for all $0 \leqq \alpha \leqq \alpha' \leqq 1$. Thus, from Lemma 2, $G^{\phi}(\Omega_{B_m(\alpha)}) \subset G^{\phi}(\Omega_A)$ for all $0 \leqq \alpha \leqq 1$ and all $m \geqq 1$, and it is clear that the set $G^{\phi}(\Omega_{B_m(\alpha)})$ increases monotonically with α. We shall show that $B_m(0)$ has exactly s_j^{ϕ} eigenvalues in the bounded component $G_j^{\phi}(\Omega_A)$, and the theorem will follow by continuously increasing α from zero to unity.

From (4.4), the only possibly nonzero entries of the matrix $B_m(0)$ are $b_{i,i}(0)$ and $b_{i,\phi(i)}(0)$ where $\phi(i) \neq i$. Hence, by considering the disjoint cycles of the permutation ϕ, we can find an $n \times n$ permutation matrix P such that

$$(4.5)\qquad PB_m(0)P^T = \begin{bmatrix} B_{1,1} & & 0 \\ & B_{2,2} & \\ 0 & & \ddots \\ & & & B_{N,N} \end{bmatrix}, \quad 1 \leqq N < n\,.$$

Here, $B_{1,1}$ is a diagonal matrix corresponding to all disjoint cycles with $\phi(i) = i$. The other matrices $B_{j,j}$ have the cyclic form

$$(4.6)\qquad B_{j,j} = \begin{bmatrix} b_{1,1}^{(j)} & b_{1,2}^{(j)} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & b_{r_j-1,r_j}^{(j)} \\ & 0 & & \\ b_{r_j,1}^{(j)} & & & b_{r_j,r_j}^{(j)} \end{bmatrix}, \quad 2 \leqq j \leqq N\,,$$

where the off-diagonal entries of $B_{j,j}$ are, from (4.4), given by $mb_{i,\phi(i)}$, $\phi(i) \neq i$. Obviously, the eigenvalues of all the $B_{j,j}$ are the eigenvalues of $B_m(0)$.

The spectrum of matrices of the form (4.6) is discussed in Example 1 of the next section, and in § 6 of [6]. We now assert that

$$(4.7)\qquad |\, b_{1,2}^{(j)} b_{2,3}^{(j)} \cdots b_{r_j,1}^{(j)} \,| \neq 0 \quad \text{for any} \quad 2 \leqq j \leqq N\,.$$

Otherwise, $b_{k,\phi(k)} = 0$ for some integer k, where $\phi(k) \neq k$, and, as shown in the next section, this implies that $G^{\phi}(\Omega_A)$ is the entire complex plane. This contradicts the hypothesis that $G^{\phi}(\Omega_A)$ has a bounded component. From (4.4), we can write the product in (4.7) as $m^{r_j} \cdot K_j$, where K_j is independent of m and α. Then, it is readily verified that the eigenvalues λ of $B_{j,j}$ satisfy

$$\prod_{k=1}^{r_j} | b_{k,k}^{(j)} - \lambda | = m^{r_j} \cdot K_j , \qquad 2 \leqq j \leqq N , \tag{4.8}$$

for *any* $B_m(0)$ derived from $B \in \Omega_A$. Since $B_m(0) \in \Omega_A^{\phi}$ for all $m \geqq 1$, we may choose m to be arbitrarily large, and it is clear from (4.8) that the eigenvalues of $B_{j,j}$ must lie in an unbounded component of $G^{\phi}(\Omega_A)$ for any $2 \leqq j \leqq N$. Hence, the number of eigenvalues of $B_m(0)$ which lie in the bounded component $G_j^{\phi}(\Omega_A)$ is just the number of diagonal entries of $B_{1,1}$ in $G_j^{\phi}(\Omega_A)$, which by definition is precisely s_j^{ϕ}. Now, increasing α continuously from zero to unity, it follows that B has exactly s_j^{ϕ} eigenvalues in $G_j^{\phi}(\Omega_A)$, which completes the proof.

