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THE DEDEKIND COMPLETION OF $C(\mathscr{X})$

J. E. MACK AND D. G. JOHNSON

The question to which this study addresses itself is the following: given a completely regular space $\mathscr{X}$, is the Dedekind completion of $C(\mathscr{X})$ isomorphic to $C(\mathscr{Y})$ for some space $\mathscr{Y}$? Here, $C(\mathscr{X})$ denotes the ring of continuous real-valued functions on $\mathscr{X}$ under pointwise order. Affirmative answers were provided by Dilworth for the class of compact spaces in 1950 and by Weinberg for the class of countably paracompact and normal spaces in 1960. It remained an open question whether there were any spaces for which a negative answer held. In this paper, we provide a necessary and sufficient condition that the Dedekind completion of $C(\mathscr{X})$, for $\mathscr{X}$ a realcompact space, be isomorphic to $C(\mathscr{Y})$ for some $\mathscr{Y}$. Using this, we are able to provide an example of a space $\mathscr{X}$ for which the Dedekind completion of $C(\mathscr{X})$ is not isomorphic to $C(\mathscr{Y})$ for any space $\mathscr{Y}$.

Specifically, we define and characterize a class of spaces which we call *weak cb-spaces*: those spaces $\mathscr{X}$ with the property that every locally bounded, lower semicontinuous function on $\mathscr{X}$ is bounded above by a continuous function. We then prove that for an arbitrary (completely regular) space $\mathscr{X}$, the Dedekind completion of $C(\mathscr{X})$ is isomorphic to some $C(\mathscr{Y})$ if and only if $\nu\mathscr{X}$ (the Hewitt realcompactification of $\mathscr{X}$) is a weak *cb*-space. The sufficiency of this condition actually generalizes Weinberg's result, as is shown by examples; the necessity provides the negative result referred to above.

The preliminary investigation of the Dedekind completion is done in Section 1, in the setting of an arbitrary Φ-algebra. In Section 2, we study the connection between the lattice of normal upper semicontinuous functions on a completely regular space $\mathscr{X}$ and the minimal projective extension of $\mathscr{X}$. This leads to the observation, in Section 4, that for a weak *cb*-space $\mathscr{X}$ the Dedekind completion of $C(\mathscr{X})$ is isomorphic to $C(\mathscr{Y})$, where $\mathscr{Y}$ is the minimal projective extension of $\mathscr{X}$. Weak *cb*-spaces are studied in Section 3, and Section 4 contains our main result.

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The Dedekind completion of an arbitrary Φ-algebra. A *Φ-algebra* is a real archimedean lattice-ordered algebra with identity element 1 that is a weak-order unit. A *homomorphism* of a Φ-algebra is an algebra homomorphism that also preserves lattice operations. The kernel of a homomorphism is an $\mathscr{L}$*-ideal*: a ring ideal I which satisfies the condition: if $a \in I$ and $|b| \leqq |a|$, then $b \in I$. If I is an $\mathscr{L}$-ideal in A, then A/I is a lattice-ordered algebra and the natural algebra homomorphism $A \rightarrow A/I$ is a homomorphism: for $a \in A$, we let $I(a)$ denote the image of under this natural homomorphism. The set of positive elements of A is denoted A_+.

Φ-algebras are studied in [7], where it is shown that for any Φ-algebra A, the set $\mathscr{M}(A)$ of maximal $\mathscr{L}$-ideals of A is a compact (Hausdorff) space under the hull-kernel topology. Moreover, A is isomorphic to a Φ-algebra of extended real-valued functions on $\mathscr{M}(A)$ (i.e., continuous functions on $\mathscr{M}(A)$ into the two-point compactification of the real line that are real-valued on dense subsets of $\mathscr{M}(A)$). The set of real ideals of A (those $M \in \mathscr{M}(A)$ with A/M the real field) is denoted $\mathscr{R}(A)$; A is a *Φ-algebra of real-valued functions* if $\cap \mathscr{R}(A) = \{0\}$ (i.e., if $\mathscr{R}(A)$ is dense in $\mathscr{M}(A)$).

Let A be a Φ-algebra. In [8], it is shown that A can be embedded in a complete Φ-algebra. Precisely, there is a complete Φ-algebra $\hat{A}$ and an isomorphism α of A onto a subalgebra αA of $\hat{A}$ such that the following conditions hold.

(i) αA is *dense* in $\hat{A}$, in the sense that each $a \in \hat{A}$ satisfies $\sup\{\alpha b: b \in A \text{ and } \alpha b \leqq a\} = a = \inf\{\alpha b: b \in A \text{ and } \alpha b \geqq a\}$. It follows from this that α preserves *all* suprema and infima in A.

(ii) $\hat{A}$ is unique in the following sense: if β is an isomorphism of A onto a dense subalgebra of a complete lattice-ordered algebra B, then there is an isomorphism γ of B onto $\hat{A}$ such that $\gamma\beta = \alpha$.

The complete Φ-algebra $\hat{A}$ (together with the mapping α) is called the *Dedekind completion* of A.

