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QUASI DIMENSION TYPE. I. TYPES IN THE REAL LINE

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Fréchet defined the concept of dimension type in attempting to obtain a reasonable way of comparing two abstract topological spaces. The Fréchet dimension type of a topological space X is said to be less than or equal to the Fréchet dimension type of the topological space Y if and only if there is a homeomorphism between X and a subset of Y. In this case we write $dX \leqq dY$. Notice that the statement, $dX \leqq dY$, is equivalent to the statement, X can be (topologically) embedded in Y.

Fréchet dimension type also is a more delicate way of comparing two spaces (when it applies) than covering dimension (denoted by dim). One difficulty is that many spaces are not comparable with respect to Fréchet dimension type because of the strong restriction of requiring that one be embeddable in the other. In this paper we will relax this restriction somewhat to obtain a new dimension type called quasi dimension type. Under quasi dimension type many more spaces are comparable and yet many of the properties of Fréchet dimension type are retained. Kuratowski showed that there are 2^c Fréchet dimension types represented by subsets of the real line. We will show that there are only denumerably many quasi dimension types represented by subsets of the real line. Furthermore, we completely determine the partial ordering of these types and give a topological characterization of the linear sets having a given type.

For two topological spaces X and Y, we will say X is *quasi embeddable in* Y, if, for each covering α of X, there is a closed α-map, f_α, of X into Y. (Recall that a continuous function $f: X \rightarrow Y$ is an α-map if there is a covering β of Y such that $f^{-1}[\beta[> \alpha$, i.e., $f^{-1}[\beta]$ refines α, and f is closed if it takes closed subsets of X into closed subsets of Y.) We use "covering" to mean "open covering".

Furthermore, we will say that *the quasi dimension type of a topological space X is less than or equal to the quasi dimension type of a topological space Y* if and only if X is quasi embeddable in Y. If this case we write $qX \leqq qY$. We say $qX < qY$ if $qX \leqq qY$ but it is false that $qY \leqq qX$. Further if $qX \leqq qY$ and $qY \leqq qX$ then $qX = qY$. So for two spaces X and Y, one and only one of the following conditions holds: (1) $qX < qY$, (2) $qY < qX$, (3) $qX = qY$ and (4) X and Y are not comparable with respect to quasi dimension type. Clearly, if $qX \leqq qY$ and $qY \leqq qZ$ then $qX \leqq qZ$. Note also that quasi dimension type is a topological invariant of Y and is monotone on closed subsets, (i.e., if X is a closed subset of Y then $qX \leqq qY$).

1. Preliminaries. All spaces considered in the remainder of this paper are metric. The diameter of A will be denoted by $\delta(A)$.

DEFINITION 1.1. A mapping $f: X \rightarrow Y$ is a *strong ε-map* if there is an $\eta > 0$ such that: if $A \subset Y$ and $\delta(A) < \eta$, then $\delta(f^{-1}[A]) < \varepsilon$. A mapping $f: X \rightarrow Y$ is an *ε-map* provided that, for each $y \in Y$, we have $\delta(f^{-1}(y)) < \varepsilon$.

DEFINITION 1.2. A space X is *strongly ε-embeddable in a space* Y provided that, for each $\varepsilon > 0$, there is a closed strong ε-map of X into Y. If this is the case we write $sX \leqq sY$.

REMARK. Note that X and Y may be homeomorphic without $sX \leqq sY$ being true (see Example 1.1). In general, neither quasi embeddability nor strong ε-embeddability implies the other (see Examples 1.1 and 1.2). However, we do have (1) if Y is compact, then $qX \leqq qY$ implies $sX \leqq sY$ and (2) if X is compact, then $sX \leqq sY$ implies $qX \leqq qY$. If X and Y are compact then the following statements are equivalent: (3) $qX \leqq qY$, (4) $sX \leqq sY$ and (5) there are ε-maps of X into Y for all $\varepsilon > 0$. If $qX \leqq qY$, then (6) if for each α, and for each α-map of X into Y, f_α, we have that $f_\alpha[X]$ is compact (connected), then X is compact (connected) and (7) $\dim X \leqq \dim Y$ (see [8] and [9] for more details on these mappings). It should be pointed out that in contrast to (7) that the existence of ε-maps of X into Y for all $\varepsilon > 0$ does not, in general, imply that $\dim X \leqq \dim Y$ (see [12]).

DEFINITION AND NOTATION 1.3. If $\alpha = \{A_b\}$ is a collection of sets and X is a set, then $\alpha \cap X = \{A_b \cap X \mid A_b \in \alpha, A_b \cap X \neq \varnothing\}$. A set A is *countable* if and only if card $A \leqq \aleph_0$. A set A is *denumerable* if and only if card $A = \aleph_0$. A set is *dense in itself* if each of its points is a limit point of it. We define the sets (where R^1 denotes the real (Euclidean) line):

$$Q = \{x \in R^1 \mid x \text{ is rational}\}\,,$$
$$J_0 = \varnothing = \text{empty set}$$
$$J = \{x \in R^1 \mid x \text{ is a positive integer}\}\,,$$
$$J_n = \{x \in J \mid 1 \leqq x \leqq n\}\,,$$
$$C = \text{Cantor discontinuum on } [0, 1]\,,$$
$$D = \{x \in R^1 \mid x = 0 \text{ or } 1/n \text{ for } n = 1, 2, \cdots\}\,,$$
$$I = [-1/2, 0]\,,$$

$$J^* = \bigcup \{[i,\ i + 1/2] \mid i \in J\}\ ,$$
$$L^* = \bigcup \{[1/(n + 1), 1/n) \mid n = 1, 3, 5, \cdots\} \bigcup \{0\}\ ,$$
$$M^* = \bigcup \{(1/(n + 1), 1/n) \mid n = 1, 3, 5, \cdots\} \bigcup \{0\}\ ,$$
$$L_n = \bigcup \{(-2j - 1/2,\ -2j] \mid j \in J_n\}\ ,$$
$$L_\infty = \bigcup \{(-2j - 1/2,\ -2j] \mid j \in J\},\ L_0 = \emptyset$$
$$M_n = \bigcup \{(-2j + 1/2,\ -2j + 1) \mid j \in J_n\}\ ,$$
$$M_\infty = \bigcup \{(-2j + 1/2,\ -2j + 1) \mid j \in J\},\ M_0 = \emptyset$$

EXAMPLE 1.1. ($qX \leqq qY$ does not imply $sX \leqq sY$.) Let $X = J$, $Y = D - \{0\}$ and α be any cover of X. Then $\alpha^* = \{\{i\} \mid i \in J\}$ refines α and $f: X \to Y$ defined by $f(i) = 1/i$ is a closed map. Let $\beta = \{\{1/i\} \mid i \in J\}$. Then $f^{-1}[\beta] > \alpha^* > \alpha$ and so f is an α-map. Thus we have $qX \leqq qY$. Now we will assume $sX \leqq sY$ and show it leads to a contradiction. Let g be a closed (1/2)-map of X and Y. Then, for any $\eta > 0$, there are distinct positive integers i and j such that $|(1/i) - (1/j)| < \eta$ and $1/i, 1/j \in g[X]$. But $\delta(g^{-1}[\{1/i\} \cup \{1/j\}]) \geqq 1 > 1/2$. Thus $sX \leqq sY$ must be false.