We remark that the order s_j^{ϕ} of a bounded component $G_j^{\phi}(\Omega_A)$ is a positive integer. For, if s_j^{ϕ} were zero, no $B \in \Omega_A$ would have an eigenvalue in $G_j^{\phi}(\Omega_A)$, so that $S(\Omega_A) \cap G_j^{\phi}(\Omega_A)$ would be empty, which is a contradiction.

5. Some examples. We now give three examples to illustrate our results concerning the sets $S(\Omega_A)$, $G^{\phi}(\Omega_A)$, and $H(\Omega_A)$.

EXAMPLE 1. It was previously shown [6] for the matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & & & 0 \\ 0 & a_{2,2} & a_{2,3} & & \\ & & \ddots & \ddots & \\ 0 & & & & a_{n-1,n} \\ a_{n,1} & & & & a_{n,n} \end{bmatrix} , \tag{5.1}$$

where

$$| a_{1,2} a_{2,3} \cdots a_{n,1} | = 1 , \tag{5.2}$$

that $\partial G^{I}(\Omega_A) = S(\Omega_A)$, I being the identity permutation. Let ψ be the permutation[1] $(1\,2\,3 \cdots n)$. If ϕ is any permutation other than ψ or I, there is a positive integer k, $1 \leqq k \leqq n$, such that $\phi(k) \neq k$, and $\phi(k) \neq \psi(k)$, so that $a_{k,\phi(k)} = 0$. Thus, from (1.7′),

$$r_k^{\phi}(\sigma; \boldsymbol{x}) = | \sigma - a_{k,k} | + | a_{k,\psi(k)} | \, x_{\psi(k)}/x_k > 0 \tag{5.2}$$

for *all* $\boldsymbol{x} > \boldsymbol{0}$, and for *all* complex numbers σ. Hence, we deduce from

[1] That is, in this section we are describing a permutation by its disjoint cycles.

(2.2), (2.3), and Theorem 1 that $G^{\phi}(\Omega_A)$ is the entire complex plane. This argument shows more generally for an arbitrary matrix A that any permutation ϕ which places a zero on the diagonal of $Q^{\phi}(\sigma)$ yields a minimal Gerschgorin set $G^{\phi}(\Omega_A)$ which is the entire complex plane.

For $\phi = I$, it was shown [6] for the matrix of (5.1) that

$$G^{I}(\Omega_A) = \left\{\sigma \,\middle|\, \prod_{i=1}^{n} |\sigma - a_{i,i}| \leqq 1\right\} \tag{5.3}$$

and in an identical fashion, we can show that

$$G^{\psi}(\Omega_A) = \left\{\sigma \,\middle|\, \prod_{i=1}^{n} |\sigma - a_{i,i}| \geqq 1\right\}. \tag{5.4}$$

Hence, it follows that

$$S(\Omega_A) = H(\Omega_A) = G^{I}(\Omega_A) \cap G^{\psi}(\Omega_A) = \partial G^{I}(\Omega_A) . \tag{5.5}$$

EXAMPLE 2. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \tag{5.6}$$

In this case, there are only three permutations, corresponding to $\phi = I$, $\phi = (13)$, and $\phi = (23)$, for which $G^{\phi}(\Omega_A)$ is not the entire complex plane, and it is readily verified that

$$\begin{cases} G^{I}(\Omega_A) = \{\sigma \mid |2 - \sigma|^2 \cdot |1 - \sigma| \leqq |1 - \sigma| + |2 - \sigma|\} , \\ G^{(13)}(\Omega_A) = \{\sigma \mid |2 - \sigma|^2 \cdot |1 - \sigma| \geqq |1 - \sigma| - |2 - \sigma|\} , \\ G^{(23)}(\Omega_A) = \{\sigma \mid |2 - \sigma|^2 \cdot |1 - \sigma| \geqq - |1 - \sigma| + |2 - \sigma|\} . \end{cases} \tag{5.7}$$

The boundaries $\partial G^{\phi}(\Omega_A)$ are obviously determined by choosing the equality signs in (5.7). The spectrum $S(\Omega_A)$ in this case is a multiply connected region and is illustrated in Figure 1.

