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# AN INEQUALITY FOR OPERATORS IN A HILBERT SPACE

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**Let $A$ be a self-adjoint operator on a Hilbert space $H$ satisfying $mI \leqq A \leqq MI$, $0 < m < M$. Set $q = M/m$. Let $j$ and $k$ be real numbers, $jk \neq 0, j < k$. Then**

$$(A^k x, x)^{1/k}/(A^j x, x)^{1/j} \leqq \{j^{-1}(q^j - 1)\}^{-1/k}\{k^{-1}(q^k - 1)\}^{1/j}\{(k-j)^{-1}(q^k - q^j)(x, x)\}^{(1/k)-(1/j)}$$

**for all $x \in H (x \neq 0)$. Letting $j = -1$ and $k = 1$, this inequality reduces to $(Ax, x)(A^{-1}x, x) \leqq [(M+m)^2/4mM](x, x)^2$, the well-known Kantorovich Inequality.**

**Preliminaries.** We shall make use of the following four inequalities:

For $a > 0, b > 0$,

$$a^\alpha b^{1-\alpha} \leqq \alpha a + (1-\alpha)b \qquad \text{if } 0 < \alpha < 1 \tag{1}$$

$$a^\alpha b^{1-\alpha} \geqq \alpha a + (1-\alpha)b \qquad \text{if } \alpha < 0\,. \tag{2}$$

For $j < k, 1 \leqq y \leqq q$,

$$(q^k - 1)y^j - (q^j - 1)y^k - (q^k - q^j) \geqq 0 \qquad \text{if } jk > 0 \tag{3}$$

$$-(q^k - 1)y^j + (q^j - 1)y^k + (q^k - q^j) \geqq 0 \qquad \text{if } jk < 0\,. \tag{4}$$

(1) is the well-known inequality between the arithmetic and geometric means. Simple proofs of (2), (3) and (4) can be found in a recent paper by Goldman [3].

Let $C$ be a self-adjoint operator on a Hilbert space $H$ satisfying

$$I \leqq C \leqq qI \tag{5}$$

where $I$ is the identity operator (and (5) is understood in the usual sense that $(x, x) \leqq (Cx, x) \leqq q(x, x)$ for all $x \in H$). To the real valued function $u(\lambda)$, defined and continuous on $[1, q]$, there is associated in a natural way a self-adjoint operator on $H$ denoted by $u(C)$ (see e.g. [6] pp. 265–273).

We shall make use of the following [loc. cit.]:

LEMMA. *If $u(\lambda) \geqq 0$ for $1 \leqq \lambda \leqq q$, then $u(C) \geqq 0$,* i.e., *$u(C)$ is a positive operator.*

**Results.**

THEOREM 1. *Let $C$ be a self-adjoint operator on a Hilbert space $H$ satisfying $I \leqq C \leqq qI$. Let $j$ and $k$ be real numbers, $j < k, jk \neq 0$. The operator*

$$(q^k - 1)C^j - (q^j - 1)C^k - (q^k - q^j)I \tag{6}$$

*is positive if $jk > 0$; while the operator*

$$-(q^k - 1)C^j + (q^j - 1)C^k + (q^k - q^j)I \tag{7}$$

*is positive if $jk < 0$.*

*Proof.* The theorem follows directly from (3) and (4) by virtue of the Lemma.

Letting $j = -1$ and $k = 1$, Theorem 1 yields an inequality that is equivalent to one given by Diaz and Metcalf [2].

The following theorem, which is the main result of this paper, is a Hilbert space generalization of Cargo and Shisha [1] and Mond [5].