EXAMPLE 1.2. ($sX \leqq sY$ does not imply $qX \leqq qY$). Let $X = D - \{0\}$ and $Y = D$. Then, for any $\varepsilon > 0$, there is a closed strong ε-map, $f_\varepsilon: X \to Y$, defined by

$$f_\varepsilon(1/i) = \begin{Bmatrix} 0,\ 0 < 1/i < \varepsilon/2 \\ 1/i,\ \varepsilon/2 \leqq 1/i \leqq 1 \end{Bmatrix}\ .$$

Let $\eta = \varepsilon/2$ and suppose $A \subset Y$ and $\delta(A) < \eta$. Then if $0 \in A$ we have $A \subset [0, \eta)$ and so $\delta(f^{-1}[A]) < \varepsilon$. If $0 \notin A$ we have $\delta(f^{-1}[A]) < \eta < \varepsilon$. Hence $sX \leqq sY$. Now we will assume that $qX \leqq qY$ and show it leads to a contradiction. The image under each closed α-map of X is closed in Y and hence compact. This implies that X is compact which is a contradiction.

EXAMPLE 1.3. It is possible for X to admit closed ε-mappings into Y, for all $\varepsilon > 0$, and yet not admit ε-mappings onto a fixed subset of Y for all $\varepsilon > 0$. For example, let $X = D$ and $Y = J$. Just defined f_n by

$$f_n(x) = \begin{Bmatrix} 1,\ \text{for}\ 0 \leqq x < 1/n\ \text{and}\ x \in X \\ 1 + (1/x),\ \text{for}\ 1/n \leqq x \leqq 1\ \text{and}\ x \in X \end{Bmatrix}\ .$$

Then f_n is a $(1/n)$-mapping of X onto $J_{n+1} \subset Y$, (i.e., $qD \leqq qJ$). Suppose that there is a subset Y_0 of Y such that X admits ε-mappings onto Y_0 for all $\varepsilon > 0$. Since X is compact so is Y_0. But the only compact subsets of Y are finite. Obviously X cannot be ε-mapped onto a finite set for all $\varepsilon > 0$.

EXAMPLE 1.4. Now while two spaces of the same Fréchet dimension type have the same cardinality, this may not be the case for two spaces of the same quasi dimension type. For example, consider the spaces C and D. It is clear that $qD \leqq qC$ and we wish to show that $qC \leqq qD$. Recall that C is obtained by first dividing $[0, 1]$ into three equal subintervals and deleting the interior of the middle one. Then each of the two remaining intervals is divided into three equal intervals and the interiors of the middle ones are deleted. Then each of the remaining four intervals is treated similarly and so on indefinitely. The set of points remaining after this process is C. Now consider the 2^n disjoint closed intervals, $\{K_i \mid i = 1, \cdots, 2^n\}$, remaining after the nth step in this construction of C. Define $f_n: C \to D$ by $f_n[K_i \cap C] = 1/i$, $i = 1, \cdots, 2^n$. Then f_n is an $(1/2^n)$-mapping since

$$f_n^{-1}(1/i) = \begin{Bmatrix} K_i, \ i = 1, \cdots, 2^n \\ \varnothing, \ i > 2^n \end{Bmatrix}$$

and $\delta(K_i) = 1/3^n < 1/2^n$. Thus we have that $qC \leqq qD$ since C and D are compact. Hence we may conclude that $qC = qD$.

Throughout this paper a compact connected space of more than one point will be called a *continuum*. A *chain* is a finite collection of open sets $U_1, \cdots, U_n$ such that U_i intersects U_j if and only if $i = j - 1, j$, or $j + 1$. If the links of a chain are of diameter less than ε, the chain is called an *ε-chain*. A continuum is called *snake-like* if for each $\varepsilon > 0$ it can be covered by an ε-chain (see [1]).

EXAMPLE 1.5. Let X be a snake-like continuum and Y an arc. Then X admits ε-mappings onto Y for all $\varepsilon > 0$ (see [5, p. 229] or [2, Lemma 1.6]). So $qX \leqq qY$. If we specialize X further to be a snake-like continuum which does not contain an arc then $qY \leqq qX$ is false. So in this we have $qX < qY$. However, X and Y are not comparable with respect to Fréchet dimension type.

We now give some general results which will be of use in the next section.

THEOREM 1.1. *If $qX \leqq qY$ and Y is compact (totally disconnected), then X is compact (totally disconnected).*

COROLLARY 1.1. *If $qX \leqq qY$ and each component of Y is compact, then each component of X is compact.*

THEOREM 1.2. *If* (1) *X is the union of two disjoint closed sets X_1 and X_2,* (2) *Y is the union of two disjoint closed sets Y_1 and Y_2, and* (3) *$qX_i \leqq qY_i$, for $i = 1, 2$, then $qX \leqq qY$.*

Proof. Let α be a covering of X. Then $\alpha_1 = \alpha \cap X_1$ and $\alpha_2 = \alpha \cap X_2$ are coverings of X_1 and X_2, respectively. Let f_i be a closed α_i-mapping of X_i into Y_i for $i = 1, 2$. Then, for $i = 1, 2$, there is a covering β_i of Y_i such that $f_i^{-1}[\beta_i] > \alpha_i$ and since Y_i is open in Y, so is each element of β_i. Therefore, $\beta = \beta_1 \cup \beta_2$ is a covering of Y and the mapping which is f_1 on X_1 and f_2 on X_2 is a closed α-mapping of X into Y.

REMARK. Theorem 1.2 is not true with "$\leqq$" replaced everywhere by "$<$". To see this let $X_1 = I, X_2 = J, Y_1 = I \cup J$ and $Y_2 = \{x + 3/4 \mid x \in (I \cup J)\}$. Then $qX_i < qY_i$, $i = 1, 2$, but $qX = qY$.

2. Quasi dimension type in the real line. We show in this section that there are only denumerably many quasi dimension types represented by subsets of R^1. This is in contradistinction to Kuratowski's result [6] that there are 2^c Frechet dimension types represented by subsets of R^1. Furthermore, we obtain a representative of each type, determine completely the ordering of these types and give a topological characterization of the linear sets having a given type. *For the remainder of the paper all sets considered are subsets of R^1.*

THEOREM 2.1. [10] *If A and B are denumerable sets, each dense in itself, then A and B are homeomorphic.*

COROLLARY 2.1. *If X is a countable closed set then $qX \leqq qQ$.*

Proof. The set $X \cup Q$ is dense in itself and denumerable so $q(X \cup Q) = qQ$. Since X is a closed set it is also a closed subset of $X \cup Q$. Thus we have $qX \leqq q(X \cup Q) = qQ$.