Let A be a Φ-algebra and $\hat{A}$ its Dedekind completion. Thus, we may view A as a dense subring of $\hat{A}$ which contains the identity element 1 of $\hat{A}$.

For any $\mathscr{L}$-ideal I of A, set

$$\bar{I} = \{b \in \hat{A}: |b| \leqq a \text{ for some } a \in I\} .$$

It is readily verified that $\bar{I}$ is an $\mathscr{L}$-ideal of $\hat{A}$ and that if $I(1)$ is a strong order unit in A/I, then $\bar{I}(1)$ is a strong order unit in $\hat{A}/\bar{I}$. Hence, if $M \in \mathscr{R}(A)$ and if $\hat{M}$ denotes any maximal $\mathscr{L}$-ideal of $\hat{A}$ containing M, then $\hat{M} \in \mathscr{R}(\hat{A})$. Conversely, if $\hat{M} \in \mathscr{R}(\hat{A})$, then $\hat{M} \cap A \in \mathscr{R}(A)$.

PROPOSITION 1.1. *If A is a Φ-algebra of real-valued functions, then so is $\hat{A}$.*

Proof. Let $0 \neq b \in \hat{A}$; we may suppose $b > 0$. Since A is dense in $\hat{A}$, there is an element $0 \neq a \in A$ with $0 \leqq a \leqq b$. Take $M \in \mathscr{R}(A)$ not containing a. If $\hat{M}$ is any maximal $\mathscr{L}$-ideal of $\hat{A}$ containing M, then $a \notin \hat{M}$, so $b \notin \hat{M}$. Thus, $\cap \mathscr{R}(\hat{A}) = \{0\}$.

Thus, if A is an algebra of real-valued functions, then we may view $\hat{A}$ as a subring of $C(\mathscr{R}(\hat{A}))$. By Theorem 5.6 of [7], $\hat{A} = C(\mathscr{R}(\hat{A}))$ if and only if $\hat{A}$ is closed under *inversion*: i.e., whenever $a \in \hat{A}$ with $a \notin \hat{M}$ for each $\hat{M} \in \mathscr{R}(\hat{A})$, we have $1/a \in \hat{A}$. Now, $\hat{A}$ forms an order-convex subset of the set of all extended real-valued functions on $\mathscr{M}(\hat{A})$ ([7], Lemma 3.7). Hence, in order to verify that $\hat{A}$ is closed under inversion, it is enough to suppose $a \geqq 0$ and to show that from $a \notin \hat{M}$ for each $\hat{M} \in \mathscr{R}(A)$ it follows that there is $b \in A$ with $b \geqq 1/a$. In case A is closed under inversion, it suffices to exhibit $b \in A$ with $0 \leqq b \leqq a$ and $b \notin M$ for each $M \in \mathscr{R}(A)$.

LEMMA 1.2. *If A is closed under bounded inversion* (i.e., *if $1/f \in A$ whenever $1 \leqq f \in A$*), *and if $0 \leqq a \in \hat{A}$ and $M \in \mathscr{M}(A)$ are such that $a \notin \hat{M}$ for each maximal $\mathscr{L}$-ideal $\hat{M}$ of $\hat{A}$ with $\hat{M} \supseteqq M$, then there is $b \in A$ with $0 \leqq b \leqq a$ and $b \notin M$.*

Proof. By the hypotheses on a and M, the $\mathscr{L}$-ideal of $\hat{A}$ generated by a and M is all of $\hat{A}$. Hence, there are $c \in \hat{A}$ and $d \in M$ with $1 \leqq ac + d$. Since A is dense in $\hat{A}$, we may choose $c' \in A$ with $c' \geqq c$, so $1 - d \leqq ac'$. Since we may also choose $c' \geqq 1$, we have

$$(1 - d)\frac{1}{c'} \leqq a \ ,$$

since A is closed under bounded inversion.

Set $b = ((1 - d)(1/c')) \vee 0$. Then $0 \leqq b \leqq a$, and

$$M(b) = \left[M(1 - d)M\left(\frac{1}{c'}\right)\right] \vee M(0) = M\left(\frac{1}{c'}\right) \vee M(0) \ ,$$

since $d \in M$. Now $M(1/c') > 0$, so $b \notin M$.

THEOREM 1.3. *Let A be a Φ-algebra of real-valued functions that is closed under inversion. Then $\hat{A} \simeq C(\mathscr{R}(\hat{A}))$ if and only if A satisfies the following condition.*

If S is a set of positive elements of A such that for each $M \in \mathscr{R}(A)$ there is an $s \in S$ with $s \notin M$, then there is $0 \leqq b \in A$ such that $b \notin M$ for each $M \in \mathscr{R}(A)$ and $b \leqq c$ whenever $c \geqq s$ for each $s \in S$.

Proof. By the remarks preceeding the lemma above, $\hat{A} = C(\mathscr{R}(\hat{A}))$ if and only if whenever $0 \leqq a \in \hat{A}$ satisfies $a \notin \hat{M}$ for each $\hat{M} \in \mathscr{R}(\hat{A})$ there is $b \in A$ with $0 \leqq b \leqq a$ and $b \notin M$ for each $M \in \mathscr{R}(A)$.

