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TAME SUBSETS OF SPHERES IN E^3

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Let F be a closed subset of a 2-sphere S in E^3. We define F to be tame if F lies on some tame 2-sphere in E^3. The sets F and S satisfy Property $(*, F, S)$ provided Bing's Side Approximation Theorem can be applied in such a way that the approximating 2-sphere S' misses F (that is, $S \cap S'$ lies in a finite collection of disjoint small disks in $S - F$). In this paper we show that Property $(*, F, S)$ implies that F is tame by establishing a conjecture made by Gillman. Other properties which are equivalent to Property $(*, F, S)$ are also given.

If $F_1, F_2, \cdots, F_n$ is a finite collection of closed subsets of S such that Property $(*, F_i, S)$ holds for each i, then Property $(*, \sum F_i, S)$ also holds. We use this result to show that if S is locally tame modulo $\sum F_i$, then S is tame.

Bing's Side Approximation Theorem [8, Theorem 16] can be stated as follows:

THEOREM 0. *If S is a 2-sphere in E^3, V is a component of $E^3 - S$, and $\varepsilon > 0$, then there is a polyhedral 2-sphere S' containing a finite collection $D_1, D_2, \cdots, D_n$ of disjoint disks each of diameter less than ε, and there is a finite collection $E_1, E_2, \cdots, E_r$ of disjoint disks on S, each of diameter less than ε, such that*

1. *there is a homeomorphism of S onto S' that moves no point as much as ε,*
2. $S' - \sum_{i=1}^{n} D_i \subset V$, *and*
3. $S \cap S' \subset \sum_{i=1}^{r} E_i$.

If F is a closed subset of the 2-sphere S and V is a component of $E^3 - S$, we define *Property* $(*, F, V)$ to mean that Theorem 0 can be applied relative to S and V with the additional requirement that

4. $(\sum E_i) \cap F = \varnothing$.

Property $(*, F, S)$ is satisfied if Property $(*, F, V)$ holds for each component V of $E^3 - S$.

Gillman has already established that an arc A is tame if A lies on a 2-sphere S and Property $(*, A, S)$ is satisfied; however, he comments that the "natural approach" to the problem requires a certain conjecture which he states and does not prove [13, p. 467]. Theorem 3 establishes this conjecture, and Theorem 6 shows that an arbitrary closed set F on S is tame if Property $(*, F, S)$ holds.

Hosay has announced two sufficient conditions for a closed subset

F of a 2-sphere S to be tame [14]. In the statement of his results he requires that the set of diameters of the components of F be bounded below by a positive number. Under this restriction on the size of the components of F we show that each of Hosay's sufficient conditions is equivalent to Property $(*, F, S)$; hence, Hosay's result follows from Theorem 6. For these results, see § 3.

If $(*, F, S)$ is satisfied it follows from Theorem 1 that there is a nondegenerate continuum M on S such that $F \subset M$ and $(*, M, S)$ holds. Using Theorem 1, Theorem 8, and Hosay's result [14], we see that $(*, F, S)$ implies that F is tame. Working independently, Ernest Milton also made this observation and established a theorem similar to Theorem 1. His work has not yet been published. As mentioned above, we do not depend on Hosay's result in this paper.

Theorems 17 and 18 are generalizations of Theorems 8.4 and 8.5 of [9]. We use these generalizations to show that the union F of a finite collection of closed subsets $F_1, F_2, \cdots, F_n$ of a 2-sphere S satisfies $(*, F, S)$ provided Property $(*, F_i, S)$ holds for each i (Theorem 21). It follows from this result and Theorem 15 that S is tame if S is locally tame modulo the union of a finite collection of closed subsets F_i where $(*, F_i, S)$ holds for each i. We also use some of the results of Sections 3, 4, and 5 in another paper [16] where we give some conditions under which a 2-sphere is tame in E^3.

We use the prefix in "ε-disk", "ε-set", etc., to imply that the point set in question has diameter less than ε. However, the prefix in "2-sphere" refers to the dimension of the sphere. It should be clear in which context the prefix is to be taken. The distance function in E^3 is denoted by "ρ". The symbol "$N(R, \varepsilon)$", where $\varepsilon > 0$ and R is a set, is synonymous with "ε-neighborhood of R" and is defined as the set of all points that are within a distance ε of some point of R. We indicate that a point set R has diameter less than ε by writing "diam $R < \varepsilon$". If $\{D_i\}$ is a countable sequence of disks such that $\lim \operatorname{diam} D_i = 0$, we call $\{D_i\}$ a *null sequence* of disks. The closure of a set R is denoted by $\mathrm{Cl}(R)$. If S is a 2-sphere we denote the bounded and unbounded components of $E^3 - S$ by $\operatorname{Int} S$ and $\operatorname{Ext} S$, respectively. If D is a disk we let $\operatorname{Int} D = D - \operatorname{Bd} D$, where $\operatorname{Bd} D$ is the boundary of D.

Most of the definitions used here will be found in either [4] or [10]; however, we will review some of them briefly. A 2-sphere S in E^3 is *tame* if there is a homeomorphism h of E^3 onto itself such that $h(S)$ is polyhedral. The set Y is *locally simply connected* at a point p of $\mathrm{Cl}(Y)$ if for each neighborhood N of p there is an open set U containing p such that each map of a simple closed curve into $U \cap Y$ can be shrunk to a point in $N \cap Y$. A 2-sphere S is said to be *tame from a complementary domain* V of S if $S + V$ is a

3-manifold with boundary. The sphere S is *locally tame at a point p from V* if p is in S and p lies in a subset U of $S + V$ such that U is open relative to $S + V$ and $\mathrm{Cl}(U)$ is a topological cube.

2. **(*, F, S) implies that F is tame.** We now give an alternative definition of Property $(*, F, S)$.

Property $(*, F, S)$. Let F be a closed subset of a 2-sphere S in E^3. We say that Property $(*, F, \mathrm{Ext}\, S)$ holds if and only if for each $\varepsilon > 0$ there is a polyhedral 2-sphere S' containing a finite collection of disjoint ε-disks $D_1, D_2, \cdots, D_n$ and there is a finite collection of disjoint ε-disks $E_1, E_2, \cdots, E_r$ on S such that

1. there is a homeomorphism of S onto S' which moves no point as much as ε,
2. $S' - \sum D_i \subset \mathrm{Ext}\, S$,
3. $S - \sum E_i \subset \mathrm{Int}\, S'$, and
4. $F \cap (\sum E_i) = \varnothing$.

A similar definition is made for $(*, F, \mathrm{Int}\, S)$, and we define $(*, F, S)$ to mean that both $(*, F, \mathrm{Int}\, S)$ and $(*, F, \mathrm{Ext}\, S)$ hold.

The property defined by $(*, F, S)$ in the introduction follows directly from this definition. The converse is also true as can be seen using the technique illustrated in [6, p. 585]. Since the two definitions of Property $(*, F, S)$ are equivalent we will use whichever definition seems appropriate.

Although we show in this section that Property $(*, F, S)$ implies that F is tame, it is not true that S is locally tame at points of F if Property $(*, F, S)$ holds. To see this, let F be an arc on the 2-sphere S described in [5]. Then F is tame, so Property $(*, F, S)$ holds [13, Theorem 10]. Yet S is not locally tame at any point.

THEOREM 1. *If $\varepsilon > 0$ and F is a closed subset of a 2-sphere S in E^3 such that $(*, F, S)$ is satisfied, then there is a continuum M on S and a null sequence $\{D_i\}$ of disjoint ε-disks on S such that*

1. $M = S - \sum \mathrm{Int}\, D_i$,
2. $(*, M, S)$ *is satisfied, and*
3. $F \subset M - \sum D_i = S - \sum D_i$.

Proof. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots$ be a sequence of positive numbers (subject to restrictions to be mentioned later), and let $S_1, S_2, S_3, \cdots$ be a sequence of polyhedral 2-spheres, where S_i is obtained relative to ε_i using Property $(*, F, S)$, such that for each i

(1) S_i is homeomorphically within ε_i of S,

(2) S contains a finite collection of disjoint ε_i-disks $E_{i1}, E_{i2}, \cdots, E_{in(i)}$

such that $\left(\sum_{j=1}^{n(i)} E_{ij}\right) \cap F = \varnothing$,

(3) S_i contains a finite collection of disjoint ε_i-disks $D_{i1}, D_{i2}, \cdots, D_{ir(i)}$ such that

(a) $S_i - \sum_{j=1}^{r(i)} D_{ij} \subset \operatorname{Int} S$ and $S - \sum_{j=1}^{n(i)} E_{ij} \subset \operatorname{Ext} S_i$ if i is an odd integer, and

(b) $S_i - \sum_{j=1}^{r(i)} D_{ij} \subset \operatorname{Ext} S$ and $S - \sum_{j=1}^{n(i)} E_{ij} \subset \operatorname{Int} S_i$ if i is an even integer,

We will impose restrictions on the ε_i's to insure that $S - \sum_{i=1}^{\infty} \sum_{j=1}^{n(i)} E_{ij}$ contains a continuum M which satisfies the requirements of Theorem 1. First we insist that $\varepsilon_i < \varepsilon/i$ for each i. We need an inductive procedure for defining the remaining restrictions to be imposed on the ε_i's. In this inductive procedure we also define an array of ε-disks

$$
\begin{aligned}
&G_{11} G_{12} \cdots G_{1t(1)} \\
&G_{21} G_{22} \cdots G_{2t(1)} \cdots G_{2t(2)} \\
&G_{31} G_{32} \cdots G_{3t(1)} \cdots G_{3t(2)} \cdots G_{3t(3)} \\
&\cdots
\end{aligned}
$$

where the k^{th} row is a finite collection of disjoint ε-disks on S such that $\sum_{i=1}^{n(k)} E_{ki} \subset \sum_{i=1}^{t(k)} \operatorname{Int} G_{ki}$; $G_{k+1i} \subset G_{ki}$ for $k = 1, 2, \cdots$ and $1 \leqq i \leqq t(k)$ (that is, the columns are nested); and no G_{kj} intersects F. We will also do the construction so that for each fixed row k, diam $G_{kj} < \varepsilon_f$ if $t(f-1) < j \leqq t(f)$ (where $t(0)=0$ and $1 \leqq f \leqq t(k)$).