REMARK. So qJ_i, qD and qJ are all $\leqq qQ$. Using previous examples we have

$$qJ_i < qD = qC < qJ = q(D - \{0\}) \leqq qQ .$$

THEOREM 2.2. *$qX \leqq qJ$ if and only if X is totally disconnected.*

Proof. We first assume that X is totally disconnected. If α is a covering of X, then, since $\dim X \leqq 0$, there is a covering, α', of X

which refines α and is such that no two elements α' meet. Then by Lindelöf's theorem there is a countable subcollection α'' of α' which covers X. Let $f: X \rightarrow J$ be defined by $f[A_i] = i$ for $A_i \in \alpha''$. Then f is a closed α-map since any subset of J is closed and $\beta = \{\{i\} \mid i \in J\}$ is such that $f^{-1}[\beta] > \alpha'' > \alpha' > \alpha$. Thus we have $qX \leqq qJ$. The converse follows from Theorem 1.1.

THEOREM 2.3. *$qX \leqq qC$ if and only if X is compact and totally disconnected. A space X is infinite (finite), compact and totally disconnected if and only if $qX = qC$ (qJ_i, for some nonnegative integer i).*

Proof. Any compact, totally disconnected, metric space X is homeomorphic to a closed subset of C. So $qX \leqq qC$. If X is infinite then it contains a copy of D. So $qD \leqq qX$ and it follows that $qX = qC$. The converses are obvious from Theorem 1.1.

COROLLARY 2.2. *$qX = qJ$ if and only if X is totally disconnected and not compact.*

Proof. Suppose X is totally disconnected and not compact. By Theorem 2.2 we have $qX \leqq qJ$. If α is any covering of J, then $\alpha' = \{\{i\} \mid i \in J\}$ refines it. There is a sequence of points $x_1, x_2, \cdots$ of X such that no subsequence converges to a point of X. Define $f: J \rightarrow K$ by $f(i) = x_i$. Then f is a closed map since each subset of $\{x_i \mid i \in J\}$ is closed in X. There is a covering β of X, each set of which contains at most one point x_i. Then $f^{-1}[\beta] > \alpha' > \alpha$ and so $qJ \leqq qX$. Thus we have $qJ = qX$. The converse follows immediately from the two previous theorems.

We state as a separate corollary some of the information embodied in the proof of Corollary 2.2.

COROLLARY 2.3. *If X is not compact, then X has a closed subset Z such that $qJ \leqq qZ$.*

DEFINITION 2.1. We say *qA immediately preceeds qB*, written $qA \ll qB$, if $qA < qB$ and there is no set X such that $qA < qX < qB$.

REMARK. For totally disconnected sets we have the following ordering:

$$qJ_0 \ll qJ_1 \ll qJ_2 \ll \cdots \ll qJ_i \ll qJ_{i+1} \ll \cdots < qC \ll qJ .$$

Note that qC is the least upper bound of the qJ_i's. This follows from

Theorem 2.3. That $qC \ll qJ$ follows from Theorems 2.2 and 2.3 and Corollary 2.2. Moreover, if X is a compact (noncompact) infinite totally disconnected set then $qX = qC(qJ)$. Thus there are only denumerably many quasi dimension types of totally disconnected subsets of R^1.

THEOREM 2.4. *$qX = qI$ if and only if X is compact and not totally disconnected.*

Proof. Assume X is compact and not totally disconnected. Since X contains a nondegenerate component, K, which is closed (in X) and hence compact we have $K = [a, b]$. So $qI = q[a, b] \leqq qX$. Now X being compact is homeomorphic to a closed subset of I, so $qX \leqq qI$. The converse follows from Theorem 1.1.

THEOREM 2.5. *$qJ \leqq qX$ if and only if X contains a closed subset homeomorphic to J.*

Proof. One implication follows from the definition of quasi dimension type. To show the converse we proceed as in the proof of Theorem 2.3. Let $\alpha = \{\{i\} \mid i \in J\}$ denote the finest covering of J. Then there is a closed α-map $f: J \rightarrow X$. Since f is closed, one-to-one and continuous it is a homeomorphism.

THEOREM 2.6. *The space J and any compact, not totally disconnected space X are not comparable. In particular, J and I are not comparable.*

Proof. Theorem 1.1 implies $qX \nleqq qJ$ (since X is not totally disconnected) and $qJ \nleqq qX$ (since J is not compact).

THEOREM 2.7. $qC \ll qI$.

Proof. Since C is homeomorphic to a closed subset of I and $qI \nleqq qC$ we have $qC < qI$. Suppose there is a set X such that $qC < qX < qI$. Then by Theorem 1.1 we have that X is compact. *Case* 1: X is not totally disconnected; then $qX = qI$ which is a contradiction. *Case* 2: X is totally disconnected. Then $qX \leqq qC$ which again is a contradiction. Thus we may conclude that $qC \ll qI$.

THEOREM 2.8. $qJ \ll q(I \cup J)$.

Proof. Since J is a closed subset of $I \cup J$ and $q(I \cup J) \nleqq qJ$ we have $qJ < q(I \cup J)$. Suppose that there is a set X such that

$qJ < qX < q(I \cup J)$. Then X is not compact. *Case* 1: X is not totally disconnected. Then X contains nondegenerate arc components. If one of these, K, is not compact, then $q(I \cup J) \leqq qK \leqq qX$ (this last inequality is due to the fact that K is closed in X). This is a contradiction. If, on the otherhand, K is compact, then we take a closed, totally disconnected, noncompact subset Z of $X - K$ (which is possible because X is not compact) and have $q(I \cup J) \leqq q(K \cup Z) \leqq qX$. Again this is a contradiction. *Case* 2: X is totally disconnected. Then $qX = qJ$ which is a contradiction.

THEOREM 2.9. $qI \ll q(I \cup J)$.

Proof. Since I is closed in $I \cup J$ and $q(I \cup J) \not\leqq qI$ we have $qI < q(I \cup J)$. Suppose that there is a set X such that $qI < qX < q(I \cup J)$. Then X must be noncompact. If X is totally disconnected, then $qX = qJ$ which is impossible. Hence X contains nondegenerate arc components. If one of these (K) is not compact, K has a closed subset homeomorphic to $I \cup J$, so $q(I \cup J) \leqq qK \leqq qX$. This is impossible. If, on the other hand, K is compact, then we get a contradiction as before.

