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ON THE FUNCTIONAL EQUATION $F(mn)F((m, n)) = F(m)F(n)f((m, n))$

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Let f be a multiplicative arithmetic function, $f(1) = 1$. Necessary and sufficient conditions on f will be found so that the functional equation

$$F(mn)F((m, n)) = F(m)F(n)f((m, n))$$

will have a solution F with $F(1) \neq 0$ and all solution F will be determined. It will be shown that two different types of solutions may exist and that one of these requires that f have a property similar to complete multiplicativity.

The special case of the equation

$$(1) \qquad F(mn)F((m, n)) = F(m)F(n)f((m, n))$$

with f completely multiplicative and $F(1) \neq 0$ was solved completely by Apostol and Zuckerman [1]. Specialized results were also given for the case $F(1) = 0$, but this case was not solved in general.

We note that if $f(1) = 0$ then f is identically zero and is thus completely multiplicative. This case was solved completely in [1] and will not be considered here. Thus in the sequel we will assume that f is multiplicative and is not identically zero (which implies that $f(1) = 1$).

1. If F is a solution of (1) with $F(1) \neq 0$ then any constant multiple of F is also a solution. Thus we may reduce the problem of solving (1) with $F(1) \neq 0$ to that of solving

$$(2) \qquad F(mn)F((m, n)) = F(m)F(n)f((m, n)) , \quad F(1) = 1 .$$

The proof of Theorem 1 is essentially the same as that of Theorem 2 of [1] and will be omitted.

THEOREM 1. *F is a solution of (2) if, and only if, F is a nonzero multiplicative function and for each prime p*

$$(3) \qquad F(p^{a+b})F(p^a) = F(p^b)F(p^a)f(p^a) \quad \text{if} \quad b \geqq a \geqq 1 .$$

The problem is thus reduced to that of determining the form of F on the powers of each prime p.

THEOREM 2. *Let F be a solution of (2). If $F(p^m) \neq 0$ for some*

$m \geqq 1$ *then* $F(p^{km+n}) = F(p^{m+n})f(p^m)^{k-1}$ $(k = 2, 3, \cdots)$, $(n = 0, 1, 2, \cdots)$.

The proof follows easily from (3) by induction on k.

COROLLARY 2.1. *Let* F *be a solution of* (2). *If* $F(p^m) \neq 0$ *and* $f(p^m) = 0$ *for some* $m \geqq 1$, *then* $F(p^n) = 0$ *for* $n \geqq 2m$.

COROLLARY 2.2. *Let* F *be a solution of* (2). *If* $F(p^m) \neq 0$ *and* $f(p^m) \neq 0$ *for some* $m \geqq 1$, *then* $F(p^{km}) \neq 0$ $(k = 2, 3, \cdots)$.

THEOREM 3. *Let* F *be a solution of* (2). *If* $F(p^m) \neq 0$ *and* $f(p^m) \neq 0$ *for some* $m \geqq 1$, *then* $f(p^n) \neq 0$ *whenever* $F(p^n) \neq 0$. *If, for some* $m \geqq 1$, $F(p^m) \neq 0$ *and* $f(p^m) = 0$ *then* $f(p^n) = 0$ *whenever* $F(p^n) \neq 0$.

Proof. To prove the first proposition we observe that if $F(p^n) \neq 0$ and $f(p^n) = 0$ then $F(p^t) = 0$ for $t \geqq 2n$ which contradicts Corollary 2.2. The proof of the second proposition is similar.

We are now in a position to determine the form of the solution F on the powers of p if $F(p^m) \neq 0$ and $f(p^m) = 0$.

THEOREM 4. *Let* m *be a positive integer such that* $f(p^m) = 0$. *Let* $m_1 = m, m_2, \cdots, m_k$, *be the integers* t *on the interval* $[m, 2m)$ *for which* $f(p^t) = 0$. *The function* F *is a solution of* (3) *with* m *the smallest positive integer such that* $F(p^m) \neq 0$ *if, and only if,* $F(p^m) \neq 0$ *and* $F(p^n) = 0$ *whenever* $n \neq m_i$.