Assuming we have defined such an array of disks, we let A_j be the intersection of the j^{th} column; that is $A_j = \bigcap_{i=1}^{\infty} G_{ij}$ (where we let $G_{ij} = S = \operatorname{Int} G_{ij}$ $j > t(i)$). Since A_j is the intersection of a nested collection of disks it follows that A_j is a continuum. In fact we have defined a sequence of disjoint continua $A_1, A_2, A_3, \cdots$ on S such that for each i

(4) A_i does not separate S,

(5) $\operatorname{diam} A_i < \varepsilon$,

(6) $\lim \operatorname{diam} A_i = 0$, and

(7) $A_i \cap F = \varnothing$.

In the inductive definition of the G_{ij}'s we will insure that

(8) if E_{mf} (where m is a positive integer and $1 \leqq f \leqq n(m)$) lies in G_{mk} (where k is a positive integer such that $1 \leqq k \leqq t(m)$), then $E_{mf} \subset \bigcap_{i=1}^{\infty} G_{ik} = A_k$.

Condition (8) implies that

$$(9) \qquad \sum_{i=1}^{\infty} \sum_{j=1}^{n(i)} E_{ij} \subset \sum_{i=1}^{\infty} A_i .$$

Assuming that all these conditions are satisfied, let us show how the proof can be completed. From (6) the decomposition G of S whose only nondegenerate elements are the A_i's is upper semicontinuous. It follows from (4) and [18] that the decomposition space is a 2-sphere. Since there are only a countable number of nondegenerate elements in G, the image of these nondegenerate elements forms a countable point set in the decomposition space. Also from (7) this countable set fails to intersect the image of F in the decomposition space. Using (5) and (6) we can find an infinite collection of disks in the decomposition space such that the pre-images of these disks form a null sequence of disjoint ε-disks $D_1, D_2, D_3, \cdots$ on S, where no D_i intersects F and $\sum_{i=1}^{\infty} A_i \subset \sum_{i=1}^{\infty} \operatorname{Int} D_i$. The continuum M which is required in the conclusion of the statement of Theorem 1 is $S - \sum_{i=1}^{\infty} \operatorname{Int} D_i$.

Obviously M contains F. Let α be a positive number, and let V be a complementary domain of S. Since the ε_i's converge to zero we can find an odd integer x and an even integer y such that ε_x and ε_y are each less than α. If $V = \operatorname{Int} S$, then S_x will satisfy the conditions of $(*, M, \operatorname{Int} S)$ relative to α. If $V = \operatorname{Ext} S$, then S_y will satisfy the conditions of $(*, M, \operatorname{Ext} S)$ relative to α. For example, consider $V = \operatorname{Int} S$. Then S_x satisfies Conditions (1), (2), and (3a) as stated in the first paragraph of this proof. All we need to show is that M does not intersect $(\sum_{j=1}^{n(x)} E_{xj})$. But this follows immediately from (9) and the fact that the A_i's lie in the union of the interiors of the D_j's. Hence we have $(*, M, S)$.

Now let us show how the induction is carried out to define the ε_i's and the infinite array of G_{ij}'s. We indicate the inductive procedure by illustrating the first three steps.

Step 1. No further restriction is placed on ε_1, so we let S_1 be a polyhedral 2-sphere satisfying (1), (2), and (3a). There is a finite collection of disjoint ε_1-disks $G_{11}, G_{12}, \cdots, G_{1n(1)}$ so that $E_{1i} \subset \operatorname{Int} G_{1i}$ and $G_{1i} \cap F = \varnothing$. For convenience in our inductive procedure we let $n(1) = t(1)$, and we let $H_{1i} = E_{1i}$ for $1 \leqq i \leqq t(1)$. We choose ε_2 to be less than $\rho(H_{1i}, S - G_{1i})$ for $1 \leqq i \leqq t(1)$. Then S_2 is defined to satisfy (1), (2), and (3b).

Step 2. Let $K_{21}, K_{22}, \cdots, K_{2t(1)}$ be the components of $(\sum_{i=1}^{t(1)} H_{1i} + \sum_{i=1}^{n(2)} E_{2i})$ such that $H_{1i} \subset K_{2i}$. From our choice of ε_2, $K_{2i} \subset \operatorname{Int} G_{1i}$. Let H_{2i} be K_{2i} plus the sum of all components of $S - K_{2i}$ which lie in G_{1i}, for $1 \leqq i \leqq t(1)$. Then

$$E_{1i} = H_{1i} \subset K_{2i} \subset H_{2i} \subset \operatorname{Int} G_{1i} , \quad \text{for} \quad 1 \leqq i \leqq t(1) . \tag{10}$$

For each i such that $1 \leqq i \leqq t(1)$, we let G_{2i} be a disk in $\operatorname{Int} G_{1i}$ such that $G_{2i} \cap (\sum_{i=1}^{t(1)} H_{1i} + \sum_{i=1}^{n(2)} E_{2i}) = H_{2i} \subset \operatorname{Int} G_{2i}$. Let $t(2)$ be a non-negative integer so that exactly $t(2) - t(1)$ of the E_{2i}'s are not covered by $\sum_1^{t(1)} G_{2i}$. We define $H_{2\,t(1)+1}, \cdots, H_{2\,t(2)}$ to be these uncovered E_{2i}'s. Now expand each $H_{2i}(t(1) < i \leqq t(2))$ slightly to obtain a collection of disjoint ε_2-disks $G_{2\,t(1)+1}, \cdots, G_{2\,t(2)}$ covering $\sum_{t(1)+1}^{t(2)} H_{2i}$ such that no G_{2i} intersects F. The G_{2i}'s are selected so that $G_{21}, G_{22}, \cdots, G_{2\,t(2)}$ is a finite collection of disjoint ε-disks satisfying

$$\left(\sum_{i=1}^{n(1)} E_{1i} + \sum_{i=1}^{n(2)} E_{2i}\right) \subset \sum_{i=1}^{t(2)} \operatorname{Int} G_{2i} , \tag{11}$$

$$H_{2i} \subset \operatorname{Int} G_{2i} \subset G_{2i} \subset \operatorname{Int} G_{1i} , \quad \text{for} \quad 1 \leqq i \leqq t(2) , \tag{12}$$

$$\operatorname{diam} G_{2i} < \varepsilon_2 \quad \text{for} \quad t(1) < i \leqq t(2) , \tag{13}$$

$$\left(\sum_{i=1}^{t(2)} G_{2i}\right) \cap F = \varnothing , \quad \text{and} \tag{14}$$

$$G_{2i} \cap \left(\sum_{i=1}^{t(1)} H_{1i} + \sum_{i=1}^{n(2)} E_{2i}\right) = H_{2i} , \quad \text{for} \quad 1 \leqq i \leqq t(2) . \tag{15}$$

It follows from (12) that

$$\operatorname{diam} G_{2i} < \varepsilon_1 \text{ if } 1 \leqq i \leqq t(1) . \tag{16}$$

Choose ε_3 less than $\rho(H_{2i}, S - G_{2i})$ for $1 \leqq i \leqq t(2)$. Then S_3 is chosen relative to ε_3 so that S_3 satisfies (1), (2), and (3a).

Step 3. Let $K_{31}, K_{32}, \cdots, K_{3t(2)}$ be the components of $(\sum_{i=1}^{t(2)} H_{2i} + \sum_{i=1}^{n(3)} E_{3i})$ such that $H_{2i} \subset K_{3i}$. From the definition of ε_3, $K_{3i} \subset \operatorname{Int} G_{2i}$. Define H_{3i} to be K_{3i} plus the sum of the components of $S - K_{3i}$ which lie in $\operatorname{Int} G_{2i}$ $(1 \leqq i \leqq t(2))$. Then

$$H_{2i} \subset K_{3i} \subset H_{3i} \subset \operatorname{Int} G_{2i} \quad \text{for} \quad 1 \leqq i \leqq t(2) . \tag{17}$$

For each i such that $1 \leqq i \leqq t(2)$, we let G_{3i} be a disk in $\operatorname{Int} G_{2i}$ such that $G_{3i} \cap (\sum_{i=1}^{t(2)} H_{2i} + \sum_{i=1}^{n(3)} E_{3i}) = H_{3i} \subset \operatorname{Int} G_{3i}$. Let $t(3)$ be a non-negative integer so that there are exactly $t(3) - t(2)$ of the E_{3i}'s which are not covered by $\sum_{i=1}^{t(2)} G_{3i}$, and let $H_{3\,t(2)+1}, \cdots, H_{3\,t(3)}$ be these uncovered E_{3i}'s. Now we expand each $H_{3i}(t(2) < i \leqq t(3))$ so slightly that we obtain a collection of disjoint ε_3-disks $G_{3\,t(2)+1}, \cdots, G_{3\,t(3)}$ covering $\sum_{i=t(2)+1}^{t(3)} H_{3i}$ such that no G_{3i} intersects F and no $G_{3i}(t(2) < i \leqq t(3))$ intersects a $G_{3j}(1 \leqq j \leqq t(2))$. The collection $G_{31}, G_{32}, \cdots, G_{3t(3)}$ can be selected to form a disjoint set of ε-disks satisfying

$$\left(\sum_{i=1}^{n(1)} E_{1i} + \sum_{i=1}^{n(2)} E_{2i} + \sum_{i=1}^{n(3)} E_{3i}\right) \subset \sum_{i=1}^{t(3)} \operatorname{Int} G_{3i} , \tag{18}$$

$$H_{3i} \subset \operatorname{Int} G_{3i} \subset G_{3i} \subset \operatorname{Int} G_{2i} \quad \text{for} \quad 1 \leqq i \leqq t(3) , \tag{19}$$

$$\operatorname{diam} G_{3i} < \varepsilon_3 , \quad \text{for} \quad t(2) < i \leqq t(3) , \tag{20}$$

$$\left(\sum_{i=1}^{t(3)} G_{3i}\right) \cap F = \varnothing , \quad \text{and} \tag{21}$$

$$G_{3i} \cap \left(\sum_{i=1}^{t(2)} H_{2i} + \sum_{i=1}^{n(3)} E_{3i}\right) = H_{3i} \quad \text{for} \quad 1 \leqq i \leqq t(3) . \tag{22}$$

From (13), (16), and (19) it follows that

$$\operatorname{diam} G_{3i} < \varepsilon_2 \text{ if } t(1) < i \leqq t(2) , \tag{23}$$

and

$$\operatorname{diam} G_{3i} < \varepsilon_1 \text{ if } 1 \leqq i \leqq t(1) .$$

Choose ε_4 less than $\rho(H_{3i}, S - G_{3i})$ for $1 \leqq i \leqq t(3)$, and let S_4 be defined to satisfy Conditions (1), (2), and (3b). Now we are ready to proceed with Step 4 where ε_5 is defined.

