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ON A THEOREM BY HOFFMAN AND RAMSAY

STELIOS NEGREPONTIS

Hoffman and Ramsay recently constructed, using the continuum hypothesis, a proper subalgebra of the algebra $C(\beta N)$ of all complex-valued continuous functions on βN which is uniformly closed and separates points of βN. (This example settles in the negative a conjecture by Badé and Curtis). The purpose of this note is to indicate that this result does not need the continuum hypothesis.

Their construction, in [3], naturally falls into two parts, the first of which is devoted to the proof that the space X of maximal ideals of the algebra L^∞ of all essentially bounded measurable complex-valued functions on the unit circle is homeomorphic to a subset of $\beta N - N$. It is in this part that they are forced to use the continuum hypothesis.

1. We have a simple proof of this fact without appeal to the continuum hypothesis.

THEOREM. *The space X is homeomorphic to a subset of $\beta N - N$.*

Proof. We first show that there is a compact separable space S in which X is embedded. This can be seen in at least two distinct ways. By a theorem in [2, p. 174] we can regard X as a subset of $\mathscr{M}(H^\infty)$, the space of maximal ideals of H^∞. (For a definition of H^∞, the reader should consult [2]). According to the "Corona" conjecture proved by Carleson, the open unit disc is dense in $\mathscr{M}(H^\infty)$, and thus, $\mathscr{M}(H^\infty)$ is a separable space. Thus, we may take $S = \mathscr{M}(H^\infty)$. An alternative, and more elementary, way is to notice that $L^\infty = C(X)$, a fact proved in [2, p. 170], and hence that $C(X)$ has cardinality $\boldsymbol{C} = 2^{\aleph_0}$; but this clearly implies that we can embed X in the compact "cube" $[0, 1]^{\mathrm{C}}$, which according to [4, $3N$] is separable. Thus, we may take $S = [0, 1]^{\mathrm{C}}$.

In any case, let $\varphi : N \rightarrow S$ be any mapping onto a dense subset of S. The Stone-Cech extension of φ, denoted by $\bar{\varphi}$, provides a mapping of βN onto S. As it is well-known, using Zorn's lemma, we can find a closed subset A of $\bar{\varphi}^{-1}(X)$ such that $\bar{\varphi} : A \rightarrow X$ is irreducible, i.e. $\bar{\varphi}$ is onto, and $\bar{\varphi}(B) \neq X$ for every proper closed subset B of A. Gleason's lemma [1, 2.3], which applies since X is extremally disconnected, states that $\bar{\varphi}$ is a homeomorphism. We can embed βN into $\beta N - N$, and hence, the theorem is proved.

2. REMARKS. (i) The initial attempt to prove the above theorem, which failed, was the following: Let S^1 be the unit circle in the plane, and let $\sigma : X \to S^1$ be the natural projection (as described in [2, p. 171]). A quick proof of the theorem would follow, if it could be shown that σ were irreducible; but this is not the case.

(ii) The space X is the only example known to the author of an extremally disconnected subspace of βN which is not separable. Surely X is not a retract of βN. It follows that although the projective objects (in the category of compact spaces and continuous maps) are "the retracts of the free objects" (in the terminology of Rainwater [5]), it is not true that a projective object is a retract of every free object in which it is embedded.

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INDIANA UNIVERSITY
AND
McGILL UNIVERSITY