Suppose A satisfies the condition of the Theorem, and let $0 \leqq a \in \hat{A}$ satisfy $a \notin \hat{M}$ for each $\hat{M} \in \mathscr{R}(\hat{A})$. Let $S = \{s \in A: 0 \leqq s \leqq a\}$. By 1.2, for each $M \in \mathscr{R}(A)$ there is an $s \in S$ with $s \notin M$. By hypothesis, there is $0 \leqq b \in A$ such that $b \notin M$ for each $M \in \mathscr{R}(A)$ and $b \leqq c$ whenever $c \in A$ and $c \geqq s$ for each $s \in S$. Since $a = \sup_{\hat{A}} S = \inf_{\hat{A}}\{c \in A: c \geqq a\}$, we have in particular that $a \geqq b$. Thus, $\hat{A} \cong C(\mathscr{R}(\hat{A}))$.

Conversely, suppose $\hat{A} \cong C(\mathscr{R}(\hat{A}))$, and let S be a set of positive elements of A such that for each $M \in \mathscr{R}(A)$ there is $s \in S$ with $s \notin M$. We may suppose that S is bounded above and let $a = \sup_{\hat{A}} S$. Then $0 \leqq a \in \hat{A}$ and $a \notin \hat{M}$ for each $\hat{M} \in \mathscr{R}(\hat{A})$, so there is $b \in A$ with $0 \leqq b \leqq a$ and $b \notin M$ for each $M \in \mathscr{R}(A)$. Clearly, if $c \in A$ and $c \geqq s$ for each $s \in S$, then $c \geqq a$, whence $c \geqq b$. Thus, the condition of the theorem is fulfilled.

2. The lattice of normal upper semicontinuous functions.

Throughout this paper, $\mathscr{X}$ will denote a completely regular (Hausdorff) space and $C = C(\mathscr{X})$ the Φ-algebra of continuous real-valued functions on $\mathscr{X}$. As usual, $C^* = C^*(\mathscr{X})$ represents the set of bounded elements of C and $\hat{C}$ denotes the Dedekind completion of C.

A real-valued function f on $\mathscr{X}$ is *locally bounded* if it is bounded on a neighborhood of each point of $\mathscr{X}$. The upper and lower limit functions of f will be denoted by f^* and f_*, respectively: for each $x \in X$

$$f^*(x) = \inf\{\sup_{y\in\mathscr{U}} f(y) \colon \mathscr{U} \text{ is a neighborhood of } x\}\ ,$$

and

$$f_*(x) = \sup\{\inf_{y\in\mathscr{U}} f(y) \colon \mathscr{U} \text{ is neighborhood of } x\}\ .$$

Then the extended real-valued functions f^* and f_* are, respectively, upper and lower semicontinuous; they are both real-valued if and only if f is locally bounded. Since $\mathscr{X}$ is completely regular, f^* is a pointwise infimum of continuous functions if and only if f is bounded above by a continuous function. An analogous statement holds for f_*. A real-valued function f is *normal upper semicontinuous* if f_* is real-valued and $f = (f_*)^*$; it is *normal lower semicontinuous* if f^* is real-valued and $f = (f^*)_*$. The properties of f^* and f_* given in [3, Section 3] hold for locally bounded functions as well. Also, the properties of the star elements listed in § 9 of [10] are valid in the present context.

In [5], Gleason showed that in the category of compact spaces and continuous maps the projectives are the (compact) extremally disconnected spaces and that for each compact space there is a unique minimal projective extension.

Here we consider the category of completely regular spaces and fitting maps (a map τ from $\mathscr{Y}$ to $\mathscr{X}$ is *fitting* if it is continuous, closed, and $\tau^{-1}(x)$ is compact for each $x \in \mathscr{X}$). In view of [6; 1.5], the program outlined in § 4 of [5] carries through for the category of completely regular spaces and fitting maps. In particular, every completely regular space $\mathscr{X}$ has a *minimal projective extension*; i.e., an extremally disconnected space $\mathscr{Y}$ and a tight fitting map τ from $\mathscr{Y}$ onto $\mathscr{X}$ (the mapping τ is *tight* if it maps no proper closed subspace of $\mathscr{Y}$ onto $\mathscr{X}$). Moreover, $\mathscr{Y}$ is essentially unique.

THEOREM 2.1. *If $\mathscr{X}$ is a completely regular space, $\mathscr{Y}$ its minimal projective extension and τ is the tight fitting map of $\mathscr{Y}$ onto $\mathscr{X}$, then $f \to (f \circ \tau)_*$ is an isomorphism of the lattice of normal upper semicontinuous functions on $\mathscr{X}$ onto $C(\mathscr{Y})$.*

Proof. First, we shall prove the following lemmas.