We assume that the inductive process is completed, so we have defined an array of G_{ij}'s which satisfies the conditions required at the beginning of the proof. Perhaps we should elaborate on the reason that Condition (8) is satisfied. From the inductive procedure, as illustrated by Conditions (10) and (17), we see that $H_{1j} \subset H_{2j} \subset H_{3j} \cdots$ for each j, provided we let $H_{ij} = \varnothing$ if H_{ij} is not yet defined. Also from Conditions (12) and (19), carried through the inductive process, we have $\sum_{i=1}^{\infty} H_{ij} \subset \bigcap_{i=1}^{\infty} G_{ij} = A_j$. Suppose now that E_{mf} lies in G_{mk} (see (8)). Then the intersection of G_{mk} with $(\sum_{i=1}^{t(m-1)} H_{m-1\,i} + \sum_{i=1}^{n(m)} E_{mi})$ is H_{mk} (see Condition (22)), so E_{mf} lies in H_{mk}. Since $\sum_{i=1}^{\infty} H_{ik} \subset \bigcap_{i=1}^{\infty} G_{ik}$, Condition (8) holds.

The other conditions on the array of G_{ij}'s are easily verified, so the proof is complete.

THEOREM 2. *If U is an open subset of a 2-sphere S in E^3 and F is a closed set in U such that $(*, F, S)$ is satisfied, then there is a closed set F' containing F such that*

1. *F' is a subset of U,*
2. *$(*, F', S)$ is satisfied, and*
3. *the set of diameters of the components of F' has a positive lower bound.*

Proof. Let V be an open set whose closure lies in U such that F lies in V. Choose a positive number ε less than $\rho(F, \operatorname{Bd} V)$. From Theorem 1, F lies in a nondegenerate continuum M in S such that $(*, M, S)$ is satisfied. But M might not lie in U so we suppose that

M intersects Bd U.

Let Q be the set of all components of $M \cap \mathrm{Cl}(V)$ which intersect F. For each element C in Q, $\operatorname{diam} C > \varepsilon$ since C intersects both F and Bd V. Let F' be the sum of the elements of Q. Then F' satisfies Conditions 1, 2, and 3 of Theorem 2. We need only show that F' is closed.

Suppose F' is not closed and let $p \in \mathrm{Cl}(F') - F'$. Let L be the component of $M \cap \mathrm{Cl}(V)$ which contains p. Then L does not intersect F. There is a sequence of points $\{p_i\}$ converging to p such that each p_i lies in some C_i in Q. It follows that $\limsup C_i$ is a subcontinuum L' of L which contains p. Then L' fails to intersect F, so we let N be an open set containing L' which does not intersect F. For each i, we can find a point q_i in $C_i - N$, since otherwise some C_i would lie in N and consequently N would intersect F. Then the set of q_i's has a limit point q in $S - N$. The point q must be in L' by the definition of L'. Hence we have a contradiction. Then F' is closed.

The following lemma, which we state without proof, has been used by Gillman in the proof of Theorem 2 in [13].

LEMMA 1. *If S is a 2-sphere in E^3 and $\varepsilon > 0$, then there is a positive number δ so that if f is a homeomorphism of S which moves no point more than δ, then any δ-subset of $f(S)$ lies in a disk in $f(S)$ of diameter less than ε.*

THEOREM 3. *If U is an open subset of a 2-sphere R in E^3, F is a closed subset of U such that $(*, F, R)$ is satisfied, and S is a 2-sphere in E^3 containing U, then $(*, F, S)$ is also satisfied.*

Proof. Let α be a positive number, and let V be a complementary domain of S. We will construct a 2-sphere $h(S)$ which satisfies the conditions of $(*, F, V)$ relative to α. The construction of $h(S)$ is similar to Bing's construction of $h(S)$ in his proof of Theorem 1 in [4], but the construction here could be considered simpler in the sense that we do not parallel his third approximation (the one which is obtained using Dehn's lemma). For convenience we assume $V = \operatorname{Int} S$, and Theorem 2 allows us to assume that the diameters of the components of F are bounded below by α. If $R = U$ the theorem is trivial, so we assume this is not the case.

Let ε be a positive number such that each ε-subset of S lies in a disk in S of diameter less than α. Then let ε_1 be a positive number satisfying the two conditions

$$3\varepsilon_1 < \rho(F, R - U), \quad \text{and} \tag{1}$$

$$14\varepsilon_1 < \varepsilon. \tag{2}$$

(a) **A special cellular decomposition of S.** Let T be a decomposition of S into disks so that

(3) the diameter of each disk of T is less than ε_1,

(4) the collection of disks of T is the sum of three finite subcollections A_1, A_2, and A_3 such that no two elements of A_i intersect $(i = 1, 2, 3)$, and

(5) if D and D' are two elements of T, then $\operatorname{Int} D$ and $\operatorname{Int} D'$ do not intersect.

To see how to obtain such a decomposition, see [4, p. 296]. We let K_1 be the 1-skeleton of T; that is, K_1 is the sum of the boundaries of the disks of T.

(b) **Pulling the disks of T partially into $\operatorname{Int} S$.** Let δ be a positive number so small that the distance between two disks of T without a common point is more than δ. We also require that

$$\delta < \varepsilon_1 . \tag{6}$$

Using Lemma 1, we let ε_2 be a positive number so that if f is a homeomorphism of R which moves points no more than ε_2, then each ε_2-subset of $f(R)$ lies in a $\delta/6$-disk on $f(R)$. This, together with (6), implies that

$$\varepsilon_2 < \delta/6 < \varepsilon_1/6 . \tag{7}$$

Let S_1 be a polyhedral 2-sphere and let h_1 be a homeomorphism of S onto S_1 such that

(8) h_1 moves no point as much as ε_2,

(9) S_1 contains a finite collection of disjoint ε_2-disks $H_1, H_2, \cdots, H_z$ such that $S_1 - \sum \operatorname{Int} H_i \subset \operatorname{Int} S$,

(10) S contains a finite collection of disjoint ε_2-disks such that S minus these disks lies in $\operatorname{Ext} S_1$, and

$$h_1(K_1) \subset S_1 - \sum H_i . \tag{11}$$

For details on how to obtain S_1 and h_1, see [4, p. 296] and [8].

(c) **The next approximation to elements of T.** For each disk D in T, $h_1(D)$ is a first approximation to D. Notice that $h_1(D) \cap S$ lies in the sum of a finite collection of disjoint ε_2-disks in $h_1(\operatorname{Int} D)$ (this collection of disks is a subcollection of the disks H_i of (9)), and

$h_1(D) - \sum \operatorname{Int} H_i$ lies in $\operatorname{Int} S$. It follows from (7) and (8) that

(12) $h_1(D)$ lies in a $\delta/6$-neighborhood of D, for each disk $D \in T$.

We will now construct a second approximation $h_2(D)$ to each disk D in T. Let C be the set of all disks D in T such that $h_1(D)$ intersects F. For each disk D in $T - C$, we choose a homeomorphism h_2 which agrees with h_1 on D; that is, for D in $T - C$, the second approximation to D is just $h_1(D)$. We must show how $h_2(D)$ is obtained if D is in C.

Let C' be the set of all disks D' in T such that there is a disk D in C which intersects D'. Then for D' in C' it follows that there is a disk D in C such that $h_1(D)$ and $h_1(D')$ intersect. Notice that $C \subset C'$, and h_2 is already defined on the disks in $C' - C$.

Using (3), (7), and (8), it follows that

(13) $\operatorname{diam} h_1(D) < \varepsilon_1 + 2\varepsilon_2 < \varepsilon_1 + \delta/3 < 2\varepsilon_1$, for each $D \in T$.

Consider a disk D' in C'. There is a disk D in C such that $h_1(D)$ and $h_1(D')$ intersect. From (7) we see that $2(\varepsilon_1 + \delta/3)$ is less than $3\varepsilon_1$. Since $h_1(D)$ intersects F, we may use (13) to see that

(14) $h_1(D')$ lies in a $3\varepsilon_1$-neighborhood of F, for D' in C'.

From (14) and (1) it follows that

(15) $h_1(D') \cap R \subset U$, for each disk D' in C'.

Then from (9) and (15) we have

(16) $h_1(D') - \sum \operatorname{Int} H_i \subset E^3 - R$, if D' is a disk in C'.

Thus we may choose a positive number ε_3 such that it is less than $\rho(h_1(D') - \sum \operatorname{Int} H_i, R)$ for each disk D' in C'. We also require that

(17) $\varepsilon_3 < \varepsilon_2$.

Let C_1' be the set of all disks D in C' such that $h_1(D) - \sum \operatorname{Int} H_i \subset \operatorname{Ext} R$ and let $C_1 = C \cap C_1'$. Let C_2' be $C' - C_1'$, and let $C_2 = C \cap C_2'$. Notice that $C = C_1 + C_2$, and $C' = C_1' + C_2'$. If D is in C_2, then $h_1(D)$ intersects F and $h_1(D) - \sum \operatorname{Int} H_i$ lies in $\operatorname{Int} R$. We will first show how to obtain $h_2(D)$ for the disks in C_1.

Using $(*, F, \operatorname{Ext} R)$, we let S' be a polyhedral 2-sphere and we let h' be a homeomorphism of R onto S' such that

(18) h' moves no point as much as ε_3,

(19) S' contains a finite collection of disjoint ε_3-disks such that S' minus these disks lies in $\operatorname{Ext} R$,

(20) R contains a finite collection of disjoint ε_3-disks $Y_1, Y_2, \cdots, Y_b$ such that $R - \sum \operatorname{Int} Y_i \subset \operatorname{Int} S'$, and

$$(\sum Y_i) \cap F = \varnothing \ . \tag{21}$$

Consider a disk D in C_1. From our choice of ε_3, $S' \cap h_1(D) \subset \sum \operatorname{Int} H_i$. We assume with no loss in generality that $h_1(D) \cap S'$ consists of a finite number of disjoint simple closed curves, and we let Q be the component of $h_1(D) - S'$ which contains $\operatorname{Bd} h_1(D)$. The simple closed curves in $\operatorname{Cl}(Q) \cap S'$ will be denoted by $J_1, J_2, \cdots, J_r$. Since each J_i lies in some H_j and $\operatorname{diam} H_j < \varepsilon_2$, it follows that $\operatorname{diam} J_i < \varepsilon_2$. Using (17), (18), and the definition of ε_2, we see that each J_i bounds a disk F_i in S' such that

$$\operatorname{diam} F_i < \delta/6 \ . \tag{22}$$

From (18), the definition of ε_3, and [15, p. 97] it follows that $h_1(D) - \sum \operatorname{Int} H_i \subset \operatorname{Ext} S'$ (this is true for all disks D in C_1'). We fill the holes in Q with the F_i's, moving each F_i slightly into $\operatorname{Ext} S'$ as we add it, to obtain a polyhedral disk $h_2(D)$ in $\operatorname{Ext} S'$ [4, p. 297]. The F_i's are moved to slightly that the new F_i's also have diameter less than $\delta/6$. Henceforth we use the symbol F_i to denote the disk F_i after it has been moved into $\operatorname{Ext} S'$. Notice that $h_2(D)$ does not intersect F from (20) and (21). The homeomorphism h_2 is selected to agree with h_1 on $h_1^{-1}(h_1(D) - \sum \operatorname{Int} H_i)$. Thus we have obtained $h_2(D)$ for each disk D in C_1.