NOTATION. Let Y be a subset of X. We use $N(Y)$ to denote the closure (in X) of the union of the nondegenerate components of Y.

LEMMA 2.1. *If* $qX \leqq q(I \cup J)$, *then* $qN(X) \leqq qI$.

Proof. If $N(X)$ is compact we are done by Theorem 2.4. Suppose $N(X)$ is not compact. Then, since each component of $N(X)$ is compact (Corollary 1.1), there is a sequence $K_1, K_2, \cdots$ of closed intervals which are components of $N(X)$, such that (1) $K = \bigcup_{i=1}^{\infty} K_i$ is a closed subset of $N(X)$, (2) each K_i is open in K, and (3) no subsequence of the K_i's converges to a point of X. Now any cover β of K can be refined by a cover α of K such that for each K_i, $\alpha \cap K_i$ is a chain, each link of which is a proper subset of K_i. Since K is a closed subset of $N(X)$, $qK \leqq qN(X) \leqq q(I \cup J)$; so there is a closed α-map f_α of K into $I \cup J$. But $f_\alpha [K] \cap J = \emptyset$, for otherwise f_α would map some K_i into a point $p \in J$ and so, contrary to our assumption, f_α would not be an α-map (since, for any open set U about p, $f_\alpha^{-1}[U]$ would contain K_i and therefore would not be contained in any element of α.). So f_α is a closed α-map of K into I. This implies that K is compact which is a contradiction.

THEOREM 2.10. *If each component of* X *is compact, then* $qX \leqq qJ^*$.

Proof. If X is totally disconnected, then $qX \leqq qJ < qJ^*$ and we are done. So assume otherwise and write $T = X - \bigcup_{i=1}^{n} C_i$ where the C_i are the nondegenerate components of X and $n \leqq \infty$. Let γ be any covering of X. Since $\dim X \leqq 1$, γ has a refinement γ_1 of order $\leqq 2$ (i.e., no 3 elements of γ_1 meet); say γ_1 consists of sets $W \cap X$ where W is open in R^1. Let the components of the W's be denoted by A_i; they are open intervals, there is countably many of them, and no point of X belongs to any 3 of them. The covering $\beta = \alpha \cap X$, where $\alpha = \{A_i \mid i \in J\}$, refines γ. Denote $\alpha \cap C_i$ by α_i. Let Δ be a denumerable dense subset of $R^1 - X$ and let

$$U = \{(a, b) \mid a, b \in \Delta, C_i \subset (a, b) \subset \bigcup \{A_j \mid A_j \in \alpha_i\} \text{ for some } i\ (1 \leqq i \leqq n)\}$$
$$\bigcup \{(a, b) \mid a, b \in \Delta, x \in (a, b) \subset A_j \text{ for some } x \in T \text{ and } A_j \in \alpha\}\ .$$

Then since Δ is denumerable so is U and we may write $U = \{U_i \mid i \in J\}$. Note that $U \cap X$ covers X.

Now we define the following:

(1) $\tilde{U}_1 = U_1$, $\tilde{U}_i = U_i - \bigcup \{U_j \mid j = 1, \cdots, i-1\}$ for $i = 2, 3, \cdots$,

(2) $\tilde{U} = \{\tilde{U}_i \mid \tilde{U}_i \neq \varnothing, i \in J\}$,

(3) $\tilde{V} = \{\tilde{K} \mid \tilde{K}$ is a component of some $\tilde{U}_i \in \tilde{U}\}$.

Note that the elements of $\tilde{V}$ are disjoint intervals (open, closed or half-open) having their end points in Δ. So that $V = \tilde{V} \cap X$ is a countable collection of disjoint, open and closed subsets of X which covers X. Actually the elements of V are strongly disjoint, i.e., if $K_1, K_2 \in V$, then $(\text{glb } K_1, \text{lub } K_1) \cap (\text{glb } K_2, \text{lub } K_2) = \varnothing$.

We now index the elements of V as follows: even positive integers are used to index the totally disconnected elements of V and odd positive integers are used to index the elements of V which are not totally disconnected. For any element K_i of V, there are $a_i, b_i \in \Delta$, such that $K_i = (a_i, b_i) \cap X$. We now define a closed β-mapping, f_β, of X into J^* as follows:

(1) if i is even, $f_\beta\, [K_i] = i$,

(2) if i is odd, $(f_\beta \mid K_i)$ is an order preserving homeomorphism of $C_j = [c_j, d_j]$ (some nondegenerate component of X contained in K_i) onto $[i, i + 1/2]$; $f_\beta\, [(a_i, c_j) \cap X] = i$, $f_\beta\, [(d_j, b_i) \cap X] = i + 1/2$.

To show that f_β is a β-mapping let:

$$\theta = \bigcup \{f_\beta[K_i] \mid i = 1, 3, 5, \cdots\}\ ,$$
$$\lambda_1 = \{[i, i + 1/2] \mid \theta \cap [i, i + 1/2] = \varnothing\}\ ,$$
$$\lambda_2 = \{f_\beta[\beta_j \cap K_i] \mid \beta_j \cap K_i \in \beta \cap K_i \text{ and } i = 1, 3, 5, \cdots\}\ .$$

Then $\lambda = \lambda_1 \cup \lambda_2$ is a covering of J^*. If $[i, i + 1/2] \in \lambda_1$, then $f_\beta^{-1}[[i, i + 1/2]] = K_i$ (or $\varnothing$) and K_i is totally disconnected. By construction, K_i is a subset of some $U_h \in U$, and either

$$C_k \subset U_h \subset \bigcup \{A_j \mid A_j \text{ meets } C_k\}$$

for some C_k, or $x \in U_h \subset A_j$ for some $x \in T$ and $A_j \in \alpha$. In the latter case, K_i is contained in a single A_j. But this also follows in the other case, since otherwise some point of X will belong to 3 distinct sets A_j. Thus $K_i \subset A_j \cap X \in \beta$. If $f_\beta[\beta_j \cap K_i] \in \lambda_2$, then $f_\beta^{-1}[f_\beta[\beta_j \cap K_i]] \subset \beta_j \in \beta$. Hence $f_\beta^{-1}[\lambda] > \beta$, i.e., f is a β-mapping. Since $\beta > \gamma$ we have that f_β is also a γ-mapping.

