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SUB-STATIONARY PROCESSES

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This note supplements the longer paper [3]. It is proved in § 2 that if T is a bounded Schwartz distribution on R^n, e.g. an L^∞ function, then its Fourier transform $\mathscr{F}T$ is of the form $\partial^n f/\partial t_1 \cdots \partial t_n$ where f is integrable over any bounded set to any finite power. This follows from the main theorem of [3], but the proof here is much shorter.

Secondly, § 3 shows that a p-sub-stationary random (Schwartz) distribution has sample distributions of bounded order. This generalizes a result of K. Ito for the stationary case.

Third, in § 4 it is shown that p-sub-stationary stochastic processes define p-sub-stationary random distributions if $p \geqq 1$.

In [5], K. Ito introduced stationary random Schwartz distributions L with second moments. He obtained the "spectral measure" representation of the covariance of L. Using this, he proved for each such L:

(I) There is a finite n such that almost all the sample distributions of L are nth Schwartz derivatives of continuous functions.

The spectral measure also yields

(II) Almost all the sample distributions of L are tempered distributions, and their Fourier transforms are first Schwartz derivatives of locally square-integrable functions.

In [3], (II) was proved for random distributions L which are "p-sub-stationary" for some $p > 1$, i.e. for each f in the Schwartz space $\mathscr{D}$,

$$\sup_h E\,|\,L(\tau_h f)\,|^p < \infty\ ,$$

where $(\tau_h f)(t) = f(t-h)$. Also, "locally square-integrable" was strengthened to "locally integrable to any finite power". In § 2, we shall give corollaries of this result for fixed distributions and stochastic processes with much easier proofs. In § 3, we first prove (I) in the p-sub-stationary case for any $p > 0$, using some lemmas from [3] but no Fourier analysis. Then we obtain a result on the Fourier transform of the covariance for $p = 2$. In § 4, we show that for $p \geqq 1$ a p-sub-stationary stochastic process is also a p-sub-stationary random distribution.

2. Fourier transforms of bounded functions and distributions. All three theorems in this section are immediate corollaries of the main theorem of [3], but perhaps the easier proofs here will make that result more accessible.

We use the notations of L. Schwartz [8], e.g. $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{S}$, $\mathscr{S}'$. $\mathscr{F}$ is the Fourier transform operator. The results say that if a distribution B is bounded or belongs to a suitable "stochastically bounded" class, then $\mathscr{F}B$ is of the following type:

DEFINITION. A distribution C in $\mathscr{D}'(R^k)$ is an *FB-distribution* ($C \in FB$) if and only if there is a measurable function f on R^k such that

$$C = \partial^k f/\partial t_1 \cdots \partial t_k$$

in the sense of distributions, and

$$\int_K |f(t)|^r \, dt_1 \cdots dt_k < \infty$$

whenever $0 < r < \infty$ and K is compact.

Beurling [1] has called a distribution on R a "pseudomeasure" if it is the first derivative of a locally integrable function. Thus the pseudomeasures include the class FB on the real line. The work of Beurling, Kahane and Salem [6] and others on pseudomeasures has apparently been primarily devoted to the question of which compact sets carry pseudomeasures of certain types. I do not know of any mutual implications between our results.

A distribution B in $\mathscr{D}'(R^k)$ is called *bounded* ($B \in \mathscr{B}'$) if for every f in $\mathscr{D}$,

$$\sup \{| B(\tau_h f) | : h \in R^k\} < \infty$$

(cf. Schwartz [8, tome I, Théorème IX(b) p. 72; tome II, "Autre définition des distributions bornées", p. 61]). It follows immediately from the main theorem of [3] that if $B \in \mathscr{B}'$, then $\mathscr{F}B \in FB$.

We shall use here the Hausdorff-Young inequality for Fourier transforms rather than for series as in [3]. Suppose $1 < p \leqq 2$, $q = p/(p-1)$, and $f \in L^p(R)$. Let

$$f_n(t) = \begin{cases} f(t), & |t| \leqq n \\ 0, & |t| > n \,. \end{cases}$$

Then the functions $\mathscr{F}f_n$ are in $L^q(R)$, and for some h in $L^q(R)$, $\mathscr{F}f_n \rightarrow h$ in L^q (Zygmund [9, 12.41 p. 316]). In the sense of tempered distributions, we have simply $\mathscr{F}f = h$.

