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EVERY ABELIAN GROUP IS A CLASS GROUP

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Let T be the set of minimal primes of a Krull domain A. If S is a subset of T, we form $B = \cap A_P$ for $P \in S$ and study the relation of the class group of B to that of A. We find that the class group of B is always a homomorphic image of that of A. We use this type of construction to obtain a Krull domain with specified class group and then alter such a Krull domain to obtain a Dedekind domain with the same class group.

Let A be a Krull domain with quotient field K. Thus A is an intersection of rank 1 discrete valuation rings; and if $x \in K$, x is a unit in all but a finite number of these valuation rings. If P is a minimal prime ideal of A, then A_P is a rank 1 discrete valuation ring and must occur in any intersection displaying A as a Krull domain. In fact, if T denotes the set of minimal prime ideals of A, then $A = \bigcap_{P \in T} A_P$ displays A as a Krull domain.

Choose a subset S of T ($S \neq \emptyset$) and form the domain $B = \bigcap_{P \in S} A_p$. It is immediate that B is also a Krull domain which contains A and has quotient field K. If one of the A_P were eliminable from the intersection representing B, it would also be eliminable from that representing A. Thus the A_P for $P \in S$ are exactly the rings of the type B_Q, where Q is a minimal prime ideal of B. If Q is minimal prime ideal of B, then $Q \cap A = P$ for the $P \in S$ such that $B_Q = A_P$.

Let A and B be generic labels throughout this paper for a Krull domain A and a Krull domain B formed from A as above. We recall that the valuation rings A_P are called the essential valuation rings, and we will denote by V_P the valuation of A going with A_P. We summarize and add a complement to the above.

PROPOSITION 1. *With A and B as above, B is a Krull domain containing A, and the A_P for $P \in S$ are the essential valuation rings of B. Every ring B is of the form A_M for some multiplicative set M if and only if the class group of A is torsion.*

Proof. Everything in the first assertion has been given above.

Suppose the class group of A is torsion; then for each Q_i in $T - S$ choose an integer n_i such that $Q_i^{(n_i)}$ is principal, say $Q_i^{(n_i)} = As_i$. Let M be the multiplicative set generated by all s_i. Then [3, 33.5,

p. 116] shows that $B = A_M$. On the other hand, if Q is a prime ideal of A whose class is not torsion, then the same reference shows that $B = \bigcap_{P \neq Q} A_P$ cannot be of the form A_M.

Let $C(A)$ denote the class group of A for any Krull domain A. Samuel [4] has shown how to define a homomorphism of $C(A)$ into $C(B)$ when B is a Krull domain such that $A \subseteqq B$ and B is A-flat. Of course, our rings B are not necessarily A-flat, but we have nonetheless:

PROPOSITION 2. $C(B)$ is a homomorphic image of $C(A)$.

Proof. Let I be an ideal of A defined by essential valuation conditions. Although IB may not be defined by essential valuation conditions, $B\colon (B\colon IB)$ is and is quasi-equal to IB [5, p. 92]. If $P(A)$ and $P(B)$ denote the ideals of A and B defined by essential valuation conditions, let $g\colon P(A) \rightarrow P(B)$ be defined by $g(I) = B\colon (B\colon IB)$. It is easy to see that if I is the ideal $\{x\colon V_P(x) \geqq n_P;\ P \in T\}$, then $g(I)$ is the ideal $\{x\colon V_P(x) \geqq n_P,\ P \in S\}$. Since the product $I \circ I'$ (see [4]) is $A\colon (A\colon II')$, the above description yields immediately that $g(I \circ I') = g(I)^{\circ} g(I')$. If $x \in A$, then $g(xA) = B\colon (B\colon xB) = xB$, so g induces a homomorphism $\bar{g}\colon C(A) \rightarrow C(B)$ which is obviously onto.

COROLLARY 3. *With $\bar{g}$ as defined in Proposition 2, the kernel of $\bar{g}$ is the subgroup of $C(A)$ generated by all minimal primes Q of A for which $Q \notin S$.*

Proof. If $Q \notin S$, then $g(Q) = B$. If on the other hand $g(I)$ is principal, we have $g(I) = xB$ or $g(x^{-1}I) = B$. Thus $x^{-1}I$ is in the subgroup of $P(A)$ generated by certain $Q \notin S$.

In the next two propositions, we generalize to Krull domains certain results of [1] and [2].

PROPOSITION 4. If A is a Krull domain, then $A[X]$ is a Krull domain. $C(A)$ is isomorphic to $C(A[X])$, and every class of $C(A[X])$ contains a prime ideal of $A[X]$.

Proof. Everything but the last assertion is [4, Prop. 3., p. 158]. Let c be an element of $C(A[X])$; then c^{-1} can be represented by $IA[X]$, where I is an integral ideal of A defined by essential valuation conditions. Choose a_0 and a_1 in A so that I is quasi-equal to (a_0, a_1) [5, Exer. 4., p. 95]. Consider the prime ideal $P = (a_0 + a_1X)\, K[X] \cap A[X]$. It is clear that $P = I^{-1}A[X] \cdot (a_0 + a_1X)A[X]$ [6, p. 85]. So P is in c.

