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A THEOREM ON LATTICE ORDERED GROUPS, RESULTS OF PTAK, NAMIOKA AND BANACH, AND A FRONT-ENDED PROOF OF LEBESGUES THEOREM

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The main theorem in this paper is on (not necessarily commutative) lattice ordered groups, and is a generalization of a result on finitely additive set functions due to Namioka. Our result can be used to prove Ptak's combinatorial theorem on convex means, to give a short non measure-theoretic proof of Lebesgue's dominated convergence theorem for a sequence of continuous functions on a countably compact topological space, and to give a short proof of Banach's criteria for the weak convergence of a sequence in the Banach space of all bounded, real functions on an abstract set.

We shall prove the following result.

THEOREM 1. *If L is a lattice ordered group, $\{g_1, g_2, \cdots\}$ is a sequence of positive elements in L, and φ is an order-preserving homomorphism of L into the real numbers such that the sequence $\varphi(g_1 \vee \cdots \vee g_p)(p = 1, 2, \cdots)$ is bounded above and $\lim \sup_p \varphi(g_p) > 0$ then there exist integers $0 < r(1) < r(2) < \cdots$ such that, for each s, $\varphi(g_{r(1)} \wedge \cdots \wedge g_{r(s)}) > 0$.*

The idea for this stems from the following result of Namioka.

Namioka's Theorem. If X is a nonvoid set, S is a field of subsets of X, $\{A_1, A_2, \cdots\}$ is a sequence in S and μ is a positive, finitely additive function on S such that $\lim \sup_p \mu(A_p) > 0$ then there exist integers $0 < r(1) < r(2) < \cdots$ such that, for each s, $\mu(A_{r(1)} \cap \cdots \cap A_{r(s)}) > 0$.

Namioka's Theorem can be found in [2, 17.9, p. 157] and [4, Lemma 2, p. 714]—it can clearly be deduced from our result by taking L to be the set of all S-simple functions and $\varphi(\cdot) = \int \cdot \, d\mu$. Namioka's theorem was proved in order to give a proof of Krein's Theorem on weak compactness that avoids measure theory. This has also been done, in a superficially very different way, by Ptak, using his combinatorial theorem on the existence of convex means, which appears in print in [3, §24, No. 6, p. 331], [5, 1.3, p. 439] and [6]. Ptak shows that Namioka's theorem can be deduced from his ([5, 5.9, p. 447]—Ptak proves a slightly weaker form in which the conclusion "$\mu(A_{r(1)} \cap \cdots \cap A_{r(s)}) > 0$" is replaced by "$A_{r(1)} \cap \cdots \cap A_{r(s)} \neq \phi$").

Ptak's Theorem. We suppose that K is an infinite set and that X is a nonvoid family of subsets of K. We write $P(K)$ for the collection of all positive, real valued functions λ on K such that $\{k: k \in K, \lambda(k) > 0\}$ is finite and $\sum_{k \in K} \lambda(k) = 1$; for $x \subset K$ we write $\lambda(x) = \sum_{k \in x} \lambda(k)$. If

$$\inf_{\lambda \in P(K)} \sup_{x \in X} \lambda(x) > 0$$

then there exist $x_1, x_2, \cdots \in X$ and distinct $k_1, k_2, \cdots \in K$ such that, for each s, $\{k_1, \cdots, k_s\} \subset x_s$.

We shall show that Ptak's Theorem can be deduced from Theorem 1 in a natural way and that the "convexity" is a consequence of the result that weak and the norm closures of a convex subset of a normed linear space coincide.

If X is a nonvoid set we write $B(X)$ for the set of all bounded, real functions on X. $B(X)$ is a Banach space under the norm $\|f\| = \sup_{x \in S} |f(x)|$. We shall show that Theorem 1 can be used to give criteria for the weak convergence of a sequence in $B(X)$.

Finally, we shall show how Theorem 1 can be used to give a short non measure-theoretic proof of Lebesgue's dominated convergence theorem for a sequence of *continuous* functions on a countably compact topological space.

Theorem 1 was first proved in a different context. I would like to thank Professor K. Fan for reading the manuscript and suggesting the possibility of an application to Lebesgue's theorem.

2. Proof of Theorem 1. Using the identities $f - f \vee g + g \leqq f \wedge g$ and $(f \vee g) \wedge h (= (f \wedge h) \vee (g \wedge h)) \leqq f \wedge h + g \wedge h$, valid for any positive f, g, and h in L, we see that, if q, p are integers and $0 < q < p$,

$$\begin{aligned} \varphi(g_p) + \varphi(g_1 \vee \cdots \vee g_q) &\leqq \varphi((g_1 \vee \cdots \vee g_q) \vee g_p) \\ &\quad + \varphi((g_1 \vee \cdots \vee g_q) \wedge g_p) \\ &\leqq \varphi(g_1 \vee \cdots \vee g_p) + \sum_{1 \leqq r \leqq q} \varphi(g_r \wedge g_p). \end{aligned}$$