For D in C_2 we obtain $h_2(D)$ in just the same way as for the disks in C_1. The only difference is that we use $(*, F, \operatorname{Int} R)$ to obtain a polyhedral 2-sphere S'', homeomorphically within ε_3 of R, which "lies almost in $\operatorname{Int} R$ and misses F'' (that is, S'' satisfies conditions similar to (18), (19), (20), and (21)). For each disk D in C_2, $h_1(D)$ will lie "almost" in $\operatorname{Int} S''$, so we can "pull $h_1(D)$ and S'' apart" just as we "pulled $h_1(D)$ and S' apart" in the preceding paragraph to obtain a polyhedral disk $h_2(D)$ in $\operatorname{Int} S''$. Again h_2 is selected to agree with h_1 on $D - \sum h_1^{-1}(\operatorname{Int} H_i)$. Thus h_2 is defined for all disks D in T in such a way that

$$h_2(D) \cap F = \varnothing \ , \quad \text{for each disk } D \text{ in } T \ . \tag{23}$$

Now consider a disk D in C. We will prove that the disks F_i, which replaced disks in $h_1(D)$ to form $h_2(D)$, do not intersect $S_1 - \sum \operatorname{Int} H_i$. First we observe from the construction of $h_2(D)$ and (22) that

(24) $h_2(D)$ lies in a $\delta/6$-neighborhood of $h_1(D)$, for each $D \in T$.

From (12) and (24) we have

(25) $h_2(D)$ lies in a $\delta/3$-neighborhood of D for each $D \in T$.

If D' does not intersect D, it follows from (25) and the definition of δ that $h_2(D)$ and $h_2(D')$ fail to intersect. Thus if D is in C, F_i is a disk associated with $h_2(D)$, and F_i intersects $h_2(D')$, then D' is in C'. Consider a disk D' in C'. It follows from the definition of ε_3 and (18) that F_i does not intersect $h_1(D') - \sum \operatorname{Int} H_i$. Recalling that $h_1 = h_2$ on $h_1^{-1}(h_1(D) - \sum \operatorname{Int} H_i)$, we see that no F_i intersects $S_1 - \sum \operatorname{Int} H_i$. Summarizing;

(26) If $h_2(D)$ and $\operatorname{Int} h_2(D')$ intersect and $D \neq D'$, then D and D' are both in C'. Furthermore if $D \neq D'$, then $h_2(D')$ cannot intersect $\operatorname{Int} h_2(D)$ in the set $(h_1(D) - \sum H_i)$.

(d) **Untangling the $h_2(D)$'s.** This section is the same as §(e) in [4, p. 298] provided we substitute our $h_2(D)$'s for the $h_3(D)$'s in that paper. In order to be sure we have a 2-sphere $h(S)$ after we use Bing's untangling process, we must be sure that if D and D' are two elements of T which do not intersect, then $h_2(D)$ and $h_2(D')$ also fail to intersect. However this follows directly from (25) and the definition of δ.

Notice that from (26) we know the intersection of $h_2(D)$ with $\operatorname{Int} h_2(D')$ does not intersect $S_1 - \sum \operatorname{Int} H_i$ (unless D is D'). This means that the untangling process does not involve $S_1 - \sum \operatorname{Int} H_i$, so we may choose h equal to h_2 (and equal to h_1) on $h_1^{-1}(S_1 - \sum \operatorname{Int} H_i)$. Thus $h(S)$ contains the set $S_1 - \sum \operatorname{Int} H_i$, which is a 2-sphere minus a finite collection of disjoint disks. To be sure that $h(S)$ is homeomorphically within α of S, we insure that the disks which are added to $S_1 - \sum \operatorname{Int} H_i$ to form $h(S)$ are of small diameter. We show in §(e) that this has been accomplished. From (23) we see that the untangling can be done so that $h(S) \cap F = \varnothing$.

(e) **The homeomorphism h moves no point as much as α.** From (3), (25), and (6) it follows that $\operatorname{diam} h_2(D) < \operatorname{diam} D + 2(\delta/3) < 2\varepsilon_1$, for each disk D in T. Since $h = h_2$ on each disk of A_1, we have

$$\operatorname{diam} h(D) < 2\varepsilon_1 , \quad \text{if } D \text{ is a disk in } A_1 . \tag{27}$$

Let D' be a disk in A_2. To form $h(D')$ we added to $h_2(D')$ disjoint disks near $h(D)$ where D is in A_1. Then from (27),

$$\operatorname{diam} h(D') < \operatorname{diam} h_2(D') + 2(2\varepsilon_1) < 6\varepsilon_1 , \quad \text{if } D' \in A_2 . \tag{28}$$

Let D'' be a disk in A_3. In forming $h(D'')$, we added to $h_2(D'')$ disks near $h(D)$ for D in A_1 and disks near $h(D')$ for D' in A_2. Using (27) and (28) we see that each disk added to $h_2(D'')$ has diameter less than

$6\varepsilon_1$. Since each added disk intersects $h_2(D'')$, it follows that

(29) $h(D)$ lies in a $6\varepsilon_1$-neighborhood of $h_2(D)$, for each $D \in T$, and

(30) $\operatorname{diam} h(D) < \operatorname{diam} h_2(D) + 2(6\varepsilon_1) < 14\varepsilon_1$, for each $D \in T$.

From (29) and (25), each point of $h(D)$ lies within a distance $6\varepsilon_1$ of a point of $h_2(D)$ which lies within a distance $\delta/3$ of a point of D. Hence no point is moved more than $6\varepsilon_1 + \delta/3 + \operatorname{diam} D < 8\varepsilon_1 < \varepsilon < \alpha$; using (6), (3), and $\varepsilon < \alpha$.

(f) **Property** $(*, F, \operatorname{Int} S)$ **is satisfied.** Since the homeomorphism h was selected to agree with h_1 on the set $S - \sum h_1^{-1}(\operatorname{Int} H_i)$, each $\operatorname{Bd} H_i$ lies in some $h(D)$ and bounds a disk D_i in that $h(D)$. From (30) and (2), $\operatorname{diam} D_i < \varepsilon$. The collection of disjoint ε-disks $D_1, D_2, \cdots, D_z$ on $h(S)$ has the property that $h(S) \cap S \subset \sum \operatorname{Int} D_i$ (see Condition (9)). It follows from (9) and (11) that $h(S) - \sum D_i \subset \operatorname{Int} S$.

We must exhibit a finite collection of disjoint α-disks on S so that none of these disks intersect F and so that $S \cap h(S)$ lies in the union of these disks. Since each component Z of $h(S) \cap S$ lies in some D_j, $\operatorname{diam} Z < \varepsilon$. From the definition of ε, Z lies in an α-disk on S. Assuming 2α is less than $\operatorname{diam} S$ and recalling that α is a lower bound on the diameters of the components of F, we see that F lies in the large component of $S - Z$ ($Z \cap F = \varnothing$ because $h(S) \cap F = \varnothing$), for each component Z of $h(S) \cap S$. Following the procedure in the proof of Theorem 9 of [8], we obtain a finite collection of disjoint α-disks $G_1, G_2, \cdots, G_t$ on S such that $(\sum G_i) \cap F = \varnothing$ and $h(S) \cap S \subset (\sum G_i)$. *Then* $(*, F, \operatorname{Int} S)$ *is satisfied.*

A similar procedure is used to establish $(*, F, \operatorname{Ext} S)$. Then it follows that $(*, F, S)$ is satisfied.

REMARK. The proofs of the next three theorems are modifications of Bing's proofs of Theorems 4, 5, and 1 in [6]. Consequently we do not carry out the details of the proofs, but merely outline the differences between his proofs and ours.

For what is meant by "a 2-sphere S can be ε approximated from $\operatorname{Int} S$ (or $\operatorname{Ext} S$)", see [6]. We define "$H(A, B) < \varepsilon$" to mean that A and B are homeomorphically within ε of each other.

THEOREM 4. *If* $\varepsilon > 0$ *and* F *is a closed subset of a 2-sphere* S *in* E^3 *such that* $(*, F, \operatorname{Int} S)$ *is satisfied, then there is a 2-sphere* S' *such that*

1. S' *is obtained from* S *by removing a finite collection of disjoint* ε*-disks from* $S - F$ *and replacing them with* ε*-disks*

and

2. *S' can be ε approximated from* $\operatorname{Int} S'$.

Proof. Because of the definition of $(*, F, \operatorname{Int} S)$, we are able to accomplish in one paragraph all the essential elements in the first three paragraphs of Bing's proof of Theorem 4 in [6]. We know there is a polyhedral 2-sphere S'' containing a finite collection of disjoint $\varepsilon/6$-disks $D_1, D_2, \cdots, D_n$ and there is a finite collection of disjoint $\varepsilon/6$-disks $E_1, E_2, \cdots, E_m$ on S such that $S' - \sum D_i \subset \operatorname{Int} S$, $S - \sum E_i \subset \operatorname{Ext} S''$, no E_i intersects F, and $H(S, S'') < \varepsilon/6$. In fact since F is closed, we may assume with no loss in generality that $S'' - \sum \operatorname{Int} D_i$ lies in $\operatorname{Int} S$ and $S - \sum \operatorname{Int} E_i \subset \operatorname{Ext} S''$.

The proof is completed by following Bing's paragraphs 4, 5, 6, and 7 in his proof of Theorem 4 in [6].