To illustrate our method, we first prove

THEOREM 2.1. *If* $f \in L^\infty(R)$, *then* $\mathscr{F} f \in FB$.

Proof. Let $g(t) = f(t)$ for $|t| \leqq 1$, $g(t) = 0$ elsewhere, and $h = f - g$. Then by the Paley-Wiener theorem, $\mathscr{F} g$ is an entire analytic function, hence so is its indefinite integral, and $\mathscr{F} g \in FB$.

Let $j(t) = h(t)/t$. Then $j \in L^p(R)$ for all $p > 1$, so $\mathscr{F} j \in L^q$ for all $q \geqq 2$. Thus

$$D(\mathscr{F} j) = \mathscr{F}(-2\pi i t j) = \mathscr{F}(-2\pi i h) \in FB ,$$

so $\mathscr{F} h \in FB$. Hence $\mathscr{F} f \in FB$.

In [3], there was an example of a bounded function f (the Heaviside function) with $\mathscr{F} f = D\phi$, so that $\phi \in L^r$ on each bounded set for r finite, but with ϕ unbounded near zero.

Next suppose $(\Omega, \mathscr{B}, P)$ is a probability space. A jointly measurable map

$$\langle t, \omega \rangle \to x(t, \omega)$$

of $R^k \times \Omega$ into R will be called a *measurable stochastic process* on R^k, which is *p-sub-stationary* if

$$\sup_t \int |x(t, \omega)|^p \, dP(\omega) = M < \infty .$$

We let $X_\omega(t) = x(t, \omega)$, and E = integral with respect to P.

THEOREM 2.2. *Suppose* $x(\cdot, \cdot)$ *is a p-sub-stationary process on* R *and* $p > 1$. *Then for P-almost all* ω, $\mathscr{F} X_\omega \in FB$.

Proof. Let $Y_\omega(t) = X_\omega(t)$ for $|t| \leqq 1$, $Y_\omega(t) = 0$ elsewhere, and $Z_\omega = X_\omega - Y_\omega$. Then for $1 < r \leqq p$,

$$E\int_{-\infty}^{\infty} |Z_\omega(t)/t|^r \, dt \leqq \int_{|t| \geqq 1} (E\,|X_\omega(t)|^p)^{r/p} / |t|^r \, dt \leqq 2M^{r/p}/(r - 1) .$$

Thus $Z_\omega(t)/t \in L^r$ for almost all ω, so

$$\mathscr{F}(Z_\omega(t)/t) \in L^s \quad \text{for} \quad p/(p-1) \leqq s < \infty .$$

Thus $D\mathscr{F}(Z_\omega(t)/t) \in FB$, and hence $\mathscr{F} Z_\omega \in FB$. Now Y_ω is almost surely integrable with compact support, so $\mathscr{F} Y_\omega$ and its indefinite integral are entire functions, $\mathscr{F} Y_\omega \in FB$, and $\mathscr{F} X_\omega \in FB$ for almost all ω.

Now we generalize Theorem 2.1:

THEOREM 2.3. *If* $T \in \mathscr{B}'(R^k)$, *then* $\mathscr{F} T \in FB$.

Proof. T is a finite sum of partial derivatives of bounded functions (Schwartz [8, tome II, Théorème XXV p. 57]). Clearly FB is closed under multiplication by polynomials. Thus we may assume T is a function f in $L^\infty(R^k)$.

For each subset A of the finite set $\{1, 2, \cdots, k\}$, let S_A be the set of all t in R^k such that $|t_j| > 1$ if and only if $j \in A$. Let $f_A = f$ on S_A, $f_A = 0$ elsewhere. Then for each A,

$$g_A = f_A / \prod_{j \in A} t_j \in L^p(R^k) \quad \text{for all } p > 1 ,$$

so that $\mathscr{F} g_A \in L^q(R^k)$ for all $q \geqq 2$. Taking indefinite integrals in the x_j for $j \notin A$, we obtain $\mathscr{F} f_A = \partial^k h_A / \partial x_1 \cdots \partial x_k$, where

$$\int_k |h_A(x)|^r dx_1 \cdots dx_k < \infty$$

whenever $0 < r < \infty$ and K is compact. Thus

$$\mathscr{F} f = \sum_A \mathscr{F} f_A \in FB .$$

The converse of Theorem 2.3 is not true, since it is easy to construct examples of 2-sub-stationary stochastic processes whose sample functions are unbounded (as distributions) with probability 1.