Letting $p \to \infty$,

$$\begin{aligned} \limsup_p \varphi(g_p) + \varphi(g_1 \vee \cdots \vee g_q) &\leqq \lim_p \varphi(g_1 \vee \cdots \vee g_p) \\ &\quad + \sum_{1 \leqq r \leqq q} \limsup_p \varphi(g_r \wedge g_p). \end{aligned}$$

Letting $q \to \infty$,

$$\begin{aligned} \limsup_p \varphi(g_p) + \lim_q \varphi(g_1 \vee \cdots \vee g_q) &\leqq \lim_p \varphi(g_1 \vee \cdots \vee g_p) \\ &\quad + \sum_{r \geqq 1} \limsup_p \varphi(g_r \wedge g_p). \end{aligned}$$

By hypothesis, $\lim_p \varphi(g_1 \vee \cdots \vee g_q) = \lim_p \varphi(g_1 \vee \cdots \vee g_p) < \infty$ and

$\lim\sup_p \varphi(g_p) > 0$. Hence

$$0 < \sum_{r \geqq 1} \lim\sup_p \varphi(g_r \wedge g_p)$$

and thus there exists $r(1)$ such that $\lim\sup_p \varphi(g_{r(1)} \wedge g_p) > 0$ — and, from this, $\varphi(g_{r(1)}) > 0$.

Performing a similar argument on the elements $g_{r(1)} \wedge g_{r(1)+1}$, $g_{r(1)} \wedge g_{r(1)+2}, \cdots$ we can show that there exists $r(2) > r(1)$ such that $\lim\sup_p \varphi((g_{r(1)} \wedge g_{r(2)}) \wedge (g_{r(1)} \wedge g_p)) > 0$, i.e.,

$$\lim\sup_p \varphi(g_{r(1)} \wedge g_{r(2)} \wedge g_p) > 0 \text{ — and, from this } \varphi(g_{r(1)} \wedge g_{r(2)}) > 0 .$$

Continuing this process inductively, we complete the proof of the theorem.

3. Application of Theorem 1 to Ptak's theorem. If ψ is a bounded linear functional on $B(X)$ then there exists a positive linear functional φ on $B(X)$ such that, for all $f \in B(X)$, $|\psi(f)| \leqq \varphi(|f|)$. (If $f \geqq 0$ we define $\varphi(f)$ to be $\sup_{|g| \leqq f} \psi(g)$ and extend φ to the whole of $B(X)$ by linearity).

COROLLARY 2. *Let X be a nonvoid set and Y a bounded subset of $B(X)$. Let Z be the convex extension of Y in $B(X)$. Then* (a) *implies* (b) *and* (b) *implies* (c).

(a) $\operatorname{Inf}_{z \in Z} \|z\| > 0$.

(b) *There exists a positive linear functional φ on $B(X)$ such that $\inf_{y \in Y} \varphi(|y|) > 0$.*

(c) *For each sequence $\{y_1, y_2, \cdots\}$ in Y there exist integers $0 < r(1) < r(2) < \cdots$ and $x_1, x_2, \cdots \in X$ such that $y_{r(t)}(x_s) \neq 0$ whenever $0 < t \leqq s$.*

Proof of Corollary 2. If (a) is true then $0 \notin$ norm-closure of Z. Thus, since Z is convex, $0 \notin$ weak-closure of Z, hence $0 \notin$ weak-closure of Y. Thus there exists a bounded linear functional ψ on $B(X)$ such that $\inf_{y \in Y} |\psi(y)| > 0$. If the positive linear functional φ on $B(X)$ is chosen as in the remarks preceding this Corollary then (b) is satisfied.

If (b) is true, φ is as in (b) and $y_1, y_2, \cdots \in Y$ then $\lim\sup_p \varphi(|y_p|) > 0$. We apply Theorem 1 to $L = B(X)$ and $g_p = |y_p|$ and find that there exist integers $0 < r(1) < r(2) < \cdots$ such that, for each s, $\varphi(|y_{r(1)}| \wedge \cdots \wedge |y_{r(s)}|) > 0$. It follows from this that (c) is satisfied.

REMARK. It can easily be seen that, if X, Y and Z are as above and φ is a positive linear functional on $B(X)$ then

$$\inf_{y\in Y}\varphi(y) \leqq \varphi(1)\ \inf_{z\in Z}\|z\|$$

and so, if all the functions in Y are *positive*, condition (b) of the Corollary implies condition (a). This is, essentially, the proof used in [5, 5.9, p. 447].

Proof of Ptak's Theorem. If K is any nonvoid set and X is any nonvoid family of subsets of K then, for each $k \in K$, we define $y_k \in B(X)$ by $y_k(x) = 0$ if $k \notin x$ and $y_k(x) = 1$ if $k \in x$. We apply Corollary 2 ((a) implies (c)) to $Y = \{y_k : k \in K\}$ and obtain: if

$$\inf_{\lambda\in P(K)} \sup_{x\in X} \lambda(x) > 0$$

then, for all $k_1, k_2, \cdots \in K$, there exist integers $0 < r(1) < r(2) < \cdots$ and $x_1, x_2, \cdots \in X$ such that, for each s, $\{k_{r(1)}, \cdots, k_{r(s)}\} \subset x_s$. This is formally stronger than, though in fact equivalent to, Ptak's Theorem.