Proof. If F is a solution of (3) and m is the smallest positive integer such that $F(p^m) \neq 0$ then by Corollary 2.1 we see that $F(p^n) = 0$ for $n \geqq 2m$ and by Theorem 3 we see that $F(p^n) = 0$ if $n \neq m_i$ $(m < n < 2m)$. To prove the converse we substitute in (3).

2. The case $F(p^m) \neq 0$ and $f(p^m) \neq 0$. We will first show that in this case f cannot be defined arbitrarily on the powers of p if a solution of (2) is to exist.

THEOREM 5. *Let* F *be a solution of* (2). *If* $F(p^m) \neq 0$ *and* $f(p^m) \neq 0$ *for some positive integer* m, *then*

$$f(p^{mk}) = f(p^m)^k \qquad (k = 1, 2, \cdots) .$$

Proof. From Corollary 2.2 and Theorem 3 we see that $F(p^{km}) \neq 0$ and that $f(p^{km}) \neq 0$ $(k = 1, 2, \cdots)$. Taking $a = b = km$ in (3) and using Theorem 2 we obtain

$$F(p^{2km}) = F(p^{km})f(p^{km}) = F(p^m)f(p^m)^{k-1}f(p^{km}) .$$

If we now take $n = 0$ and replace k by $2k$ in Theorem 2 we obtain

$$F(p^{2km}) = F(p^m)f(p^m)^{2k-1} .$$

Comparing the last two equations we see that $f(p^m)^k = f(p^{km})$.

THEOREM 6. *Let F be a solution of* (2). *Suppose $f(p^m) \neq 0$ and $F(p^m) \neq 0$ for some $m \geqq 1$, and let d be the smallest positive integer such that $F(p^{m+d}) \neq 0$. Then $d \mid m$. Furthermore, if n is a positive integer then $F(p^{m+n}) \neq 0$ if and only if $n \equiv 0 \pmod{d}$.*

Proof. (A) Such an integer d must exist by Corollary 2.2. Suppose $d \nmid m$. Let t be the smallest positive integer such that $td > m$. We can write $m < td = m + j < m + d$. From Theorem 2

$$F(p^{t(m+d)}) = F(p^{(t+1)m+j}) = F(p^{m+j})f(p^m)^t = 0$$

since $0 < j < d$; similarly, by Theorem 2 and 3 we have

$$F(p^{t(m+d)}) = F(p^{m+d})f(p^{m+d})^{t-1} \neq 0$$

which is impossible. Thus $d \mid m$.

(B) If $n \not\equiv 0 \pmod{d}$ there exist positive integers K and L such that $Km < Ln = Km + j < Km + d$. By considering $F(p^{Ln})$ we obtain a contradiction similar to that in part (A) if we assume $F(p^{m+n}) \neq 0$.

(C) Suppose $n \equiv 0 \pmod{d}$, say $n = kd$. Applying Theorem 2 twice we see that

$$F(p^{km+n}) = F(p^{k(m+d)}) = F(p^{m+d})f(p^{m+d})^{k-1} \neq 0$$

and

$$F(p^{km+n}) = F(p^{m+n})f(p^m)^{k-1} .$$

Thus

$$F(p^{m+n}) = F(p^{m+kd}) = \frac{F(p^{m+d})f(p^{m+d})^{k-1}}{f(p^m)^{k-1}} \neq 0 . \tag{4}$$

We now extend the result of Theorem 5 to completely characterize the solutions of (2) with $f(p^m) \neq 0$ and $F(p^m) \neq 0$.