3. p-sub-stationary random distributions are of finite order. Let $(\Omega, \mathscr{B}, P)$ be a probability space and let $M(\Omega)$ be the linear space of $\mathscr{B}$-measurable complex-valued functions on Ω modulo functions which vanish P-almost everywhere. On $M(\Omega)$, let $T(P)$ be the topology of convergence in probability. $T(P)$ is metrizable, e.g. by the metric

$$d(f, g) = \int |f(x) - g(x)| / (1 + |f(x) - g(x)|) dP(x) ,$$

but it is not locally convex in general.

DEFINITION. A *random distribution* is a sequentially continuous linear map from $\mathscr{D}(R)$ into some $M(\Omega)$ with topology $T(P)$.

It follows from a theorem of R. A. Minlos [7] (see [4, Chapter 4, § 2, ♯ 4, Theorem 6]) that for any random distribution L there is a countably additive measure Q on $\mathscr{D}'$ such that for any $f_1, \cdots, f_n$ in $\mathscr{D}$ and Borel set $B \subset C^n$,

$$Q\{M: \langle M(f_1), \cdots, M(f_n) \rangle \in B\} = P\{\omega: \langle L(f_1)(\omega), \cdots, L(f_n)(\omega) \rangle \in B\} .$$

The subsets of $\mathscr{D}'$ on which Q is given form an algebra (the "cylinder sets"). The unique countably additive extension of Q to the generated σ-algebra will be called the *Minlos measure* of L.

For any f in $\mathscr{D}(R)$ and integer $n \geqq 0$ we let

$$\|f\|_n = \left(\sum_{j=0}^{n} \int_{-\infty}^{\infty} |D^j f(x)|^2 \, dx\right)^{1/2} .$$

Also, for any finite interval (a, b), $\mathscr{D}[a, b]$ will denote the space of C^∞ functions vanishing outside (a, b), with its relative topology from $\mathscr{D}$. This relative topology is defined by the countably many norms $\| \;\|_n$ (although that of $\mathscr{D}$ is not). For A and B in $\mathscr{D}'$ we say "$A = B$ on (a, b)" if $A(f) = B(f)$ for all f in $\mathscr{D}[a, b]$. The distribution defined by a locally integrable function f or derivative $D^p f$ will be written $[f]$ or $[D^p f]$ respectively.

Clearly a continuous linear functional A on $\mathscr{D}[a, b]$ for $\| \;\|_n$ has the form

$$A(f) = \sum_{j=0}^{n} \int_a^b D^j f(x) \bar{g}_j(x) dx$$

for some g_j in $L^2[a, b]$. Thus, integrating by parts and adding, we have

$$A(f) = [D^n g](f) = [D^{n+1} h](f)$$

for some g in $L^2(a, b)$ and absolutely continuous h on (a, b).

THEOREM 3.1. *Let L be a p-sub-stationary random distribution for some $p > 0$. Then there is a positive integer n such that the Minlos measure of L is concentrated in the set of M in $\mathscr{D}'$ such that $M = D^n f$ for some continuous function f (depending on M).*

Proof. The hypothesis becomes stronger as p increases. Thus we may assume $0 < p \leqq 1$. For each g in $\mathscr{D}$ let

$$A(g) = \sup_t (E\,|\,L(\tau_t g)\,|^p)^{1/p} < \infty .$$

Note that A will not generally be a pseudo-norm for $p < 1$. By Lemma 4 of [3], there exist K and $n \geqq 0$ such that $A(g) \leqq K \|g\|_n$ for all g in $\mathscr{D}[0, 1]$, hence for g in $\mathscr{D}[b, b+1]$ for any real b.