THEOREM 2. *Let $A$ be self-adjoint operator on a Hilbert space $H$ satisfying $mI \leqq A \leqq MI, 0 < m < M$. Set $q = M/m$. Let $j$ and $k$ be real numbers $jk \neq 0, j < k$. Then*

$$\begin{aligned}(8)\qquad &(A^k x, x)^{1/k}/(A^j x, x)^{1/j}\\ &\quad \leqq \{j^{-1}(q^j - 1)\}^{-1/k}\{k^{-1}(q^k - 1)\}^{1/j}\{(k-j)^{-1}(q^k - q^j)(x, x)\}^{(1/k)-(1/j)}\end{aligned}$$

*for all $x \in H (x \neq 0)$.*

*Proof.* Set $C \equiv A/m$. It obviously suffices to prove

$$\begin{aligned}(9)\qquad &(C^k x, x)^{1/k}/(C^j x, x)^{1/j}\\ &\quad \leqq \{j^{-1}(q^j - 1)\}^{-1/k}\{k^{-1}(q^k - 1)\}^{1/j}\{(k-j)^{-1}(q^k - q^j)(x, x)\}^{(1/k)-(1/j)}\,.\end{aligned}$$

Since $C$ satisfies (5), by Theorem 1,

$$(q^k - 1)(C^j x, x) - (q^j - 1)(C^k x, x) \geqq (q^k - q^j)(x, x) \qquad \text{if } jk > 0 \tag{10}$$

and

$$(q^k - 1)(C^j x, x) - (q^j - 1)(C^k x, x) \leqq (q^k - q^j)(x, x) \qquad \text{if } jk < 0\,. \tag{11}$$

Rewrite (10) as

$$\begin{aligned}(12)\qquad &\{-j(k-j)^{-1}\}\{j^{-1}(q^j - 1)(C^k x, x)\} + \{k(k-j)^{-1}\}\{k^{-1}(q^k - 1)(C^j x, x)\}\\ &\quad \geqq (k-j)^{-1}(q^k - q^j)(x, x)\end{aligned}$$

if $jk > 0$, and (11) as

$$
\{-j(k-j)^{-1}\}\{j^{-1}(q^j-1)(C^kx,x)\}+\{k(k-j)^{-1}\}\{k^{-1}(q^k-1)(C^jx,x)\} \leqq (k-j)^{-1}(q^k-q^j)(x,x) \tag{13}
$$

if $jk < 0$.

Assume $k > 0$. Set

$$
a = j^{-1}(q^j-1)(C^kx,x),\ b = k^{-1}(q^k-1)(C^jx,x),\ \alpha = -j(k-j)^{-1}.
$$

If $j > 0$, applying (2) and combining with (12), we obtain

$$
\{j^{-1}(q^j-1)(C^kx,x)\}^{-j/(k-j)}\{k^{-1}(q^k-1)(C^jx,x)\}^{k/(k-j)} \geqq (k-j)^{-1}(q^k-q^j)(x,x) \tag{14}
$$

which when raised to the power $(k-j)/(-kj)$ yields

$$
\{j^{-1}(q^j-1)(C^kx,x)\}^{1/k}\{k^{-1}(q^k-1)(C^jx,x)\}^{-1/j} \leqq \{(k-j)^{-1}(q^k-q^j)(x,x)\}^{(1/k)-(1/j)}. \tag{15}
$$

If $j < 0$ $(k > 0)$, applying (1) and combining with (13) yields the reverse of (14) which, when raised to the power $(k-j)/(-kj)$, yields (15).

Finally, if $j < k < 0$, set

$$
a = k^{-1}(q^k-1)(C^jx,x),\ b = j^{-1}(q^j-1)(C^kx,x),\ \alpha = k(k-j)^{-1}.
$$

Applying (2) and combining with (12) yields (14) which, when raised to the power $(k-j)/(-kj)$ yields (15). In all cases, therefore, we have (15), a rearrangement of (9). (Compare the method of proof of Theorem 2 with Goldman [3].)

The well-known [4] Kantorovich inequality, $(Ax,x)(A^{-1}x,x) \leqq [(m+M)^2/4mM](x,x)^2$, is the special case of Theorem 2 with $j=-1$, $k=1$.

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