To show that f_β is closed let Z be a closed subset of X. Note that a set is closed in J^* if and only if its intersection with each component of J^* is closed. Since distinct K_i's are mapped into distinct components of J^*, it will suffice to prove that each $f_\beta[Z \cap K_i]$ is closed. If i is even, $f_\beta[Z \cap K_i]$ has at most one point, so we need only consider the case when i is odd. Since

$$Z \cap K_i = (Z \cap C_j) \cup (Z \cap (K_i - C_j))$$

(where C_j is the nondegenerate component of K_i used in defining f_β), we can write

$$f_\beta[Z \cap K_i] = f_\beta[Z \cap C_j] \cup f_\beta[Z \cap (K_i - C_j)] \text{ .}$$

Then, since $Z \cap C_j$ is closed in X and f_β is a homeomorphism on C_j, we have that $f_\beta[Z \cap C_j]$ is a closed subset of $[i, i + 1/2]$. Furthermore, $f_\beta[Z \cap (K_i - C_j)] \subset \{i\} \cup \{i + 1/2\}$ and hence is also closed. Therefore, we may conclude that $f_\beta[Z \cap K_i]$ is a closed subset of $[i, i + 1/2]$ when i is odd. So that $f_\beta[Z] \cap [i, i + 1/2]$ is closed, for each $i \in J$, and hence $f_\beta[Z]$ is a closed subset of J^*.

Thus, for each covering γ of X, we have shown the existence of a closed γ-mapping of X into J^*. Therefore, we have the desired result, i.e., $qX \leqq qJ^*$.

THEOREM 2.11. $q(I \cup J) \ll qJ^*$.

Proof. Let α be a covering of $I \cup J$. Then a map which is a homeomorphism of I onto $[1, 3/2]$ and takes $j \in J$ into $j + 1$ is a closed α-map of $I \cup J$ into J^*. So $q(I \cup J) \leqq qJ^*$. Now suppose $qJ^* \leqq q(I \cup J)$. Then Lemma 2.1 implies that $qJ^* \leqq qI$. But this is impossible by Theorem 1.1. So $q(I \cup J) < qJ^*$.

Suppose there is a set X such that $q(I \cup J) < qX < qJ^*$. Then by Theorem 1.1 X is not compact and not totally disconnected. In addition, each component of X is compact (by Corollary 1.1). Suppose $N(X)$ is not compact. Then there is a sequence $K_1, K_2, \cdots$ of closed intervals which are components of $N(X)$, such that (1) $K = \bigcup_{i=1}^\infty K_i$ is a closed subset of $N(X)$, (2) each K_i is open in K, and (3) no subsequence of the K_i's converges to a point of X. Since there is a homeomorphism of J^* onto K (just send $[i, (i+1)/2]$ homeomorphically onto K_i, for each $i \in J$) and K is closed it X, it follows that $qJ^* \leqq qX$. This contradicts our assumption that $qX < qJ^*$.

Now suppose $N(X)$ is compact. Let γ be any covering of X. Proceeding as in the proof of Theorem 2.10 one obtains a closed β-map f_β of X into J^* (where $\beta > \gamma$). Under this mapping the image of $N(X)$ is the union of a finite number of closed intervals which are components of J^*. We follow f_β by a homeomorphism g_β of $f_\beta[X]$ onto a closed subset of $I \cup J$. The homeomorphism g_β can be defined as one which takes $f_\beta[N(X)]$ homeomorphically into I and $f_\beta[X] - f_\beta[N(X)]$ homeomorphically into J. Consequently, $qX \leqq q(I \cup J)$ which contradicts our assumption that $q(I \cup J) < qX$. So this together with the previous paragraph imply that $q(I \cup J) \ll qJ^*$.

LEMMA 2.2. *If X is not compact and not totally disconnected, then $q(I \cup J) \leqq qX$.*

Proof. If X contains a component K which is an open or half-open interval, then $q(I \cup J) \leqq qK \leqq qX$ and we are done. Assume now that the components of X are compact and that K is one such nondegenerate component. Since X is not compact, it has a closed subset Z which is homeomorphic to J and disjoint from K. Now we claim that the hypotheses of Theorem 1.2 are satisfied, for (1) $qI \leqq qK$, (2) $qJ \leqq qZ$, (3) I and J are disjoint closed subsets of $I \cup J$; (4) K and Z are disjoint closed subsets of $K \cup Z$. So applying Theorem 1.2 we conclude that $q(I \cup J) \leqq q(K \cup Z) \leqq qX$.

THEOREM 2.12. $qX = q(I \cup J)$ $(qX = qJ^*)$ *if and only if* (1) *X is not compact*, (2) *not totally disconnected*, (3) *each of its components is compact, and* (4) *$N(X)$ is* (*is not*) *compact.*

Proof. Assume that conditions (1)-(4) hold. As consequences of Theorem 2.10 and Lemma 2.2 we have $q(I \cup J) \leqq qX \leqq qJ^*$. So since $q(I \cup J)$ immediately precedes qJ^* we have that qX equals one or the other. Now if $N(X)$ is compact, it follows from Theorem 1.1 that $qJ^* \nleqq qX$. Hence in this case we have that $qX = q(I \cup J)$. Now consider the other case, i.e., when $N(X)$ is not compact. Suppose $qX \leqq q(I \cup J)$. Then as a result of Lemma 2.1 we have that $qN(X) \leqq qI$. However, this imlpies that $N(X)$ is compact contrary to our assumption. Hence in this case $qX = qJ^*$.

Now assume that $qX = q(I \cup J)$ or $qX = qJ^*$. Then in both cases by Theorem 1.1 and Corollary 1.1, we have that: (1) X is not compact,

(2) not totally disconnected and (3) each component of X is compact. In the former case, as a result of Lemma 2.1 we have that $qN(X) \leqq qI$ and hence $N(X)$ is compact. In the latter case, suppose $N(X)$ is compact. Then, by the first part of this proof $qX = q(I \cup J)$. But $q(I \cup J) < qJ^*$ so we have a contradiction. Thus in this case $N(X)$ is not compact.

THEOREM 2.13. $qJ^* \ll qL_1$.

Proof. First, $qJ^* \leqq qL_1$ since the function f defined by $f(x) = (1 - 5x)/2x$ is a homeomorphism of J^* onto a closed subset of L_1. Since L_1 is connected and not compact, Corollary 1.1 implies that $qL_1 \nleqq qJ^*$ and therefore $qJ^* < qL_1$. Now suppose there is a set X such that $qJ^* < qX < qL_1$. Then X is not compact and some component of X is not compact (a consequence of Theorem 2.10). Let K denote such a component. Then $qL_1 \leqq qK \leqq qX$ which is a contradiction and so $qJ^* \ll qL_1$.

LEMMA 2.3. *If $qX \leqq qY$, then X does not have more components than Y which are open intervals.*

Proof. First we will show that $qM_1 \nleqq qL_1$. Suppose not, i.e., $qM_1 \leqq qL_1$. Then, for any covering α of M_1, there exists a closed α-mapping, f_α, of M_1 into L_1. Let $S = (-3/2, -11/8]$, $T = [-9/8, -1)$ and $\mathscr{A}$ be the collection of coverings of M_1 with the property that none of their elements intersects both S and T. (Note that $\mathscr{A}$ is cofinal in the collection of all coverings of M_1.) For any sufficiently fine $\alpha \in \mathscr{A}$, there are points $y_1, y_2 \in L_1$ such that $f_\alpha[S] = (-5/2, y_1]$ and $f_\alpha[T] = (-5/2, y_2]$. This is the case since each image set is closed in L_1, connected and not compact (for α sufficiently fine). But this implies $f_\alpha^{-1}(\min(y_1, y_2))$ intersects both S and T which contradicts the fact that f_α is an α-mapping. Therefore, $qM_1 \nleqq qL_1$.