THEOREM 5. *If $\varepsilon > 0$ and F is a closed subset of a 2-sphere S in E^3 such that $(*, F, S)$ is satisfied, then there is a 2-sphere S' such that*

1. *S' is obtained by removing a finite number of disjoint ε-disks from $S - F$ and replacing them with ε-disks and*

2. *S' can be ε approximated from each of its complementary domains.*

Proof. The proof of Theorem 5 in [6] is followed here. We apply Theorem 4 to obtain a 2-sphere S_1 such that

(1) S_1 is obtained by removing a finite collection G of disjoint $\varepsilon/4$-disks $E_1, E_2, \cdots, E_m$ from $S - F$ and replacing them with $\varepsilon/4$-disks and

(2) there is a 2-sphere S'' in $\operatorname{Int} S_1$ such that $H(S_1, S'') < \varepsilon/4$.

Let δ be a positive number subject to four restrictions to be mentioned later. Since $S - \sum E_i$ is an open subset of $S \cap S_1$ which contains F, it follows from Theorem 3 that $(*, F, S_1)$ is satisfied. This permits us to reapply Theorem 4, this time relative to $\operatorname{Ext} S_1$ and δ, to obtain a 2-sphere S' such that

(3) S' is obtained by removing a finite collection of disjoint δ-disks $D_1, D_2, \cdots, D_n$ from $S_1 - F$ and replacing them with δ-disks and

(4) there is a polyhedral 2-sphere S''' in $\operatorname{Ext} S'$ such that $H(S''', S') < \delta$.

With suitable restrictions on δ, S' is the required 2-sphere.

Restrictions 1, 2, and 4 of the proof of Theorem 5 in [6] are also used here. We change Restriction 3 somewhat.

Restriction 3. We must identify a collection H of disjoint ε-disks on S which are to be replaced as mentioned in the first requirement in the conclusion of Theorem 5. For each i $(1 \leqq i \leqq m)$, let E_i' be an $\varepsilon/3$-disk on S such that $E_i \subset \operatorname{Int} E_i'$, $E_i' \cap F = \varnothing$, and $E_i' \cap E_j' = \varnothing$ if $i \neq j$. We choose δ so that $\delta < \rho(S - E_i', E_i)$ for each i. If we let H_i be the component of $S \cap (\sum E_i + \sum D_i)$ which contains E_i $(1 \leqq i \leqq m)$, it follows from our choice of δ that H_i lies in $\operatorname{Int} E_i'$ for each i. Then we can find a simple closed curve J_i in $E_i' - \sum D_i$ which separates $S - E_i'$ from H_i in S. Let L_i be the $\varepsilon/3$-disk in E_i' which is bounded by J_i. Then $H_i \subset L_i$. Thus we have defined a finite collection of disjoint $\varepsilon/3$-disks $L_1, L_2, \cdots, L_m$ such that for each i, $H_i \subset \operatorname{Int} L_i$, $L_i \cap F = \varnothing$, and $(\operatorname{Bd} L_i) \cap (\sum D_j) = \varnothing$.

Suppose the intersection with S of some disk D_i on S_1 is not covered by $\sum L_i$. Since D_i fails to intersect any L_j, we know that D_i lies in the intersection of S and S_1. From Restriction 1 we know that $\delta < \varepsilon/4$, so $\operatorname{diam} D_i < \varepsilon/4$. The collection H is the set of all the L_i's together with all the D_i's which are not covered by the sum of the L_i's.

THEOREM 6. *If $\varepsilon > 0$ and F is a closed subset of a 2-sphere S in E^3 such that $(*, F, S)$ is satisfied, then there is a continuum M on S and a null sequence $\{D_i\}$ of disjoint ε-disks on S such that*

1. $M = S - \sum \operatorname{Int} D_i$,
2. *$(*, M, S)$ is satisfied,*
3. $F \subset M - \sum D_i = S - \sum D_i$, *and*
4. *M lies on a tame 2-sphere in E^3.*

Proof. It follows from Theorem 1 that there is a continuum M and a sequence $\{D_i\}$ of disks satisfying Conditions 1, 2, and 3 in the statement of Theorem 6. All we need to show is that there is a tame 2-sphere in E^3 which contains M. This tame 2-sphere will be constructed as the limit of a sequence of 2-spheres $S_1, S_2, \cdots$, just as in the proof of Theorem 1 of [6]. In that proof Bing indicates three restrictions to be placed on the S_i's to insure that

(1) $\lim S_i$ is a 2-sphere,
(2) $\lim S_i$ is tame, and
(3) $\lim S_i$ shares a Sierpinski curve X with S.

Our restrictions to insure that $\lim S_i$ is a tame 2-sphere are the same as those used by Bing in his Steps 1 and 2, provided we substitute our Theorem 5 wherever he uses his Theorem 5. In the following paragraph we show that $\lim S_i$ shares M with S. Of course

(3) follows because of the construction of M.

Since $(*, M, S)$ is satisfied, we are able to use Condition 1 of Theorem 5 to obtain a 2-sphere S_1 such that S_1 is constructed by removing from S a closed set H_1 which is the sum of a finite collection of disjoint ε_1-disks in $S - M$. Then $S - H_1$ is an open set in $S \cap S_1$ which contains M. Applying Theorem 3 we see that $(*, M, S_1)$ is satisfied.

Again we apply Condition 1 of Theorem 5, this time relative to S_1, to obtain a 2-sphere S_2 such that S_2 is constructed by removing from S_1 a closed set H_2 which is the sum of a finite collection of disjoint ε_2-disks in $S_1 - M$. Then $S_1 - H_2$ is an open subset of $S_1 \cap S_2$ which contains M. From Theorem 3 we have $(*, M, S_2)$ satisfied.

We continue this procedure so that $M \subset S_i$ for each i. This insures us that $M \subset \lim S_i$.

Remark. Gillman has shown that a 2-sphere S can be pierced by a tame arc at a point p of S if and only if p lies in a tame arc on S [13, Theorem 6]. It follows that S can be pierced by a tame arc at each point of the continuum M identified in the conclusion of Theorem 6. Repeated application of Theorem 6 will thus establish that the set Y of points of S where S cannot be pierced by a tame arc is a subset of a 0-dimensional G_δ set. We state this result, without proof, as Theorem 7. Bing has already proven that Y lies in a 0-dimensional G_δ set [7, Theorem 5.2], and Gillman has shown that Y is a 0-dimensional F_σ set [13, Theorem 11].

THEOREM 7. *Each 2-sphere S in E^3 contains a sequence $\{M_i\}$ of tame continua such that for each i*

1. $M_i \subset M_{i+1}$,
2. $(*, M_i, S)$ *holds,*
3. $S - \sum M_i$ *is a 0-dimensional G_δ set, and*
4. S *can be pierced by a tame arc at each point of* $\sum M_i$.

3. Conditions which are equivalent to $(*, F, S)$. In the following definitions we are considering F to be a closed subset of a 2-sphere S *in* E^3.

Property (A, F, S). For each $\varepsilon > 0$ there is a $\delta > 0$ so that each δ-simple closed curve in $E^3 - S$ is homotopic to a constant (can be shrunk to a point) in an ε-subset of $E^3 - F$.

Property (B, F, S). There exist infinite sequences $\{S_i\}$ and $\{S_i'\}$ of 2-spheres such that S_i and S_i' are each homeomorphically within $1/i$ of S, $F \subset \operatorname{Int} S_i$, and $F \subset \operatorname{Ext} S_i'$.

Hosay has announced that if the diameters of the components of F have a positive lower bound and either of Properties (A, F, S) or (B, F, S) is satisfied, then F lies on a tame 2-sphere in E^3[14]. It is our purpose to show that if the diameters of the components of F have a positive lower bound, then not only are Properties (A, F, S) and (B, F, S) equivalent, but each is equivalent to Property $(*, F, S)$. The proof follows the pattern $(*, F, S) \Rightarrow (B, F, S) \Rightarrow (A, F, S) \Rightarrow (*, F, S)$. For convenience, we state the following two lemmas but do not include their proofs. A proof for Lemma 2 can be found in [10, Lemma 1], and Bing has stated Lemma 3 in [4, p. 294].

LEMMA 2. *If $D_1, D_2, \cdots, D_n$ is a finite collection of disjoint disks in E^3 and f is a map of a disk K into E^3 such that $f(\mathrm{Bd}\, K) \subset E^3 - \sum D_i$, then there is a map g of K into E^3 such that*

1. $g \mid \mathrm{Bd}\, K = f \mid \mathrm{Bd}\, K$,
2. $g(K) \subset f(K) + \sum \mathrm{Int}\, D_i$, *and*
3. $g(K) - \sum D_i$ *is connected.*

LEMMA 3. *If F is a closed subset of a 2-sphere S in E^3 such that for each point $p \in F$ and for each neighborhood N of p there is an open set U such that $p \in U$ and each simple closed curve in $U - S$ can be shrunk to a point in $N - F$, then for each $\varepsilon > 0$ there is a $\delta > 0$ such that each δ-simple closed curve in $E^3 - S$ can be shrunk to a point in an ε-subset of $E^3 - F$.*

THEOREM 8. *If F is a closed subset of a 2-sphere S in E^3 such that $(*, F, S)$ is satisfied, then (B, F, S) is also satisfied.*

Proof. For each positive integer i we must show the existence of 2-spheres S_i and S_i' such that each is homeomorphically within $1/i$ of S, $F \subset \mathrm{Int}\, S_i$, and $F \subset \mathrm{Ext}\, S_i'$. Using the definition of $(*, F, \mathrm{Ext}\, S)$ we obtain a polyhedral 2-sphere S_i containing a finite collection of disjoint $1/i$-disks $D_1, D_2, \cdots, D_m$ and a finite collection of disjoint $1/i$-disks $E_1, E_2, \cdots, E_r$ on S such that S and S_i are homeomorphically within $1/i$, $S_i - \sum D_j \subset \mathrm{Ext}\, S$, $S - \sum E_j$ lies in $\mathrm{Int}\, S_i$, and $(\sum E_j) \cap F = \varnothing$. It follows that F lies in $\mathrm{Int}\, S_i$.

The 2-sphere S_i' is obtained in the same manner using the definition of $(*, F, \mathrm{Int}\, S)$.

THEOREM 9. *If F is a closed subset of a 2-sphere S in E^3 such that (B, F, S) is satisfied, then (A, F, S) is also satisfied.*

Proof. All that we need to show is that for each $p \in F$ and for

each neighborhood N of p there is a neighborhood U of p such that each simple closed curve in $U - S$ is homotopic to a constant in $N - F$. The uniform condition in the definition of (A, F, S) then follows from Lemma 3.