(I) *If f and g are normal upper semicontinuous functions on $\mathscr{X}$ and $(f \circ \tau)_* \leqq (g \circ \tau)_*$, then $f \leqq g$.*

(II) *If $F \in C(\mathscr{Y})$ and $f(x) = \sup \{F(y) \colon y \in \tau^{-1}(x)\}$, then f is normal upper semicontinuous on $\mathscr{X}$.*

Proof of (I). Initially, we show that $(f \circ \tau)_* \leqq (g \circ \tau)_*$ implies $(f - f \wedge g)_* = 0$. Suppose $(f - f \wedge g)_* > 0$. Since $\mathscr{X}$ is completely regular, there exists $h \in C$ such that $0 < h \leqq f - f \wedge g$. Then $h \circ \tau + [(f \wedge g) \circ \tau]_* \leqq (f \circ \tau)_*$. By [10, (9.6)], we have

$$[(f \circ \tau) \wedge (g \circ \tau)]_* = (f \circ \tau)_* \wedge (g \circ \tau)_* ,$$

whence $h \circ \tau + (f \circ \tau)_* \wedge (g \circ \tau)_* \leqq (f \circ \tau)_*$. Since $h > 0$, this implies that $(f \circ \tau)_* \not\leqq (g \circ \tau)_*$. Now let us prove (I). Using [10; (9.8)], we get $f \wedge g = [f \wedge g)]^* = [f - (f - (f \wedge g)]^* \geqq f_* - (f - f \wedge g)_* = f_*$ when $(f \circ \tau)_* \leqq (g \circ \tau)_*$; hence $f = (f_*)^* \leqq f \wedge g \leqq g$.

Proof of (II). Since $\tau^{-1}(x)$ is compact for each $x \in \mathscr{X}$, f is a real-valued function on $\mathscr{X}$. Let $x_0 \in \mathscr{X}$ and $\varepsilon > 0$ be given, and choose $y_0 \in \tau^{-1}(x_0)$ so that $f(x_0) = F(y_0)$. To show that f is upper semicontinuous, let $\mathscr{G} = \{y \in \mathscr{Y} \colon F(y) < F(y_0) + \varepsilon\}$. Then $\mathscr{G}$ is a neighborhood of $\tau^{-1}(x_0)$. Since τ is a closed mapping, there is a neighborhood $\mathscr{V}$ of x_0 such that $\tau^{-1}(\mathscr{V}) \subseteq \mathscr{G}$. Then $f(x) < F(y_0) + \varepsilon = f(x_0) + \varepsilon$

for each $x \in \mathscr{V}$. Thus, f is u.s.c. To prove that f is normal, let x_0, y_0 and ε be given as above and let $\mathscr{U}$ be an open neighborhood of x_0. Then $\mathscr{G}_1 = \{y \in \tau^{-1}(\mathscr{U}): F(y) > F(y_0) - \varepsilon\}$ is an open neighborhood of y_0. Since τ is both closed and tight, $\mathscr{V}_1 = \mathscr{X}\backslash\tau[\mathscr{Y}\backslash \mathscr{G}_1]$ is a nonempty open subset of $\mathscr{X}$. Clearly, $\mathscr{V}_1 \subset \mathscr{U}$ and $f(x) > F(y_0) - \varepsilon = f(x_0) = \varepsilon$ on $\mathscr{V}_1$.

We now return to the proof of 2.1. For a normal upper semicontinuous function f on $\mathscr{X}$, $f \circ \tau$ is upper semicontinuous on $\mathscr{Y}$. It follows that $(f \circ \tau)_*$ is normal lower semicontinuous on $\mathscr{Y}$. Since $\mathscr{Y}$ is extremally disconnected, $(f \circ \tau)_* \in C(\mathscr{Y})$ [3, p. 431]. Clearly, the mapping $f \rightarrow (f \circ \tau)_*$ is order preserving. By (I), this mapping is one-to-one and its inverse is order preserving. It remains to show that every $F \in C(\mathscr{Y})$ is the image of some normal upper semicontinuous function on $\mathscr{X}$. Let f be the function given by (II). Clearly, $f \circ \tau \geqq F$. If $(f \circ \tau)_* \neq F$, then there exists a positive number r such that $\mathscr{U} = \{y \in \mathscr{Y}: (f \circ \tau)_*(y) > F(y) + r\}$ is nonempty. Then $\mathscr{V} = \mathscr{X}\backslash\tau[\mathscr{Y}\backslash\mathscr{U}]$ is a nonempty open subset of $\mathscr{X}$. Let $h \in C(\mathscr{X})$ vanish on $X\backslash \mathscr{V}$, while $0 < h \leqq r1$. Then $h \circ \tau + F \leqq (f \circ \tau)_* \leqq f \circ \tau$; hence for each $x \in X$ and $y \in \tau^{-1}(x)$ we have $h(x) + F(y) \leqq f(x)$. This is impossible if $h(x) > 0$. This contradiction completes the proof of Theorem 2.1.

We now consider the relation that the space $\mathscr{Y}$ in Theorem 2.1 bears to $\mathscr{R}(\hat{C})$. Observe that if $\hat{M} \in \mathscr{M}(\hat{C})$, then $\hat{M} \cap C \in \mathscr{M}(C)$. Conversely, every element of $\mathscr{M}(C)$ is contained in a maximal $\mathscr{L}$-ideal of $\hat{C}$. Since $\mathscr{M}(C)$ and $\mathscr{M}(\hat{C})$ both have the hull kernel topology, the mapping $\hat{M} \rightarrow \hat{M} \cap C$ is continuous. Also, since C is dense in $\hat{C}$, this mapping is tight. It follows from [9, 3.2] that $\mathscr{M}(\hat{C})$ is extremally disconnected. Hence $\mathscr{M}(\hat{C})$ is the minimal projective extension of $\mathscr{M}(C)$.