We now show, for any sufficiently fine $\alpha \in \mathscr{A}$, that a closed α-mapping, f_α, of M_1 into itself actually takes M_1 onto itself. Suppose not, i.e., for $\alpha \in \mathscr{B}$, a cofinal subset of $\mathscr{A}$, $f_\alpha[M_1]$ is a proper subset of M_1. Then, for α sufficiently fine, $f_\alpha[M_1] = (-3/2, y_\alpha]$ or $[y_\alpha, -1)$ where $y_\alpha \in M_1$, since $f_\alpha[M_1]$ is closed in M_1, connected and not compact. We consider only the first of the two cases since they are similar. Let g_α be a homeomorphism of $(-3/2, y_\alpha]$ onto L_1. Then, for any $\alpha \in \mathscr{B}$, $g_\alpha f_\alpha$ is a closed α-mapping of M_1 into L_1. But this is impossible since $qM_1 \nleqq qL_1$. Hence, for sufficiently fine $\alpha \in \mathscr{A}$, $f_\alpha[M_1] = M_1$.

We now prove that if X has at least n components which are open intervals, then so does Y, for $n \in J$. Let $\mathscr{K}$ denote the collection of components of X which are open intervals. Let α be any covering of X such that:

(1) if $K_i \in \mathscr{K}$, then $\alpha \cap K_i$ is a covering of K_i corresponding to a covering of M_1 in $\mathscr{A}$, under the obvious linear homeomorphism of K_i onto M_1,

(2) if $A \in \alpha$, then A intersects at most one $K_i \in \mathscr{K}$. Then, for α sufficiently fine and a closed α-mapping, f_α, we have that (1) implies $f_\alpha[K_i] = C_i$, a component of Y which is an open interval. This is the case since $f_\alpha[K_i]$ is not contained in a component of Y which is a half-open interval (otherwise $qM_1 \leqq qL_1$), nor in one which is a closed interval (otherwise K_i is compact) and so $f_\alpha[K_i]$ must be contained in one which is an open interval. But we have already shown in this situation that $(f_\alpha \mid K_i)$ maps K_i onto this open interval (for α sufficiently fine). Furthermore, since (2) implies $C_i \cap C_j = \varnothing$, for $i \neq j$, Y has at least n components which are open intervals.

LEMMA 2.4. *If $qX \leqq q(L_1 \cup J^*)$, then X has at most one component which is a half-open interval.*

Proof. Suppose X has two components which are half-open intervals, say K_1 and K_2. Let α be a covering of X in which no element intersects both K_1 and K_2. Since $q(K_1 \cup K_2) \leqq qX \leqq q(L_1 \cup J^*)$ there is a closed β-mapping f_β of $K_1 \cup K_2$ into $L_1 \cup J^*$ (where $\beta = \alpha \cap (K_1 \cup K_2)$). It follows from the fact that $f_\beta[K_1]$ and $f_\beta[K_2]$ are closed, connected and not compact (for sufficiently fine β), that $f_\beta[K_1 \cup K_2] \subset L_1$. Furthermore, there are points y_1, y_2 of L_1 such that $f_\beta[K_1] = (-5/2, y_1]$ and $f_\beta[K_2] = (-5/2, y_2]$ since each image set is closed in L_1, connected and not compact (for β sufficiently fine). But this implies that $f_\beta^{-1}(\min(y_1, y_1))$ intersects both K_1 and K_2 which contradicts the fact that f_β is a β-map. Therefore, X does not have two components which are half-open intervals.

LEMMA 2.5. *If $qL_1 \leqq qX \leqq q(L_1 \cup J^*)$, then X has exactly one component which is a half-open interval and none which is an open interval.*

Proof. By Lemma 2.3 X does not have a component which is an open interval. Suppose each component of X is compact. Then Theorem 2.10 implies that $qX \leqq qJ^*$. But $qJ^* < qL_1$ and we have a contradiction. So X has at least one component which is not compact and none which is an open interval. Hence there is at least one component of X which is a half-open interval. Finally, Lemma 2.4 implies that X has exactly one component which is a half-open interval.

THEOREM 2.14. $qX = qL_1$ *if and only if* (1) *exactly one of the* *components of X is a half-open interval,* (2) *this component, say K, has the property that the closure* (*in X*) *of $X - K$ is compact.*

Proof. We first assume that $qX = qL_1$. Then X can not be compact, nor totally disconnected. By Lemma 2.5 X has exactly one component, say K, which is a half-open interval and none which is an open interval. Let $Y = \mathrm{Cl}(X - K)$ (we assume $Y \neq \emptyset$) and let β be any covering of Y. Then β can be extended to a covering α of X (i.e., $\alpha \cap Y = \beta$ and α is a covering of X). Since $qX \leqq qL_1$ there is a closed α-map, f_α, of X into L_1. Moreover, since $f_\alpha[K]$ is closed in L_1, connected and not compact (for α sufficiently fine), $f_\alpha[K]$ must be of the form $(-5/2, c]$ where $-5/2 < c \leqq -2$. Let $g_\beta = (f_\alpha \mid Y)$ and consider β to be fine enough so that $d = \mathrm{glb}\,(g_\beta[Y]) > -5/2$. Then g_β is a closed β-map of Y into $L_1 - (-5/2, d)$, a compact set. Thus, for any sufficiently fine covering β, we have $g_\beta[Y]$ is compact and so Y is compact.

We now consider the converse. Since X has a component K which is a half-open interval, we have that $qL_1 = qK \leqq qX$. Now assume $X - K \neq \emptyset$. Then there are disjoint closed subsets X_1 and X_2 of X such that $X = X_1 \cup X_2, X_1 \supset K$, X_1 is homeomorphic to a closed subset of $(-5/2, -9/4]$ and X_2 is homeomorphic to a compact subset of $(-9/4, -2]$. Hence by Theorem 1.2 we have that $qX \leqq qL_1$.

THEOREM 2.15. $qL_1 \ll q(L_1 \cup J)$.