Let p be a point of F, and let N be a neighborhood of p such that there are points of S not in N. Let K be a disk in $N \cap S$ such that p is in $\operatorname{Int} K$, and let U be a neighborhood of p such that $\operatorname{Cl}(U) \cap S \subset \operatorname{Int} K$, $\operatorname{Bd} U$ is a tame 2-sphere, and $\operatorname{Cl}(U) \subset N$. Let J be a simple closed curve in $U - S$. Then J is homotopic to a constant in U, so we let f be a map of K into U such that f restricted to $\operatorname{Bd} K$ is a homeomorphism onto J. For convenience we assume that J is in $\operatorname{Ext} S$.

Choose a positive number δ so that

(1) $\delta < \rho(J, S)$,

(2) $\delta < \rho(S - K, U)$, and

(3) $\delta < \rho(\operatorname{Bd} N, K)$.

Using (B, F, S), we let S' be a 2-sphere which is homeomorphically within δ of S and such that $F \subset \operatorname{Int} S'$. Let h be a homeomorphism of S onto S' such that

(4) h moves no point as much as δ.

Suppose that S' intersects U in $S' - h(K)$. Then there is a point x in $S - K$ such that $h(x)$ is in U. This means that $\rho(x, h(x)) \geqq \rho(S - K, U) \geqq \delta$, using (2). Since this contradicts (4) we have

(5) $U \cap S' \subset h(K)$.

In a similar manner, using (3) and (4), it follows that

(6) $h(K) \subset N$.

Then from (1), (4) and [15, p. 97], we have

(7) $J \subset \operatorname{Ext} S'$.

Condition (7) allows us to use Lemma 2 to obtain a map g of K into E^3 such that

(8) $g \mid \operatorname{Bd} K = f \mid \operatorname{Bd} K$,

(9) $g(K) \subset f(K) + \operatorname{Int} h(K)$, and

(10) $g(K) - h(K)$ is connected.

Suppose $g(K)$ is not a subset of $S' + \operatorname{Ext} S'$. Then the connected set $g(K) - h(K)$ intersects both complementary domains of S' and consequently intersects S'. But from (5), (9), and the fact that $f(K) \subset U$, we see that this is impossible. Hence $g(K)$ lies in $S' + \operatorname{Ext} S'$, which means that $g(K) \cap F = \varnothing$. It follows directly from (6) and (9) that $g(K) \subset N$. Hence $g(K) \subset N - F$, and we have established Theorem 9.

THEOREM 10. *If F is a closed subset of a 2-sphere S in E^3 such that Property (A, F, S) is satisfied and the diameters of the com-*

ponents of F have a positive lower bound, then $(*, F, S)$ is satisfied.

Proof. The proof here is almost identical to Bing's proof that a 2-sphere in E^3 is tame if its complement is 1-*ULC* [4, Theorem 1]. Bing uses the 1-*ULC* condition to construct a 2-sphere $h(S)$ which lies in $\operatorname{Int} S$ and is homeomorphically close to S. Property (A, F, S) permits us to use the same type construction to obtain 2-sphere $h(S)$ which satisfies all the requirements of Property $(*, F, S)$, although $h(S)$ need not lie in $\operatorname{Int} S$. Bing has broken his proof into six sections-a through e. We indicate how each of these sections can be changed to obtain a proof of Theorem 10.

Let $\alpha > 0$ and let ε be a positive number so small that each ε-subset of S lies in an α-disk on S. With no loss in generality we assume that α is less than the diameter of each component of F. We choose a positive number ε_1 so that $14\varepsilon_1$ is less than ε.

We will show that the conditions of Property $(*, F, \operatorname{Int} S)$ are satisfied relative to the positive number α. We follow exactly the same procedure that Bing followed in Sections a, b, and c where he obtained a special cellular decomposition and applied the Side Approximation Theorem.

d. **Third approximation to D.** It is here that the major change is made. (Notice that "Bd D" should be changed to "$h_2(\operatorname{Bd} D)$" in the first paragraph of § d of Bing's paper). Everywhere Bing uses "$\operatorname{Int} S$" in his § d we substitute "$E^3 - F$". The remainder of his § d is followed here except we allow the open set U, which contains the singular points of $g(h_2(D))$, to intersect $S - F$, but we insist that $F \cap U = \varnothing$. This is possible since each $\operatorname{Bd} E_i'$ is shrunk to a point missing F.

e. **The fourth approximation to D.** Bing's untangling procedure can be done so that no $h(D)$ intersects F since each component of $h_3(S) \cap S$ lies in U.

f. **Epsilontics.** In the paragraph where Bing defines ε_4 we replace the second occurence of "$E^3 - S$" with "$E^3 - F$", using (A, F, S) in place of 1-*ULC*. The remainder of Bing's § f is followed here, but we must add a few comments to complete the proof.

Since $\delta_2 < \rho(S, h_1(D) - \sum E_i)$, we know that S', as identified by Bing, intersects $h_1(D)$ only in $\sum \operatorname{Int} E_i$. Then h_2 may be selected to agree with h_1 on each $D - \sum h_1^{-1}(\operatorname{Int} E_i)$. Each point of $\operatorname{Bd} E_i'$ (of § d) is within ε_4 of S and $\operatorname{Bd} E_i'$ is shrunk to a point in a $\delta_2/2$-subset of $E^3 - F$. This means that each $\operatorname{Bd} E_i'$ is shrunk to a point in a set which is within $\varepsilon_4 + \delta_2/2$ of S. Since $\varepsilon_4 < \delta_2/2$, then $g(h_2(D))$ has its

singular points within δ_2 of S. It follows from the definition of δ_2 that h_3 may be chosen to agree with h_2 on $D - \sum h_1^{-1}(\text{Int } E_i)$. Since $h_1(D) - \sum E_i$ is not involved in the untangling process described in §e, h may also be chosen to agree with h_1 on $D - \sum h_1^{-1}(\text{Int } E_i)$. It follows that each simple closed curve $\text{Bd } E_i$ lies in $h(D)$ and bounds a disk R_i in $h(D)$. Bing has shown that $h(D)$ lies in a $6\varepsilon_1$-neighborhood of $h_3(D)$ and that diam $h_3(D)$ is less than $2\varepsilon_1$. Then

$$\text{diam } R_i < \text{diam } h(D) < 2\varepsilon_1 + 2(6\varepsilon_1) = 14\varepsilon_1 ,$$

which is less than ε. Each $h(D)$ has an associated collection of R_i's, so we collect all these R_i's together to form a finite collection of disjoint ε-disks $G_1, G_2, \cdots, G_k$ on $h(S)$ so that $h(S) \cap S \subset \sum \text{Int } G_i$. Furthermore, since $h(D) - \sum \text{Int } G_i = h_1(D) - \sum \text{Int } E_i$ which lies in Int S, it follows that $h(S) - \sum G_i \subset \text{Int } S$.

To show that $h(S)$ satisfies the conditions of $(*, F, \text{Int } S)$ relative to α we must exhibit a finite collection of disjoint α-disks $H_1, H_2, \cdots, H_r$ on S such that no H_i intersects F and $S \cap h(S) \subset \sum H_i$. Since each component Z of $h(S) \cap S$ lies in an ε-disk G_i, it follows that diam $Z < \varepsilon$. Then Z lies in an α-disk on S(see the definition of ε). Since α is a lower bound on the diameters of the components of F and no Z intersects F, we may use the procedures in the proof of Theorem 9 of [8] to obtain disjoint α-disks $\{H_i\}$ such that $H_i \cap F = \varnothing$ and $h(S) \cap S \subset \sum H_i$.

Thus Property $(*, F, \text{Int } S)$ is satisfied. The proof that $(*, F, \text{Ext } S)$ holds is similar, so Property $(*, F, S)$ is satisfied.

REMARK. The requirement that F be such that the diameters of its components are bounded below by a positive number cannot be removed from the hypothesis of Theorem 10. For let S be the wild 2-sphere described by Fox and Artin [13] where the set of wild points of S consists of a single point p. It is easy to see that Property $(B, \{p\}, S)$ is satisfied and that Property $(*, \{p\}, S)$ does not hold. However the following question apparently has not been answered. If F is a closed subset of a 2-sphere S in E^3, Property (B, F, S) is satisfied, and F has no degenerate components, then will Property $(*, F, S)$ hold? Of course a similar question could be asked where Property (A, F, S) replaces Property (B, F, S).

We also note that if F is a closed subset of a 2-sphere S such that Property (A, F, S) is satisfied, then F need not lie in a nondegenerate subcontinuum M of S such that (A, M, S) holds. To see this we use the same example as in the previous paragraph and let $F = \{p\}$. If a nondegenerate continuum M exists such that $p \in M \subset S$ and (A, M, S) holds, then $(*, M, S)$ also holds, but this is impossible.

If Property (A, F, S) holds for a closed subset F of S where F has no degenerate components, then does F lie in a nondegenerate continuum M in S such that (A, M, S) holds?

The next two theorems are useful in the proofs of some theorems in [16], but they also lead to some interesting results in this paper. We define a simple closed curve in E^3 to be *unknotted* if it is the boundary of a tame disk in E^3.

THEOREM 11. *Suppose p is a point of a 2-sphere S in E^3, V is a complementary domain of S, N is a neighborhood of p, and K is a disk. Then there is an open set U containing p such that for each closed subset F of S satisfying $(*, F, V)$ and for each map f of $\operatorname{Bd} K$ into $U \cap V$ there is a finite collection of disjoint disks $H_1, H_2, \cdots, H_t$ in $(N - F) \cap S$ and there is a map g of K into N such that $g \mid \operatorname{Bd} K = f$ and $g(K) \cap S \subset \sum \operatorname{Int} H_i$.*

Proof. For convenience in notation we will assume K is a disk on S such that $p \in \operatorname{Int} K \subset K \subset N$. Let K' be a disk on S such that $p \in \operatorname{Int} K'$ and $K' \subset \operatorname{Int} K$, and let U be a neighborhood of p such that $\operatorname{Cl}(U) \cap S \subset \operatorname{Int} K'$, $\operatorname{Cl}(U) \subset N$, and $\operatorname{Bd} U$ is a tame 2-sphere. For convenience we assume $V = \operatorname{Ext} S$. Let f be a map of $\operatorname{Bd} K$ into $U \cap V$. Since $\operatorname{Bd} U$ is tame we can extend f to map all of K into U.