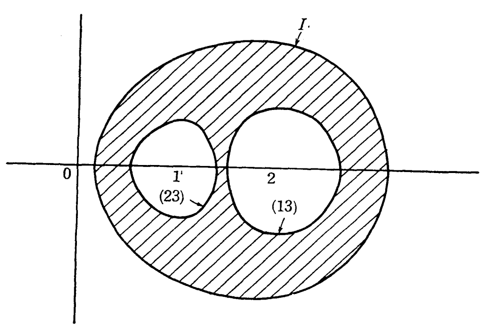

Fig. 1

EXAMPLE 3. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -5 & -1 & -1 \end{bmatrix}, \tag{5.8}$$

which is the companion matrix of the polynomial

$$p_4(z) = z^4 + z^3 + z^2 + 5z + 1 .$$

As previously shown, any permutation ϕ which places a zero on the diagonal of $Q^{\phi}(\sigma)$ yields a minimal Gerschgorin set $G^{\phi}(\Omega_A)$ which is the entire complex plane. Consequently, we need consider only the permutations I, (1234), (234), and (34). The associated minimal Gerschgorin sets are given by

$$\begin{cases} G^{I}(\Omega_A) = \{\sigma \mid |\sigma|^3 \cdot |1+\sigma| \leqq 1 + 5|\sigma| + |\sigma|^2\} , \\ G^{(1234)}(\Omega_A) = \{\sigma \mid |\sigma|^3 \cdot |1+\sigma| \geqq 1 - 5|\sigma| - |\sigma|^2\} , \\ G^{(234)}(\Omega_A) = \{\sigma \mid |\sigma|^3 \cdot |1+\sigma| \geqq -1 + 5|\sigma| - |\sigma|^2\} , \\ G^{(34)}(\Omega_A) = \{\sigma \mid |\sigma|^3 \cdot |1+\sigma| \geqq -1 - 5|\sigma| + |\sigma|^2\} . \end{cases} \tag{5.9}$$

The last minimal Gerschgorin set $G^{(34)}(\Omega_A)$ is the entire complex plane, and thus yields no boundary components of $S(\Omega_A)$. The set $G^{(234)}(\Omega_A)$ yields, however, two separate boundaries, and $G^{(234)}(\Omega_A)$ has a bounded component. Applying Theorem 4, we can assert that each matrix of the set Ω_A has exactly one eigenvalue in this component, and hence each matrix of Ω_A has exactly one eigenvalue in the inner annular region of Figure 2.

These examples have interesting common features. In each ex-

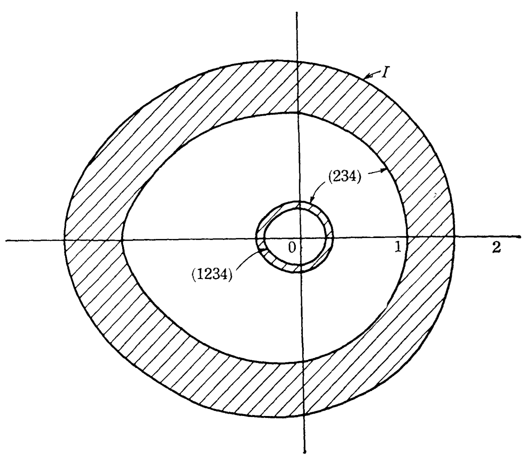

Fig. 2

ample, the minimum number of permutations necessary to define all the boundary components of $S(\Omega_A)$ does not exceed the order n of the matrix A. Similarly, the total number of boundary components of $S(\Omega_A)$ does not exceed $2n$. We conjecture this to be true in general. We do point out that examples can be constructed where these upper bounds are attained.

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