Proof. Since L_1 is closed in $L_1 \cup J$ we have $qL_1 \leqq q(L_1 \cup J)$. As a consequence of Theorem 2.14 we have $q(L_1 \cup J) \neq qL_1$ and so $qL_1 < q(L_1 \cup J)$. Suppose there is a set X such that $qL_1 < qX < q(L_1 \cup J)$. Then X cannot be compact nor totally disconnected. Moreover, Lemma 2.5 implies that X has exactly one component, say K, which is a half-open interval and none which is an open interval. So $\mathrm{Cl}\,(X - K)$ is not compact (otherwise $qL_1 = qX$). Since $\mathrm{Cl}\,(X - K)$ is not compact it contains a closed subset Z homeomorphic to J. Therefore $q(L_1 \cup J) \leqq q(K \cup Z) \leqq qX$. This is a contradiction.

LEMMA 2.6. $q(L_1 \cup J) < q(L_1 \cup J^*)$.

Proof. Since $qJ \leqq qJ^*$, by Theorem 1.2 we have that $q(L_1 \cup J) \leqq q(L_1 \cup J^*)$. Suppose $q(L_1 \cup J^*) \leqq q(L_1 \cup J)$. Any cover β of $L_1 \cup J^*$ can be refined by a subcover α such that, for each component K of $L_1 \cup J^*$, $\alpha \cap K$ is a chain, each link of which is a proper subset of K. By assumption, there is a closed α-map f_α of $L_1 \cup J^*$ into $L_1 \cup J$. But $f_\alpha[L_1 \cup J^*] \cap J = \emptyset$, for otherwise f_α would map some component K of $L_1 \cup J^*$ into a point $p \in J$ and so, contrary to our assumption, f_α would not be an α-map (since, for any open set U about p, $f_\alpha^{-1}[U]$

would contain K and therefore would not be contained in any element of α). So f_α is a closed α-map of $L_1 \cup J^*$ into L_1, which implies $q(L_1 \cup J^*) \leqq qL_1$. But then $q(L_1 \cup J^*) = qL_1$ and this contradicts Theorem 2.14. Therefore, $q(L_1 \cup J^*) \not\leqq q(L_1 \cup J)$ and we have $q(L_1 \cup J) < q(L_1 \cup J^*)$.

LEMMA 2.7. *If X contains exactly one component K which is a half-open interval, none which is an open interval, and $N(X - K)$ is not compact, then $q(L_1 \cup J^*) \leqq qX$.*

Proof. Since $N(X - K)$ is not compact, it contains a sequence $K_1, K_2, \cdots$ of nondegenerate components of X which are closed intervals and no subsequence of $K_1, K_2, \cdots$ converges to a point of X. Therefore, $\bigcup_{i=1}^{\infty} K_i$ is a closed subset of X which is homeomorphic to J^*. As K is also closed in X, $K \cup (\bigcup_{i=1}^{\infty} K_i)$ is a closed subset of X homeomorphic to $L_1 \cup J^*$. Hence $q(L_1 \cup J^*) \leqq qX$.

THEOREM 2.16. *$qX = q(L_1 \cup J)$ if and only if* (1) *exactly one of the components of X is a half-open interval,* (2) *this component, say K, has property that the closure* (*in X*) *of $X - K$ is not compact, and* (3) *$N(X - K)$ is compact.*

Proof. Assume that X satisfies conditions (1), (2) and (3). First we will show that $q(L_1 \cup J) \leqq qX$. Since $\mathrm{Cl}\,(X - K)$ (in X) is not compact, it contains a closed subset Z which is homeomorphic to J. Thus each subset of Z is closed in Z and, since Z is closed in X, each subset of Z is closed in X. Furthermore, $\mathrm{Cl}\,(X - K) = X - K$ or $(X - K) \cup \{p\}$, where p is the end point of K which is in K. Therefore, $Y = Z - p$ is closed in X, $Y \subset X - K$ and Y is homeomorphic to J. As K is also closed in X, $K \cup Y$ is a closed subset of X homeomorphic to $L_1 \cup J$. Therefore we have $q(L_1 \cup J) \leqq qX$.

Now we show that $qX \leqq q(L_1 \cup J)$. Let γ be any covering of X and $K = (c, d]$. Then γ can be refined by a covering $\beta = \alpha \cap X$, where $\alpha = \{A_j \mid j \in J\}$ is a collection of open intervals such that: (1) $\bigcup_{j=1}^{\infty} A_j \supset X$, (2) if $A_j \in \alpha_0 = \alpha \cap K$, then $c \notin A_j$ and (3) there is exactly one $A_j \in \alpha_0$ such that $d \in A_j$. Let $B = (c, b)$ where

$$b \in [(R^1 - X) \cap (\cup \{A_j \mid A_j \in \alpha_0\})] .$$

Then $W = B \cap X$ is open and closed in X, and $K \subset W \subset \bigcup \{A_j \mid A_j \in \alpha_0\}$. So W and $Y = X - W$ are closed disjoint subsets of X whose union is X. Moreover, a mapping f, which is a homeomorphism of K onto $(-5/2, -9/4]$ and takes $W - K$ into the point $-9/4$, is a closed $(\beta \cap W)$-mapping of W into $(-5/2, -9/4]$. Furthermore, since $\beta > \gamma$

we have that f is also a $(\gamma \cap W)$-mapping of W into $(-5/2, -9/4]$.

Now we consider Y. *Case* 1: Y is compact or totally disconnected, Then $qY \leqq q(I \cup J)$. *Case* 2: Y is neither compact nor totally disconnected. Then it follows from condition (3) of our hypothesis that $N(Y)$ and each comporent of Y are compact. Therefore, Theorem 2.12 applies and we have that $qY \leqq q(I \cup J)$. So, in both cases, we have $qY \leqq q(I \cup J) = q([-17/8, -2] \cup J)$. Thus there is a closed $(\gamma \cap Y)$-mapping, g, of Y into $[-17/8, -2] \cup J$. Therefore, the mapping which is f on W and g on Y is a closed γ-mapping of X into $L_1 \cup J$. Thus $qX \leqq q(L_1 \cup J)$ and since we already have shown $q(L_1 \cup J) \leqq qX$ it follows that $qX = q(L_1 \cup J)$.

Now we assume that $qX = q(L_1 \cup J)$. Then X is not compact nor totally·disconnected. By Lemma 2.6 we have that $q(L_1 \cup J) \leqq q(L_1 \cup J^*)$. So Lemma 2.5 applies and X has exactly one component, say K, which is a half-open interval and none which is an open interval. So all components of X, other than K, are compact. As a consequence of Theorems 2.14 and 2.15 we have that $\mathrm{Cl}\,(X - K)$ is not compact and so neither is $X - K$. Now $N(X - K)$ is compact, otherwise by Lemma 2.7 $q(L_1 \cup J^*) \leqq qX$ which is impossible by Lemma 2.6.

THEOREM 2.17. $q(L_1 \cup J) \ll q(L_1 \cup J^*)$.