Choose a positive number δ so that

(1) $\delta < \rho(f(\operatorname{Bd} K), S)$,

(2) $\delta < \rho(S - K, K')$,

(3) $\delta < \rho(\operatorname{Bd} N, K)$, and

(4) $\delta < \rho(S - K', U)$.

Now let F be a closed subset of S satisfying $(*, F, \operatorname{Ext} S)$.

Using the definition of $(*, F, \operatorname{Ext} S)$, we let S' be a polyhedral 2-sphere containing a finite collection of disjoint δ-disks $D_1, D_2, \cdots, D_m$ and let $E_1, E_2, \cdots, E_r$ be a finite collection of disjoint δ-disks on S such that

(5) There is a homeomorphism h of S onto S' such that h moves no point as much as δ,

(6) $S - \sum \operatorname{Int} E_i \subset \operatorname{Int} S'$,

(7) $S' - \sum D_i \subset \operatorname{Ext} S$, and

(8) $F \cap (\sum E_i) = \varnothing$.

Condition (6) does not come directly from the definition of $(*, F, S)$; however, from $(*, F, S)$ we have $S - \sum E_i \subset \operatorname{Int} S'$. Then we can find a finite collection of disjoint δ-disks $E_1', E_2', \cdots, E_r'$ on S such that $S - \sum \operatorname{Int} E_i' \subset \operatorname{Int} S'$ and no E_i' intersects F. Thus we may assume without loss in generality that Condition (6) holds.

It follows from (4) and (5) that

(9) $U \cap S' \subset h(K')$,

from (3) and (5) that

(10) $h(K') \subset N$,

and from (2) and (5) that

(11) $h(K') \cap S \subset K$.

From (1) and (5) we have $f(\mathrm{Bd}\, K) \subset E^3 - h(K')$. This allows us to use Lemma 2 to obtain a map g of K into E^3 such that

(12) $g|\, \mathrm{Bd}\, K = f|\, \mathrm{Bd}\, K$,

(13) $g(K) \subset f(K) + \mathrm{Int}\, h(K')$, and

(14) $g(K) - h(K')$ is connected.

It follows from (13) and (10) that $g(K) \subset N$. Let $H_1, H_2, \cdots, H_t$ be the subcollection of the E_i's such that each H_i intersects $h(K')$ and $h(K') \cap S \subset \sum H_i$. Since each H_i is a δ-disk that intersects $h(K')$, it follows from (11), (3), and (8) that each H_i lies in $(N - F) \cap S$.

Now we will show that $g(K) \cap S \subset \sum \mathrm{Int}\, H_i$. It follows from (12), (1), (5), and [15, p. 97] that $g(\mathrm{Bd}\, K)$ lies in $\mathrm{Ext}\, S'$. Suppose $g(K)$ intersects $S - \sum \mathrm{Int}\, H_i$. Then $g(K)$ intersects both components of $E^3 - S'$, by (6); hence, $g(K) - h(K')$ intersects each component of $E^3 - S'$. From (14), $g(K) - h(K')$ intersects S'. Then from (13) and the fact that $f(K)$ lies in U, it follows that $g(K) - h(K')$ intersects S' in U. But this contradicts (9). Then $g(K) \cap S \subset \sum \mathrm{Int}\, H_i$.

THEOREM 12. *Suppose p is a point of a 2-sphere S in E^3, V is a complementary domain of S, and N is a neighborhood of p. Then there is an open set U containing p such that for each closed subset F of S which satisfies $(*, F, V)$ and for each unknotted simple closed curve J in $U \cap V$ there is a finite collection of disjoint disks $H_1, H_2, \cdots, H_t$ in $(N - F) \cap S$ such that J bounds a tame disk D in N and $D \cap S \subset \sum \mathrm{Int}\, H_i$.*

Proof. The proof here is similar to the proof of Theorem 11. Let K', K, and U be defined as in that proof, and let J be an unknotted simple closed curve in $U \cap V$. Then J bounds a tame disk E in U. Now the proof here reads the same as the proof of Theorem 11 up to and including Condition (11) if we identify J with $f(\mathrm{Bd}\, K)$ and E with $f(K)$.

We assume that E is locally polyhedral at its interior points [2, Theorem 7] and that E and $h(K')$ are in general position. Let C be the component of $E - h(K')$ such that J lies in C, and let $J_1, J_2, \cdots J_n$ be the components of $\mathrm{Cl}(C) \cap h(K')$. Since $E \cap S' \subset h(K')$ and $J \subset \mathrm{Ext}\, S'$, we know that each J_i is a simple closed curve. Let $D_1, D_2, \cdots, D_n$ be the disks on $h(K')$ such that $\mathrm{Bd}\, D_i = J_i$ and assume that the D_i's are ordered so that if $i < j$, then D_j is not a subset of D_i. Now we add D_1 to C and move D_1 slightly into $\mathrm{Ext}\, S'$. Next we add D_2 and move it into $\mathrm{Ext}\, S'$. We continue until all the D_i's have

been added to C and adjusted, following a procedure described by Bing [4, p. 297]. Then we have a polyhedral disk D in Ext S' such that Bd $D = J$. The moving at each stage is done so that $D \subset N$ and $D \cap S$ lies in $\sum \text{Int } E_i$.

The subcollection $H_1, H_2, \cdots, H_t$ of the E_i's is chosen in the same way as in the proof of Theorem 11. Since D lies in Ext S', it follows from (6) and (8) that $D \subset N - F$.

REMARK. Theorem 12 allows us to define another property which is equivalent to $(*, F, S)$ for certain closed sets F on a sphere S.

Property (A', F, S). Let F be a closed subset of a 2-sphere S in E^3. If for each $\varepsilon > 0$ there is a $\delta > 0$ such that each unknotted simple closed curve of diameter less than δ, which lies in $E^3 - S$, bounds a tame ε-disk in $E^3 - F$, then F and S are said to satisfy Property (A', F, S).

If $(*, F, S)$ is satisfied, $p \in F$, and N is a neighborhood of p, it follows from Theorem 12 that there is a neighborhood U of p such that each unknotted simple closed curve in $U - S$ bounds a tame disk in $N - F$. Thus we can apply an argurment similar to the proof of Lemma 3 to show that $(*, F, S)$ implies (A', F, S).

The converse is also true provided the diameters of the components of F have a positive lower bound. To see this we observe that in the proof of Theorem 10 all we needed was to have certain small unknotted simple closed curves in $E^3 - S$ bound small tame disks missing F. Thus the following theorem holds.

THEOREM 13. *If the diameters of the components of the closed subset F of a 2-sphere S in E^3 have a positive lower bound, then (A', F, S) and $(*, F, S)$ are equivalent.*

THEOREM 14. *If S is a 2-sphere in E^3, V is a component of $E^3 - S$, F is a closed subset of S, $(*, F, V)$ holds, and W is an open subset of S such that V is locally simply connected at each point of $W - F$ (or equivalently, S is locally tame from V at each point of $W - F$), then S is locally tame from V at each point of W.*

Proof. Let $p \in F \cap W$, and let N be a neighborhood of p such that $N \cap S \subset W$. Let U be an open set containing p such that U satisfies the conditions of Theorem 11 relative to V, N, and p. Let f be a map of the boundary of a disk K into $U \cap V$.

Using the properties of U guaranteed by Theorem 11, we let $H_1, H_2, \cdots, H_t$ be a finite collection of disjoint disks in $(N - F) \cap S$

and we let g be a map of K into N so that $g \mid \mathrm{Bd}\, K = f$ and $g(K) \cap S \subset \sum \mathrm{Int}\, H_i$. Let X be an arc from a point $b \in f(\mathrm{Bd}\, K)$ to a point q in $S - \sum H_i$ such that $X - q$ lies in V. For each $i (1 \leqq i \leqq t)$ there is a disk H_i' such that $\mathrm{Bd}\, H_i' = \mathrm{Bd}\, H_i$, $\mathrm{Int}\, H_i' \subset V \cap N$, and $g(\mathrm{Bd}\, K) + X$ lies in $\mathrm{Ext}\,(H_i + H_i')$. This is because S is tame from V at each point of each H_i[1; 17].

Using Lemma 2 we obtain a map h of K into E^3 such that $h \mid \mathrm{Bd}\, K = f$, $h(K) \subset g(K) + \sum \mathrm{Int}\, H_i'$, and $h(K) - \sum H_i'$ is connected. Obviously $h(K) \subset N$. Since $h(K)$ is connected and $h(\mathrm{Bd}\, K) \subset V$, it will follow that $h(K) \subset V$ if we show that $h(K) \cap S = \varnothing$. To show this we use the fact that $h(K) - \sum H_i'$ is arcwise connected.

Suppose $h(K)$ intersects S. Let Y be an arc in $h(K) - \sum H_i'$ from the point b to a point d in S such that $(Y - d) \subset V$. Since $g(K) \cap S \subset \sum \mathrm{Int}\, H_i$ and $h(K) \subset g(K) + \sum \mathrm{Int}\, H_i'$, it follows that $d \in \mathrm{Int}\, H_j$ for some $j (1 \leqq j \leqq t)$. Let Z be an arc from the point q to d such that $Z - (d + q) \subset E^3 - (S + V)$. Then $X + Y + Z$ contains a simple closed curve L which links $\mathrm{Bd}\, H_j$ [7, Theorem 3.3], and it follows from Theorem 10 of [2] that L links $\mathrm{Bd}\, H_j'$. Therefore L intersects H_j'. But this is a contradiction since $X + Y + Z$ does not intersect H_j'.

Thus $h(K) \subset N \cap V$, so V is locally simply connected at each point p of $W \cap F$. Then, from the hypothesis, V is locally simply connected at each point of W. Using Theorems 1 and 2 of [10], it follows that S is locally tame from V at each point of W.

THEOREM 15. *If a 2-sphere S in E^3 is locally tame modulo a closed subset F which satisfies $(*, F, S)$, then S is tame.*

Proof. Using $W = S$ in Theorem 14, we have the result that S is locally tame from each of its complementary domains. Then S is tame from each such domain; hence S is tame.

REMARK. It follows from Theorem 15 that F cannot contain an isolated wild point of S if $(*, F, S)$ is satisfied. Also the set W of wild points of a 2-sphere S fails to satisfy $(*, W, S)$, since W is closed.