For $x \in X$, let $M_x = \{f \in C: f(x) = 0\}$. Then the subspace $\{M_x: x \in \mathscr{X}\}$ of $\mathscr{M}(C)$ is homeomorphic with $\mathscr{X}$. Let $\mathscr{Y}$ be the set of elements $\hat{M}$ in $\mathscr{M}(\hat{C})$ such that $\hat{M} \supset M_x$ for some $x \in X$. Then $\mathscr{Y} \subset \mathscr{R}(\hat{C})$. As in 1.1, it follows that $\cap\{M: M \in \mathscr{Y}\} = \{0\}$; hence $\mathscr{Y}$ is dense in $M(\hat{C})$. Therefore, $\mathscr{Y}$ is extremally disconnected and $\beta\mathscr{Y} = \mathscr{M}(\hat{C})$ [4, 6M]. Since $\mathscr{Y}$ is the preimage of $\{M_x: x \in \mathscr{X}\}$, it follows from [6, 1.5] that $\mathscr{Y}$ is the minimal projective extension of $\{M_x: x \in \mathscr{X}\}$ and hence, also, of the space $\mathscr{X}$. Similarly, it is seen that $\mathscr{R}(\hat{C})$ is the minimal projective extension of $\mathscr{R}(C) = \nu\mathscr{X}$. Thus we have proved the following:

THEOREM 2.2. *$C(\mathscr{R}(\hat{C}))$ is isomorphic with the lattice of normal upper semicontinuous functions on $\nu\mathscr{X}$.*

The question as to whether $C(\mathscr{Y})$ and $C(\mathscr{R}(\hat{C}))$ are isomorphic can be stated as follows: *If $\mathscr{Y}$ is the minimal projective extension of $\mathscr{X}$, must $\nu\mathscr{Y}$ be the minimal projective extension of $\nu\mathscr{X}$?* An alternative form of this question is: *Can every normal upper semicontinuous function on $\mathscr{X}$ be extended to a normal upper semicontinuous function on $\nu\mathscr{X}$?* An affirmative answer can be given when $\nu\mathscr{X}$ is locally compact or when $\mathscr{X}$ is a weak *cb*-space (see 3.7 below). However, the example given at the end of the next section shows that the answer to this question is, in general, negative.

We conclude this section by commenting on a problem which is related to our main question. If $\mathscr{X}$ is locally compact, then the minimal projective extension $\mathscr{Y}$ of $\mathscr{X}$ is locally compact ([4, 10.16] or [5, 4.3]). Let $C_K(C_\infty)$ denote the lattice-ordered ring of continuous functions which have compact support (which vanish at infinity, respectively). Then $\hat{C}_K$ and $\hat{C}_\infty$ are isomorphic to subrings of $\hat{C}^*$. Since $\hat{C}^* = C^*(\mathscr{Y})$ [3] and since τ and τ^{-1} both preserve compactness (where τ denotes the tight fitting map from $\mathscr{Y}$ onto $\mathscr{X}$), it follows that $\hat{C}_K = C_K(\mathscr{Y})$ and $\hat{C}_\infty = C_\infty(\mathscr{Y})$.

3. Weak *cb*-spaces. Let $\mathscr{T}$ be a topological space. Then $\mathscr{T}$ is a *cb-space* if each locally bounded function on $\mathscr{T}$ is bounded above by a continuous function. See [11] for a study of *cb*-spaces. A space $\mathscr{T}$ is *weak cb* if each locally bounded, lower semicontinuous function on $\mathscr{T}$ is bounded above by a continuous function. It follows that $\mathscr{T}$ is a *cb*-space if and only if $\mathscr{T}$ is both countably paracompact and weak *cb* [11, Theorem 10].

A subset $\mathscr{G}$ of $\mathscr{T}$ is *regular-open* if $\mathscr{G} = \operatorname{int}\operatorname{cl}\mathscr{G}$ and a set $\mathscr{F}$ is *regular-closed* if $\mathscr{F} = \operatorname{cl}\operatorname{int}\mathscr{F}$. Clearly, the interior of a closed set is regular-open and the closure of an open set is regular-closed. A *zero-set* is a set $f^{-1}(0)$ for some $f \in C(\mathscr{T})$; the complement of a zero-set is a *cozero-set*. A *regular-open* (*cozero*) *cover* is a cover consisting of regular-open (resp., cozero) sets. A countable cover will be termed *increasing* if it can be indexed so as to form an increasing sequence of sets. A family F of continuous functions is *locally finite* (*subordinate* to a cover U) if the collection of cozero-sets associated with F is locally finite (resp., is a refinement of U). A family F is a *partition of unity* if $F \subset [C(\mathscr{T})]_+$ and $\sum_{f \in F} f(x) = 1$ for each $x \in \mathscr{T}$.