4. Application of Theorem 1 to Lebesgue's theorem of dominated convergence.

DEFINITION 3. Let X be a nonvoid set and $f_1, f_2, \cdots \in B(X)$. We shall say that $\{f_1, f_2, \cdots\}$ is a *Dini sequence* if $|f_1| \wedge \cdots \wedge |f_s| \to 0$ uniformly on X as $s \to \infty$.

COROLLARY 4. *Let X be a nonvoid set, $\{f_1, f_2, \cdots\}$ be a sequence in $B(X)$ with the property that all its subsequences are Dini sequences and φ be a positive linear functional on $B(X)$ such that the sequence $\varphi(|f_1| \vee \cdots \vee |f_p|)$ $(p = 1, 2, \cdots)$ is bounded above. Then $\varphi(f_n) \to 0$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$ be given. For each p we write $g_p = (|f_p| - \varepsilon 1) \vee 0$. If $0 < r(1) < r(2) < \cdots$ are integers then, by hypothesis, for all sufficiently large s, $g_{r(1)} \wedge \cdots \wedge g_{r(s)} = 0$ hence

$$\varphi(g_{r(1)} \wedge \cdots \wedge g_{r(s)}) = 0\ .$$

Thus, from Theorem 1, $\lim\sup_p \varphi(g_p) = 0$.

Since $|f_p| - \varepsilon 1 \leqq g_p$, it follows that $\lim\sup_p \varphi(|f_p| - \varepsilon 1) \leqq 0$ and hence that $\lim\sup_p \varphi(|f_p|) \leqq \varepsilon\varphi(1)$. Since ε is arbitrary, this implies that $\lim\sup_p \varphi(|f_p|) = 0$. The required result follows since $|\varphi(f_n)| \leqq \varphi(|f_n|)$.

If X is a countably compact topological space we write $C(X)$ for the set of all real, continuous functions on X. Corollary 4 is, in fact, true with "$B(X)$" replaced everywhere by "$C(X)$"—the proof is identical. In fact any positive linear functional on $C(X)$ can be extended by the Hahn-Banach Theorem to one on $B(X)$ and so the result for $C(X)$ can also be *deduced* from the result for $B(X)$.) If $f_1, f_2, \cdots$

$\in C(X)$ and $f_n \to 0$ pointwise as $n \to \infty$ then, from Dini's Theorem, every subsequence of $\{f_1, f_2, \cdots\}$ is a Dini sequence. The following result is then immediate from the "$C(X)$" version of Corollary 4.

Lebesgue's Theorem. If X is a countably compact topological space, $f_1, f_2, \cdots \in C(X)$, $f_n \to 0$ pointwise as $n \to \infty$, and φ is a positive linear functional on $C(X)$ such that the sequence $\varphi(|f_1| \vee \cdots \vee |f_p|)(p = 1, 2, \cdots)$ is bounded above, then $\varphi(f_n) \to 0$ as $n \to \infty$.

Our final result is a slightly expanded form of a theorem due to Banach. (See [**1**, Annexe, §2, Theorem 5, p. 219] and [5, 5.4, p. 445].)

Banach's Theorem. Let X be a nonvoid set and $f_1, f_2, \cdots$ be a *bounded* sequence in $B(X)$. Then the conditions (a)—(d) are equivalent.

(a) If $x_1, x_2, \cdots \in X$ then $\lim_n \liminf_i |f_n(x_1)| = 0$.

(b) Every subsequence of $\{f_1, f_2, \cdots\}$ is a Dini sequence.

(c) $\varphi(|f_n|) \to 0$ for every positive linear functional φ on $B(X)$.

(d) $f_n \to 0$ weakly in $B(X)$.

Proof. If follows from the definition of a Dini sequence that (a) implies (b), from Corollary 4 that (b) implies (c), and from the remarks preceding Corollary 2 that (c) implies (d).

If (a) is false and $x_1, x_2, \cdots \in X$ are such that

$$\limsup_n \liminf_i |f_n(x_i)| > \varepsilon > 0$$

then there exist integers $0 < n(1) < n(2) < \cdots$ such that, for each k, $\liminf_i |f_{n(k)}(x_i)| > \varepsilon$. By the diagonal process we can find integers $0 < i(1) < i(2) < \cdots$ such that, for each k, $\lim_j (f_{n(k)})(x_{i(j)})$ exists and has absolute value greater than ε. From the Hahn-Banach Theorem, there exists a positive linear functional φ on $B(X)$ such that $\varphi(f) = \lim_j f(x_{i(j)})$ whenever $f \in B(X)$ is such that the limit exists. For this value of φ, $\varphi(f_n) \not\to 0$. Thus (d) implies (a).

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