4. Property $(*, F)$ and its relation to $(*, F, S)$. We now define Property $(*, F)$. If a closed subset F of a 2-sphere S in E^3 has no degenerate components, we suspect that Property $(*, F)$ is satisfied if and only if Property $(*, F, S)$ holds. However Theorem 16 is as close as we have come to establishing this equivalence.

Property $(*, F)$. A closed set F has Property $(*, F)$ if and only if it satisfies each of the following conditions:

1. F is a subset of some 2-sphere in E^3.

2. If S is a 2-sphere in E^3 such that S contains F and S is locally tame modulo F, then S is tame.

If each of the following sets lies on some 2-sphere in E^3, then each is an example of a set F satisfying property $(*, F)$: (1) a tame disk [2; 11], (2) a tame finite graph [11, Corollary 1], (3) a tame Sierpinski curve [9, Theorem 8.2], and (4) a set which is the union of a finite number of tame finite graphs and tame Sierpinski curves [9, Theorem 8.4]. If F consists of a single point, it is easy to see that F does not satisfy $(*, F)$ (see the example of a 2-sphere which is wild at a single point, as given in [12]). This raises a question which is related to one asked in [13, p. 464]. Does a closed set F satisfy $(*, F)$ if F lies in a tame 2-sphere and F has no degenerate components? We do not answer this question.

THEOREM 16. *A closed subset F of a 2-sphere satisfies $(*, F)$ if and only if $(*, F, S)$ is satisfied relative to each 2-sphere S containing F.*

Proof. Suppose F has Property $(*, F, S)$ relative to each 2-sphere S containing F. We want to show that $(*, F)$ holds, so we let S' be a 2-sphere such that S' contains F and S' is locally tame modulo F. We apply Theorem 15 to see that S' is tame. As suggested by Gillman, the other half of Theorem 16 is proved using the techniques in the proof of Theorem 2 in [13].

THEOREM 17. *If S^* is a 2-sphere in E^3 containing a finite collection of closed sets $F_1, F_2, \cdots, F_n$ such that each F_i satisfies $(*, F_i)$, then $(*, \sum F_i)$ also holds.*

Proof. Let S be a 2-sphere containing $\sum F_i$ such that S is locally tame modulo $\sum F_i$. We will complete the proof by showing that S is tame. First we observe from Theorem 16 that $(*, F_i, S)$ holds for each i. Now an application of Theorem 14 shows that S is locally tame from Ext S at each point of the open set $U_1 = S - \sum_{i=2}^{n} F_i$. Another application of Theorem 14 shows that S is locally tame from Ext S at each point of the open set $U_2 = S - \sum_{i=3}^{n} F_i$. We continue applying Theorem 14 until we have S locally tame from Ext S at each point of $U_n = S$. Then S is tame from Ext S. A similar argument shows that S is tame from Int S. Hence S is tame.

THEOREM 18. *If S is a 2-sphere in E^3; $F_1, F_2, \cdots, F_n$ is a finite collection of closed subsets of S such that $(*, F_i)$ holds for each i;*

and U is an open subset of S such that S is locally tame at each point of $U - (\sum F_i)$, then S is locally tame at each point of U.

Proof. Although a proof similar to the proof of Theorem 17 can also be given here, we give an alternative method. We apply Theorem 17 to see that $(*, \sum F_i)$ holds; then we use Theorem 16 to see that $(*, \sum F_i, S)$ is true. Now two applications of Theorem 14 will show that S is locally tame from each of its complementary domains at each point of U. Thus S is locally tame at each point of U.

REMARK. It is known that if F lies on some 2-sphere in E^3 and F is either a tame finite graph or a tame Sierpinski curve, then $(*, F)$ is satisfied [11; 9]. Theorem 19 shows the existence of a continuum M satisfying $(*, M)$ such that M is not a finite union of tame finite graphs and tame Sierpinski curves. It follows that Theorems 17 and 18 are generalizations of Theorems 8.4 and 8.5, respectively, in [9]. It also follows from Theorem 19 that the continuum M in the conclusion of Theorem 6 satisfies $(*, M)$.

Theorem 19. *If M is a tame continuum on a 2-sphere S in E^3 such that M is obtained by removing from S the interiors of a null sequence of disjoint disks on S, then $(*, M)$ is satisfied.*

Proof. Let $M = S - \sum \operatorname{Int} D_i$, where $\{D_i\}$ is a null sequence of disjoint disks on S. Suppose S' is a 2-sphere containing M such that S' is locally tame mod M. We will show that S' is tame.

Let E_i be the disk on S' such that $\operatorname{Bd} E_i = \operatorname{Bd} D_i$, for each i. It is easily seen that $\{E_i\}$ must be a null sequence.

Let $G = \{H_1, H_2, \cdots\}$, be an infinite collection of disjoint disks of S' such that $\lim \operatorname{diam} H_i = 0$, each E_i is in G, and $\sum H_i$ is dense in S'. Then $S' - \sum \operatorname{Int} H_i$ is a Sierpinski curve K[20], and since $K \subset M$, it follows that K is tame. Furthermore, since S' is locally tame at each point of $\sum \operatorname{Int} H_i$, it follows that S' is locally tame modulo K. Since $(*, K)$ is satisfied [9], then S' is tame.

THEOREM 20. *If S is a 2-sphere in E^3, $\{D_i\}$ is a null sequence of disjoint disks on S, and $M = S - \sum \operatorname{Int} D_i$, then the following statements are equivalent:*

1. *M is tame.*
2. *(A, M, S) is satisfied.*
3. *$(*, M, S)$ is satisfied.*
4. *$(*, M)$ is satisfied.*
5. *(B, M, S) is satisfied.*
6. *(A', M, S) is satisfied.*

Proof. Since M is a nondegenerate continuum, we have already proven that Statements 2, 3, 5, and 6 are equivalent (see Theorems 8, 9, 10, and 13). Bing's proof of Theorem 8.1 in [9] also shows that Statement 1 implies 2. Since 2 implies 3, the proof will be completed if we show that 3 implies 4 and 4 implies 1. If 3 is satisfied it follows from Theorem 6 that 1 is true. Then 4 follows from Theorem 19. Thus 3 implies 4. Section 8 of [8] shows the existence of a 2-sphere S' containing M such that S' is locally tame modulo M. Then 4 implies 1 because S' is tame if 4 is satisfied.

REMARK. Statements 1 through 6 of Theorem 20 characterize tame Sierpinski curves on 2-spheres in E^3. This follows because if K is a Sierpinski curve on S, then there is a null sequence of disjoint disks $D_1, D_2, \cdots$, on S such that $K = S - \sum \operatorname{Int} D_i$ [19].

5. Finite sums of sets F satisfying (*, F, S). The following theorem is used in [16].

THEOREM 21. *If $F_1, F_2, \cdots, F_n$ is a finite collection of closed subsets of a 2-sphere S in E^3 such that $(*, F_i, S)$ is satisfied for each i, then $(*, \sum F_i, S)$ also holds.*

Proof. We use Theorem 6 to obtain a finite collection of tame continua $M_1, M_2, \cdots, M_n$ on S such that, for each i, $(*, M_i, S)$ is satisfied, $F_i \subset M_i$, and M_i is obtained by removing from S the interiors of a null sequence of disjoint disks on S. From Theorem 19 we see that $(*, M_i)$ is satisfied for each i. Then it follows from Theorem 17 that $(*, \sum M_i)$ holds. Now we use Theorem 16 to see that $(*, \sum M_i, S)$ is satisfied, and $(*, \sum F_i, S)$ follows because $\sum F_i \subset \sum M_i$.

REMARK. We note that Theorem 21 cannot be extended to the case where the F_i's form an infinite collection of closed subsets of S. To see this, let S be a wild 2-sphere in E^3 and for each i let F_i be a continuum M_i as in the statement of Theorem 7. Then $(*, M_i, S)$ holds for each i and $S - \sum M_i$ is a 0-dimensional set. Suppose $(*, \sum M_i, S)$ holds. Since $S - \sum M_i$ contains no disk it follows that S can be homeomorphically approximated in each of its complementary domains. This is a contradiction since under these conditions S is tame [3, Theorem 2.2]. Is $(*, \sum F_i, S\}$ satisfied if $\{F_i\}$ is an infinite collection of closed subsets of S such that $\sum F_i$ is closed and $(*, F_i, S)$ holds for each i?

THEOREM 22. *If $F_1, F_2, \cdots, F_n$ is a finite collection of closed subsets of a 2-sphere S in E^3 such that for each i*

1. *either* (A, F_i, S) *or* (B, F_i, S) *is satisfied, and*
2. *the diameters of the components of* F_i *have a positive lower bound,*

then $(A, \sum F_i, S)$ *and* $(B, \sum F_i, S)$ *are satisfied.*

Proof. Because $(*, F_i, S)$, (A, F_i, S), and (B, F_i, S) are equivalent under Condition 2 of the hypothesis (see Theorem 8, 9 and 10), Theorem 22 is a direct corollary to Theorem 21.

THEOREM 23. *If* $F_1, F_2, \cdots, F_n$ *is a finite collection of disjoint closed subsets of a 2-sphere* S *in* E^3 *such that either* (A, F_i, S) *or* (B, F_i, S) *is satisfied for each* i, *then* $(A, \sum F_i, S)$ *is satisfied.*

Proof. We will show that $(A, F_1 + F_2, S)$ is satisfied, then by induction and Theorem 9 the theorem will follow. Let ε be a positive number. We assume that $3\varepsilon < \rho(F_1, F_2)$. There are positive numbers δ_1 and δ_2, obtained using the definitions of (A, F_1, S) and (A, F_2, S), respectively, such that for each i $(i = 1, 2)$ each δ_i-simple closed curve in $E^3 - S$ can be shrunk to a point in an ε-subset of $E^3 - F_i$. We choose $\delta = \min(\delta_1, \delta_2)$, and suppose that J is a δ-simple closed curve in $E^3 - S$. Let N be an ε-neighborhood of J. Since $\delta < \varepsilon$ it follows that N is a 3ε-subset of E^3. Then N cannot intersect both F_1 and F_2. Since $\delta \leqq \delta_i (i = 1, 2)$, J can be shrunk to a point in $N-(F_1+F_2)$. Thus we have established Property $(A, F_1 + F_2, S)$.

REMARK. We do not know whether Theorem 23 is true without the requirement that the F_i's be disjoint. We have an affirmative answer in the special case where the diameters of the components of each F_i are bounded below by a positive number, since in this case Properties (A, F_i, S), (B, F_i, S), and $(*, F_i, S)$ are equivalent (see Theorems 8, 9, 10, and 22).

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