Now given $c > 0$, there exist $f_1, \cdots, f_m$ in $\mathscr{D}$ such that

$$\sum_{j=1}^{m} f_j(t) = 1 \quad \text{for } |t| \leqq c ,$$

and such that the diameter of the support of each f_j is at most 1 (cf. [3, proof of Lemma 5]). Let $g \in \mathscr{D}[-c, c]$. Then for each j,

$$\begin{aligned}||gf_j||_n &= \Big(\sum_{p=0}^{n}\int_0^c |D^p(gf_j)|^2\,dt\Big)^{1/2}\\ &= \Big(\sum_{p=0}^{n}\int_0^c \Big|\sum_{q=0}^{p}\binom{p}{q}D^q g\, D^{p-q}f_j\Big|^2 dt\Big)^{1/2}\\ &\leqq (n+1)2^n\,||g||_n \max(|D^r f_j(t)| : t\in R,\ 0\leqq r\leqq n)\,.\end{aligned}$$

Thus for some $M_c > 0$,

$$\begin{aligned}A(g) &= \Big(\Big(A\sum_{j=1}^{m}(gf_j)\Big)^p\Big)^{1/p} \leqq \Big(\sum_{j=1}^{m}(A(gf_j))^p\Big)^{1/p}\\ &\leqq K\Big(\sum_{j=1}^{m}||gf_j||_n^p\Big)^{1/p} \leqq M_c\,||g||_n\end{aligned}$$

for all g in $\mathscr{D}[-c, c]$.

Now $\mathscr{D}[-c, c]$ is a nuclear space (see e.g. Gelfand and Vilenkin [4, Chapter I, §3, #6]). Thus a theorem essentially due to Minlos ([7], [4, Chapter IV, §2, #3, Theorem 4]) implies that the Minlos measure of L restricted to $\mathscr{D}[-c, c]$ is concentrated in the set of distributions continuous for $||\ \ ||_{n+r}$ for some r (actually $r = 1$). Thus the Minlos measure is concentrated in the set of all M of the form

$$M = [D^{n+r+1}f] \text{ on } (-c, c)$$

where f is continuous and depends on M. Given M, f on $(-c, c)$ is determined up to an additive polynomial of degree at most $n + r$. Fixing f on $(-1, 1)$, say, we obtain

$$M = [D^{n+r+1}f]$$

for some continuous f (not necessarily bounded on R). The proof is complete.

A simpler form of the last proof yields

THEOREM 3.2. *Let L be a random distribution, $p > 0$, and (a, b) a finite interval. Suppose $E|L(f)|^p < \infty$ for all f in $\mathscr{D}[a, b]$. Then for some n, the Minlos measure of L is concentrated in the set of all M in $\mathscr{D}'$ equal on (a, b) to $[D^n f]$ for f continuous on $[a, b]$.*

Proof. L is continuous from $\mathscr{D}[a, b]$ to $L^p(\Omega)$ [3, Lemma 2]. Thus for some n and $\varepsilon > 0$,

$$||f||_n < \varepsilon \quad \text{implies} \quad E|L(f)|^p < 1\,,$$

and

$$(E|L(f)|^p)^{1/p} \leqq ||f||_n/\varepsilon$$

for all f in $\mathscr{D}[a, b]$ by homogeneity. Now we use nuclearity of $\mathscr{D}[a, b]$ and can proceed as in the last proof.

Suppose L is a random distribution with finite second moments, i.e. its range in $M(\Omega)$ is included in $L^2(\Omega, \mathscr{B}, P)$. Then there is a unique B in $\mathscr{D}'(R^2)$ such that

$$E(L(f)\overline{L(g)}) = B(f \otimes \bar{g})$$

where $(f \otimes \bar{g})(s, t) = f(s)\bar{g}(t)$ (see e.g. [2, § 3]).

LEMMA. *If L is 2-sub-stationary, then B is bounded.*

Proof. We must show that for any h in $\mathscr{D}(R^2)$, $B(\tau_z h)$ remains bounded as z runs over R^2. We know this for h of the form $f \otimes g$, $f, g \in \mathscr{D}(R)$.