THEOREM 3.1. *The following statements are equivalent for any topological space $\mathscr{T}$.*

(a) *$\mathscr{T}$ is weak cb.*

(b) *Given a normal upper semicontinuous function h on $\mathscr{T}$, there exists $f \in C(\mathscr{T})$ such that $f \geqq h$.*

(c) *Given a positive (nonvanishing) normal lower semicontinuous* *function g on $\mathscr{T}$, there exists $f \in C(\mathscr{T})$ such that $0 < f(x) \leqq g(x)$ for each $x \in \mathscr{T}$.*

(d) *For each countable increasing regular-open cover of $\mathscr{T}$, there exists a locally finite partition of unity subordinate to that cover.*

(e) *For each countable increasing regular-open cover of $\mathscr{T}$, there is a partition of unity subordinate to that cover.*

(f) *Each countable increasing regular-open cover of $\mathscr{T}$ has a locally finite cozero refinement.*

(g) *Each countable increasing regular-open cover of $\mathscr{T}$ has a σ-locally finite cozero refinement.*

(h) *Each countable increasing regular-open cover of $\mathscr{T}$ has a countable cozero refinement.*

(i) *Given a decreasing sequence $\{\mathscr{F}_n\}$ of regular-closed sets with empty intersection, there exists a sequence $\{\mathscr{Z}_n\}$ of zero-sets with empty intersection such that $\mathscr{Z}_n \supset \mathscr{F}_n$ for each n.*

Moreover, if $\mathscr{T}$ is a normal space, then the word "cozero" may be deleted from (f) *and "closed G_δ-set" may be substituted for "zero-set" in* (i).

Each normal and countably paracompact space is weak *cb*. Also, every extremally disconnected space is weak *cb* [**11**, Theorem 11]. It follows from (i) of 3.1 that each regular-closed subspace of a weak *cb*-space is weak *cb*.

PROPOSITION 3.2. Each cozero-subspace of a weak *cb*-space is weak *cb*.

PROPOSITION 3.3. The product of a weak *cb*-space and a locally compact paracompact space is weak *cb*.

The proofs of 3.1, 3.2, and 3.3 are similar to the proofs given in [**11**] for the corresponding theorems for *cb*-spaces.

EXAMPLE. The local compactness requirement in 3.3 cannot be suppressed. To show this, we use Michael's example [**12**]. Let $\mathscr{X}$ be the reals with the usual topology refined so as to make the irrationals discrete, and let $\mathscr{Y}$ be the space of irrationals. Then $\mathscr{X} \times \mathscr{Y}$ is not a weak *cb*-space, even though it is the product of a paracompact space and a metric space. To show that $\mathscr{X} \times \mathscr{Y}$ is not weak *cb*, let $\{\mathscr{A}_n\}$ be a sequence of mutually disjoint, dense subsets of $\mathscr{Y}$. For each n, define $h(x, x) = n$ for $x \in \mathscr{A}_n$ and $h(x, y) = \min\{|x - y|^{-1}, n\}$, $x \neq y$, on $\mathscr{A}_n \times \mathscr{Y}$; let h vanish elsewhere. Then h is a locally bounded, lower semicontinuous function which is not bounded above by any continuous function.

This space can also be used to show that a closed subspace of a weak *cb*-space need not be weak *cb*. Let Γ_0 be the upper half-plane (including the horizontal axis) with the usual topology refined by allowing, for irrational x, the set

$$\{(x, 0)\} \cup \{(u, v): (u - x)^2 + (v - r)^2 < r^2\}$$

to be a neighborhood of $(x, 0)$ for each $r > 0$. Then $\mathscr{X}' = \{(x, 0): x \in \boldsymbol{R}\}$ is homeomorphic with the space $\mathscr{X}$ above. Now $\Gamma_0 \times \mathscr{Y}$ (where $\mathscr{Y}$ is, as above, the metric space of irrationals) is weak *cb* while $\mathscr{X}' \times \mathscr{Y}$ is a closed subspace that is not weak *cb*.

PROPOSITION 3.4. If $\mathscr{X}$ is a completely regular weak *cb*-space, then $\nu\mathscr{X}$ is weak *cb*.

Proof. Let h be a locally bounded, lower semicontinuous function on $\nu\mathscr{X}$. Then $h \mid \mathscr{X}$ is locally bounded and lower semicontinuous on $\mathscr{X}$. Thus, there is $f \in C(\mathscr{X})$ such that $f \geqq h \mid \mathscr{X}$. If f^ν denotes the element of $C(\nu\mathscr{X})$ which extends f, then $f^\nu - h$ is upper semicontinuous on $\nu\mathscr{X}$ and is nonnegative on the dense subspace $\mathscr{X}$. Hence, $f^\nu \geqq h$.

PROPOSITION 3.5. The product of any collection of separable complete metric spaces is a weak *cb*-space.

Proof. Let $\mathscr{P}$ be any such product and let Σ denote a Σ-product of the same spaces. In [2], it is proved that Σ is normal and countably paracompact and that $\mathscr{P} = \nu\Sigma$. Thus, $\mathscr{P}$ is a weak *cb*-space.