PROPOSITION 5. If G is the class group of a Krull domain and G' is a homomorphic image of G, then there is a Krull domain with class group G'.

Proof. Let A have class group G and let H be a subgroup of G such that $G' \cong G/H$. G is also the class group of $A[X]$; choose (Proposition 4) a minimal prime P_α of $A[X]$ representing each class c_α in H. Let T be the set of all minimal primes of $A[X]$ and let U be the set of primes $\{P_\alpha\}$. Then $B = \bigcap_{P \in T-U} A[X]_P$ has class group G' by Corollary 3.

PROPOSITION 6. If G is any abelian group, then there is a Krull domain A such that $C(A) \cong G$.

Proof. In view of Proposition 5, it is sufficient to show that there is a Krull domain whose class group is a free group on a base of given cardinality. We do so as follows:

Let J be any index set and form the polynomial ring $B = F[X_1, Y_1, Z_1, \cdots, X_i, Y_i, Z_i, \cdots]$ for $i \in J$. For each i, consider the subring $R_i = F[\cdots, X_i, Y_i, W_i, \cdots,]$ where $W_i = X_i Z_i$. Let Q_i be the ideal (X_i, Y_i) in R_i and assign an order to any element r of R_i by $v_i(r) = t$ if $r \in Q^t$ and $r \notin Q^{t+1}$. It is immediate that v_i satisfies the requirements of a valuation, and so v_i may be extended uniquely to a discrete valuation on the quotient field of R_i(= the quotient field of B).

Let V_i denote the valuation ring of v_i for all $i \in J$. Form $A = (\bigcap_{i \in J} V_i) \cap B$. We assert that A is a Krull domain, and that $C(A)$ is the free group on J.

We note first that since $A \supseteqq R_i$ for any $i \in J$, the quotient field of A is the same as the quotient field of B. Since B is a U. F. D., we can write $B = \cap B_P$ for P a minimal prime of B; this shows that A is the intersection of discrete valuation rings. If $f \in A$, f involves only a finite number of the variables, and so f can be a nonunit in only a finite number of the $\{B_P\} \cup \{V_i\}$. The set $\Sigma = \{B_P\} \cup \{V_i\}$ is in fact, the set of essential valuation rings of A. To see this, we need only produce an element of the quotient field which is not in a particular ring of Σ but is in all the other rings of Σ. For V_i, Z_i will serve. If P is a minimal prime of B and $P = X_i B$ for some $i \in J$, Y_i/X_i demonstrates that B_P is essential.

Finally, let P be a minimal prime of B not of the above type and choose an $f \in B$ such that $P = fB$. f will be a unit in any other valuation ring of Σ of the type B_Q, so let $V_{i_1}, \cdots, V_{i_k}$ be the valuation rings of Σ in which f is not a unit. Let $n_{i_j} = v_{i_j}(f)$ for $j = 1, \cdots, k$. The element $g = X_{i_1}^{\max(0, n_{i_1})} \cdots X_{i_k}^{\max(0, n_{i_k})}/f$ yields that B_P is essential in this case.

Let P be a minimal prime of B, and choose $f \in B$ such that $P = fB$. As above, f is a unit except in B_P and some rings V_i for $i \in J$. This shows that the minimal primes going with the V_i generate $C(A)$. A relation among these minimal primes alone would come from an element f of the quotient field of A which is a unit in all B_P, i.e., a unit of B. But the units of B are the elements of the field F, and this relation can only be the trivial one.

REMARK. It is fairly easy to see that the restriction of the ring A constructed above to $F[X_i, Y_i, Z_i]$ is $F[X_i, Y_i, X_iZ_i, Y_iZ_i]$; this leads to an alternative description of A as $F[\cdots, X_i, Y_i, T_i, U_i \cdots]$ subject to the relations $X_iU_i = Y_iT_i$. Indeed, the results on A may be obtained by viewing A again as a subring of $F[\cdots, X_i, Y_i, Z_i, \cdots]$ where $Z_i = T_i/X_i = U_i/Y_i$. I am indebted to the referee for suggesting this point of view on the example.

THEOREM 7. *Given any abelian group S, there is a Dedekind domain D with $C(D) \cong S$.*

Proof. We show that if A is a Krull domain with class group S, we can alter A to obtain a Dedekind domain with the same class group.

Let N denote the natural numbers and set $A_1 = A[X_1, \cdots, X_n, \cdots]$ for $n \in N$. Let Q be a prime ideal of A, which is not minimal. Choose any element a of Q and let $P_1, \cdots, P_k$ be the minimal primes of A_1 which contain a. Since $Q \not\subseteq P_1 \cup \cdots \cup P_k$ we can find b in Q such that $b \notin P_i$ for $i = 1, \cdots, k$. Let X_Q be a variable not occuring in either a or b and form $f_Q = a + bX_Q$. Then f_Q is prime in A_1 [6, Th. 29, p. 85]. Let M be the multiplicative set generated by all f_Q, where Q ranges over the nonminimal primes of A_1. Let $D = (A_1)_M$. D is a Krull domain in which minimal primes are also maximal, so D is a Dedekind domain [6, Th. 28, p. 84]. Further $C(A) \cong C(A_1) \cong C(D)$, the latter isomorphism following from [4, Prop. 2, p. 157].

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