THEOREM 7. *Let F and f be multiplicative functions. Suppose p is a prime, that m and d are the smallest positive integers such that $F(p^m) \neq 0$, $f(p^m) \neq 0$ and $F(p^{m+d}) \neq 0$. Then F is a solution of* (3) *if, and only if,*

$$d \mid m , \tag{5}$$

$$f(p^{m+kd}) = f(p^m)^{1+kd/m} \quad (k = 1, 2, 3, \cdots) , \tag{6}$$

$$F(p^{m+kd}) = F(p^m)f(p^m)^{kd/m} = \frac{F(p^m)}{f(p^m)}f(p^{m+kd})\ , \tag{7}$$

$$F(p^n) = 0 \ \textit{if} \ 0 < n < m \ \textit{or} \ \textit{if} \ n \not\equiv 0 \pmod{d}\ . \tag{8}$$

Proof. (5) and (8) were established in Theorem 6. To prove (6) we let $k_1 = 2k,\ k_2 = 2,\ m_1 = m + d,\ m_2 = m + kd,\ n_2 = 2km - 2m$, so that $k_1 m_1 = k_2 m_2 + n_2$. From Theorem 2 we obtain

$$F(p^{k_1 m_1}) = F(p^{m+d})f(p^{m+d})^{2k-1}\ . \tag{9}$$

From Theorem 2 and equations (3) and (4) we obtain

$$\begin{aligned} F(p^{k_2 m_2 + n_2}) &= F(p^{(2k-1)m+kd})f(p^{m+kd}) \\ &= F(p^{m+kd})f(p^m)^{2k-2}f(p^{m+kd}) \\ &= \frac{F(p^{m+d})f(p^{m+d})^{k-1}}{f(p^m)^{k-1}}f(p^m)^{2k-2}f(p^{m+kd})\ . \end{aligned}$$

Equating the last expression with (9) we obtain

$$f(p^{m+kd}) = \frac{f(p^{m+d})^k}{f(p^m)^{k-1}}\ . \tag{10}$$

For the special case $k = m/d$ we obtain from Theorem 5 and (10)

$$f(p^{2m}) = f(p^m)^2 = \frac{f(p^{m+d})^{m/d}}{f(p^m)^{m/d-1}}$$

so that $f(p^{m+d}) = f(p^m)^{1+d/m}$.
Substituting this in (10) we obtain (6).

To prove (7) we apply Theorem 2 twice obtaining

$$\begin{aligned} F(p^{(m+kd)m/d}) &= F(p^{m+kd})f(p^{m+kd})^{m/d-1} \\ &= F(p^{(m/d+k)m}) = F(p^m)f(p^m)^{m/d+k-1}\ . \end{aligned}$$

From this relation, along with (6) used twice we find

$$\begin{aligned} F(p^{m+kd}) &= \frac{F(p^m)f(p^m)^{m/d+k-1}}{f(p^{m+kd})^{m/d-1}} = \frac{F(p^m)f(p^m)^{m/d+k-1}}{f(p^m)^{m/d+k-1-kd/m}} \\ &= F(p^m)f(p^m)^{kd/m}\ . \end{aligned}$$

The other part of (7) follows from (6).

To prove the converse we substitute in (3).

3. Summary. We have reduced the problem of solving (2) to that of finding a multiplicative function F (not identically zero) that satisfies (3) for the powers of each prime p. Such a function F will exist if, and only if, one of the following holds for f and F on the powers of p:

1. There is a positive integer m such that $f(p^m) = 0$ and $F(p^n) = 0$ except possibly for the integers n in the interval $[m, 2m)$ for which $f(p^n) = 0$.

2. There is a positive integer m, a positive divisor d of m and a complex number C such that

$$f(p^{m+kd}) = f(p^m)^{1+kd/m} \neq 0 \quad (k = 0, 1, 2, \cdots)$$

and F has the defining properties

$$F(p^{m+kd}) = Cf(p^{m+kd}) \quad (k = 0, 1, 2, \cdots)$$

$F(p^n) = 0$ if $n \neq m + kd$ for some nonnegative integer k.

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