PROPOSITION 3.6. Let $\mathscr{X}$ be a completely regular space such that $\nu\mathscr{X}$ is locally compact. Then $\mathscr{X}$ is weak *cb* if and only if $\nu\mathscr{X}$ is weak *cb*.

This proposition is a direct consequence of the following:

LEMMA 3.7. *If h is a positive, locally bounded function on X, then the (extended real-valued) function g on $\nu\mathscr{X}$ defined by*

$$g = \text{ptwise sup}\,\{f^\nu: f \in C(\mathscr{X}), f \leqq h\}$$

is real-valued and bounded on each compact subset of $\nu\mathscr{X}$. Moreover, g is an extension of h_*.

Proof. Suppose that g is either infinite or unbounded on the compact set $\mathscr{K}$. For each positive integer n, set $\mathscr{F}_n = \{p \in \nu\mathscr{X}:$

$g(p) \leqq n\}$ and choose $x_n \in \mathscr{K} \backslash \mathscr{F}_n$. Since g is lower semicontinuous, $\mathscr{F}_n$ is a closed set. Thus there exists $f_n \in C(\mathscr{X})$ such that $f_n^\nu(x_n) = n$ while f_n^ν vanishes on $\mathscr{F}_n$. The local boundedness of h implies that $\{\mathscr{X} \backslash \mathscr{F}_n : n = 1, 2, \cdots\}$ is locally finite on $\mathscr{X}$. Hence $f = \bigvee_n f_n$ exists in $C(\mathscr{X})$. Now $f \geqq f_n$ implies that $f^\nu \geqq f_n^\nu$. In particular $f^\nu(x_n) \geqq f_n^\nu(x_n) = n$ for each positive integer n. This is impossible, since f^ν is finite and bounded on the compact set $\mathscr{K}$.

The fact that h_* is a pointwise supremum of continuous functions implies that g is an extension of h_*.

COROLLARY 3.8. *Each (completely regular) pseudocompact space is weak cb.*

Proof. If $\mathscr{X}$ is pseudocompact, then $\nu\mathscr{X}$ is compact [4, 8A.4]; hence $\mathscr{X}$ is weak *cb*, according to 3.6.

If $\mathscr{P}$ is an uncountable product of real lines, then $\mathscr{P}$ is a weak *cb*-space that is not normal. However, there is a normal and countably paracompact space Σ such that $C(\Sigma)$ is isomorphic to $C(\mathscr{P})$. On the other hand, the spaces Γ in [4, 3k], E in [1, p. 116, Ex. 4], and $S \times S$, where S is Sorgenfrey's example [15], are weak *cb*-spaces for which C is not isomorphic to the ring of continuous functions on any normal and countably paracompact space [4, 8.18 and 8A]. Proofs that these spaces are weak *cb* can be based on Theorem 3.1.

A weak *cb*-space may fail to be countably paracompact (e.g., the Tychonoff plank); the example below shows that a countably paracompact space need not be weak *cb*, even if it is locally compact. It is not known whether every normal space must be weak *cb*.

EXAMPLE. Let $\mathscr{T}$ be a completely regular space and let $\mathscr{A}$ and $\mathscr{B}$ be closed subsets of $\mathscr{T}$ such that $\mathscr{A} \cap \mathscr{B}$ is compact. In $\mathscr{T} \times \boldsymbol{N}$, identify $\mathscr{A} \times \{2n-1\}$ with $\mathscr{A} \times \{2n\}$ and $\mathscr{B} \times \{2n\}$ with $\mathscr{B} \times \{2n+1\}$. (The construction used here was suggested by the referee of [11].) Clearly, the resulting topological space $\mathscr{X}$ inherits any of the following properties that $\mathscr{T}$ may possess: normality, σ-compactness, realcompactness, paracompactness, and countable paracompactness. Moreover, if $\mathscr{A}$ and $\mathscr{B}$ are disjoint, then local compactness of $\mathscr{T}$ will imply local compactness of $\mathscr{X}$. If $\mathscr{T}$ is countably paracompact but nonnormal and if $\mathscr{A}$ and $\mathscr{B}$ are disjoint closed sets that are not contained in disjoint open sets, then $\mathscr{X}$ is not a weak *cb*-space.

In particular, let $\mathscr{W}$ and $\mathscr{W}^*$ be the spaces of ordinals less then and less than or equal to, respectively, the first uncountable ordinal ω_1, and set $\mathscr{T} = \{(\sigma, \tau) \in \mathscr{W} \times \mathscr{W}^* : \sigma \leqq \tau\}$. This space is locally compact and countably compact, but not normal: the diagonal $\mathscr{A}$ and the upper edge $\mathscr{B}$ are disjoint closed sets which cannot be separated by open sets. If we set $w = (\omega_1, \omega_1)$, then $\beta\mathscr{T} = \mathscr{T}^* = \mathscr{T} \cup \{w\}$ ($\subseteqq \mathscr{W}^* \times \mathscr{W}^*$). Next, let $\mathscr{X}$ and $\mathscr{X}^*$ be the spaces obtained from $\mathscr{T}$ and $\mathscr{T}^*$ by identifying images of $\mathscr{A}$ and $\mathscr{B}$, and those of $\mathscr{A} \cup \{w\}$ and $\mathscr{B} \cup \{w\}$, respectively, as in the above construction. Then $\mathscr{X}$ is locally compact and countably paracompact but not weak *cb*, while $\mathscr{X}^*$ is σ-compact (hence realcompact and weak *cb*). Note that $\mathscr{X}^*$ is not locally compact. Since each continuous function on $\mathscr{T}$ is constant on a deleted neighborhood of w, it follows that $\mathscr{X}$ is C-embedded in $\mathscr{X}^*$, whence $\mathscr{X}^* = \nu\mathscr{X}$. Thus, $\mathscr{X}$ is a locally compact nonweak *cb*-space such that $\nu\mathscr{X}$ is weak *cb*.