For a general h, we have $h(s, t) = 0$ outside some square C_M: $|s| \leqq M$, $|t| \leqq M$. Let $g \in \mathscr{D}(R), g(s) = 1$ for $|s| \leqq M$, and $g(s) = 0$ for $|s| \geqq 2M$. We expand h in a Fourier series:

$$h(s, t) = g(s)g(t) \sum_{m,n} a(m, n) \exp(\pi i(ms + nt)/2M)$$

for all (s, t) in R^2. Since h on C_{2M} extends to a C^∞ function periodic of period $4M$ in s and t, we know that for any polynomial p in two variables, $p(m, n)a(m, n)$ is bounded.

Now, by Lemma 4 of [3] there exist k and $N > 0$ such that

$$\sup_u (E \,|\, L(\tau_u f) \,|^2)^{1/2} \leqq N \,||\, f \,||_k$$

for all f in $\mathscr{D}[-2M, 2M]$. Let

$$h_m(s) = g(s) \exp(\pi\, ims/2M) .$$

Then

$$||\, h_m \,||_k = \Big(\sum_{j=0}^{k} \int_{-2M}^{2M} |\, D^j h_m(s) \,|^2 \, ds)^{1/2} \leqq T(1 + m^2)^k$$

for some $T > 0$ (depending on M and g, but not on m). Now

$$h(s, t) = \sum_{m,n} a(m, n) h_m(s) h_n(t)$$

and $a(m, n)(1 + m^2)^{k+1}(1 + n^2)^{k+1}$ is bounded in m and n, so

$$\begin{aligned} \sup_z |\, B(\tau_z h) \,| &\leqq \sup_{s,t} \sum_{m,n} |\, a(m, n) B(\tau_s h_m \otimes \tau_t h_n) \,| \\ &\leqq N^2 \sum_{m,n} |\, a(m, n) \,| \; ||\, h_m \,||_k \, ||\, h_n \,||_k < \infty \,. \end{aligned}$$

From the lemma just proved and Theorem 2.3, we can infer that for any 2-sub-stationary random distribution L,

$$E(L(f)\overline{L(g)}) = C(\mathscr{F}f \otimes (\mathscr{F}g)^{-})$$

for some FB-distribution C, i.e.

$$C = [\partial^2 f(x, y)/\partial x \partial y]$$

for some measurable function f integrable to any finite power over any compact set. When f is of bounded variation on R^2, L (or B) is called *harmonizable*. Clearly such a B is a bounded continuous function: $B \in \mathscr{C}(R^2)$. We have the following inclusions of subsets of $\mathscr{D}'(R^2)$:

$$\text{harmonizable} \subset \mathscr{C} \subset L^\infty \subset \mathscr{B}' \subset \mathscr{F}^{-1}(FB) \subset \mathscr{F}^{-1} \text{ (pseudomeasures)} .$$

For none of these classes do we have a simple characterization both of the distributions and of their Fourier transforms (such as the Bochner, Plancherel and Paley-Wiener theorems and their generalizations and other results of Schwartz). Thus which will yield the most useful theory remains unclear.

4. Stochastic processes and random distributions.

THEOREM 4.1. *If $p \geqq 1$, a p-sub-stationary stochastic process $x(\cdot, \cdot)$ is a p-sub-stationary random distribution.*

Proof. Let $f \in \mathscr{D}(R^k)$. For any h in R^k, let

$$A(f, h) = \int \left| \int_{R^k} f(t-h)x(t, \omega)dt \right|^p dP(\omega) = \int \left| \int_{R^k} f(s)x(s+h, \omega)ds \right|^p dP(\omega) .$$

Let C be the support of f and let λ be Lebesgue measure. We apply Hölder's inequality to the inner integral, with $q = p/(p-1)$, obtaining

$$A(f, h) \leqq \|f\|_q^p \int_C |x(s+h, \omega)|^p dsdP(\omega) \leqq \|f\|_q^p \lambda(C) \sup_s \int |x(s, \omega)|^p dP(\omega) < \infty .$$

Thus a random distribution L is defined by

$$L(f)(\omega) = \int_{R^k} x(t, \omega)f(t)dt$$

and is p-sub-stationary.

For $p < 1$, it seems unclear whether a p-sub-stationary stochastic process defines a random distribution at all.

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