4. The completion of $C(\mathscr{X})$.

PROPOSITION 4.1. If $\mathscr{X}$ is a completely regular, weak *cb*-space and $\mathscr{Y}$ is its minimal projective extension, then the Dedekind completion of $C(\mathscr{X})$ is isomorphic to $C(\mathscr{Y})$.

Proof. On a weak *cb*-space, each normal upper semicontinuous function can be identified with an element of $\hat{C}$ (Theorem 3.1), and conversely (cf. [3, p. 432]). Therefore, by Theorem 2.1, there exists a lattice isomorphism from $\hat{C}$ onto a $C(\mathscr{Y})$. Moreover, the restriction of this mapping to C preserves the ring operations. Since C is dense in $\hat{C}$, it follows that $\hat{C}$ and $C(\mathscr{Y})$ are isomorphic as Φ-algebras.

THEOREM 4.2. *Let $\mathscr{X}$ be a realcompact (completely regular) space. The Dedekind completion of $C(\mathscr{X})$ is isomorphic to an algebra $C(\mathscr{Y})$ for some space $\mathscr{Y}$ if and only if $\mathscr{X}$ is a weak cb-space.*

Proof. The Dedekind completion $\hat{C}$ of $C(\mathscr{X})$ is isomorphic to some $C(\mathscr{Y})$ if and only if $\hat{C} \cong C(\mathscr{R}(\hat{C}))$. In view of 4.1, it suffices to show that $\hat{C} \cong C(\mathscr{R}(\hat{C}))$ implies that $\mathscr{X}$ is weak *cb*. Since $\mathscr{X}$ is realcompact, we may identify $\mathscr{X}$ with $\mathscr{R}(C)$ and apply Theorem 1.3.

Suppose that $\hat{C} \cong C(\mathscr{R}(\hat{C}))$ and that g is a normal lower semi-continuous function with $g(x) > 0$ for each $x \in X$. Let $S = \{s \in C: 0 \leqq s \leqq g\}$. Then $g =$ ptwise $\sup S$. Hence, for each $x \in X$, there is an $s_x \in S$ with $s_x(x) > 0$. By 1.3, there is an $f \in C$ with $0 < f(x)$ for each $x \in X$ and $f \leqq h$ for each $h \in C$ with $h \geqq s$ for each $s \in S$. But $g^* =$ ptwise $\inf \{h \in C: h \geqq s$ for each $s \in S\}$; so $f \leqq g^*$, whence $f \leqq g$. By 3.1, $\mathscr{X}$ is a weak *cb*-space.

Thus, $\hat{C} \cong C(\mathscr{Y})$ for some $\mathscr{Y}$ if and only if $\nu\mathscr{X}$ is weak *cb*. Moreover, $\nu\mathscr{X}$ is weak *cb* if $\mathscr{X}$ is, but not conversely.

EXAMPLE. The space $\mathscr{X} \times \mathscr{Y}$, considered in the example following 3.3, is a realcompact space which is not weak *cb*.

Finally, we consider the relation that this paper bears to [3], [14] and [16]. Theorem 2.1 represents a generalization of Dilworth's characterization of the lattice of normal upper semicontinuous functions on a compact space ([3]). In [16], Weinberg proves that if $\mathscr{X}$ is normal and countably paracompact, then $\hat{C}$ is isomorphic with $C(\mathscr{Y})$ for some space $\mathscr{Y}$. Since a normal and countably paracompact space is weak *cb*, the examples given following 3.8 show that 4.1 generalizes Weinberg's result.

The characterization of the lattice of normal upper semicontinuous functions could have been developed along the lines of [14]. To see this, let $C_q(\mathscr{X})$ be the set of locally bounded functions f on $\mathscr{X}$ for which $\{x \in \mathscr{X} : (f^* - f_*)(x) > r\}$ is nowhere dense for every $r > 0$, and let N be the subset of $C_q(\mathscr{X})$ consisting of those f for which $(|f|^*)_* = 0$. Then $C_q(\mathscr{X})/N$ is a Φ-algebra which is isomorphic with the lattice of normal upper semicontinuous functions on $\mathscr{X}$.

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OHIO UNIVERSITY
THE PENNSYLVANIA STATE UNIVERSITY

UNIVERSITY OF KENTUCKY
NEW MEXICO STATE UNIVERSITY