Proof. By Lemma 2.6 we have that $q(L_1 \cup J) < q(L_1 \cup J^*)$. Now suppose there is a set X such that $q(L_1 \cup J) < qX < q(L_1 \cup J^*)$. Then by Lemma 2.5 X has exactly one component, say K, which is a half-open interval and none which is an open interval. Now $N(X - K)$ is not compact, otherwise Theorems 2.14 and 2.16 would imply $qX \leqq q(L_1 \cup J)$. But then Lemma 2.7 implies that $q(L_1 \cup J^*) \leqq qX$ which is a contradiction.

THEOREM 2.18. $qX = q(L_1 \cup J^*)$ *if and only if* (1) *exactly one of the components of* X, *say* K, *is a half-open interval*, (2) *no component of* X *is an open interval, and* (3) $N(X - K)$ *is not compact.*

Proof. Assume X satisfies conditions (1), (2) and (3). Then by Lemma 2.7 we have $q(L_1 \cup J^*) \leqq qX$. Next we show that $qX \leqq q(L_1 \cup J^*)$. Let γ be any covering of X and $K = (c, d]$. Then, as in the proof of Theorem 2.16, we have $X = W \cup Y$, where W, Y are closed disjoint subsets of X and $W \subset K$. Moreover, there is a closed $(\gamma \cap W)$-mapping, f, of W into L_1, and $qY \leqq qJ^*$ (since each component of Y is compact and Theorem 2.10 applies). Thus there is a closed $(\gamma \cap Y)$-mapping, g, of Y into J^*. So the mapping which is f on W and g on Y is a closed γ-mapping of X into $L_1 \cup J^*$. Therefore, we have $qX \leqq q(L_1 \cup J^*)$ and so $qX = q(L_1 \cup J^*)$.

Assume now that $qX = q(L_1 \cup J^*)$. Then Lemma 2.5 implies that X has exactly one component, say K, which is a half-open interval and none which is an open interval. If $\mathrm{Cl}(X - K)$ is compact, then Theorem 2.14 implies $qX = qL_1$. This is a contradiction to our assumption that $qX = q(L_1 \cup J^*)$ since $qL_1 < q(L_1 \cup J^*)$. So we now assume that $\mathrm{Cl}(X - K)$ is not compact. Then, if $N(X - K)$ is compact, Theorem 2.16 implies that $qX = q(L_1 \cup J)$. This is impossible so $N(X - K)$ is not compact.

LEMMA 2.8. *If $qY \leqq qX$ and X does not have a component which is an open interval, then Y does not have more components than X which are half-open intervals.*

Proof. We prove that if Y has at least n components which are half-open intervals, then so does X, for each $n \in J$. Let $K_1, K_2, \cdots, K_n$ denote n of these components in Y. Suppose X has only $m < n$ components which are half-open intervals, say $C_1, \cdots, C_m$. For any covering α of Y, let f_α be a closed α-mapping of Y into X. Since the components of X, other than the C_i, are compact, it follows that $f_\alpha[\bigcup_{i=1}^n K_i] \subset \bigcup_{i=1}^m C_i$, for α sufficiently fine. Therefore, some two K_i, say K_1 and K_2, are such that $f_\alpha[K_1 \cup K_2]$ is contained in some one C_i. This implies that $qL_2 \leqq qL_1$ which contradicts Lemma 2.4. Hence, X has at least n components which are half-open intervals.

THEOREM 2.19. $q(L_1 \cup J^*) \ll qL_2$.

Proof. It follows from Theorem 1.2 that $q(L_1 \cup J^*) \leqq qL_2$ and from Lemma 2.4 that $qL_2 \nleqq q(L_1 \cup J^*)$. Thus $q(L_1 \cup J^*) < qL_2$. Now suppose there is a set X such that $q(L_1 \cup J^*) < qX < qL_2$. Then, by Lemma 2.3, X does not have a component which is an open interval. Applying Lemma 2.8 to $q(L_1 \cup J^*) < qX$ we conclude that X has at least one component which is a half-open interval. On the other hand, X has fewer than two such components (otherwise $qL_2 \leqq qX$). Therefore, X has exactly one component which is a half-open interval. So X satisfies the conditions of Theorem 2.14, 2.16 or 2.18 and we may conclude that $qX \leqq qL_1$, $qX \leqq q(L_1 \cup J)$ or $qX \leqq q(L_1 \cup J^*)$. Hence, we have that $qX \leqq q(L_1 \cup J^*)$. But this is a contradiction and we have that $q(L_1 \cup J^*) \ll qL_2$.

THEOREM 2.20. *$qX = qL_n (n \in J)$ if and only if X has exactly n components which are half-open intervals, say $K_1, \cdots, K_n$, and the closure (in X) of $X - \bigcup_{i=1}^n K_i$ is compact.*

Proof. We first assume that $qX = qL_n$. Then, by Lemma 2.3, X

does not have a component which is an open interval. So Lemma 2.8 applies (twice) and X has exactly n components which are half-open intervals, say $K_1, \cdots, K_n$. Let $K = \bigcup_{i=1}^n K_i$, $Y = \mathrm{Cl}\,(X - K)$ (we assume $Y \neq \emptyset$) and let β be any covering of Y. Then β can be extended to a covering α of X (i.e., $\alpha \cap X = \beta$ and α is a covering of X). Since $qX \leqq qL_n$ there is a closed α-mapping, f_α, of X into L_n. Moreover, since $f_\alpha[K]$ is closed in L_n and, for α sufficiently fine, each of its components is not compact, $f_\alpha[K] = \bigcup \{(-2j - 1/2, c_j] \mid j \in J_n$, where $-2j - 1/2 < c_j \leqq -2j\}$. Let $g_\beta = (f_\alpha \mid Y)$ and consider β to be fine enough so that $d_j = \mathrm{glb}\,(g_\beta[Y] \cap (-2j - 1/2, -2j]) > -2j - 1/2$, for $j \in J_n$. Then g_β is a closed β-mapping of Y into

$$L_n - \bigcup \{(-5/2, d_j) \mid j \in J_n\} ,$$

a compact set. Thus, for any sufficiently fine covering β, we have $g_\beta[Y]$ is compact and so Y is compact.

Now consider the converse. So X has n components, $K_1, \cdots, K_n$, which are half-open intervals. Let $K = \bigcup_{i=1}^n K_i$. Then we have $qL_n = qK \leqq qX$. Now assume $X - K \neq \emptyset$. Then there are disjoint closed subsets, X_1 and X_2, of X such that $X = X_1 \cup X_2$, $X_1 \supset K$, X_1 is homeomorphic to a closed subset of $\bigcup \{(-2j - 1/2, -2j - 1/4] \mid j \in J_n\}$ and X_2 is homeomorphic to a compact subset of

$$\bigcup \{(-2j - 1/4, -2j] \mid j \in J_n\} .$$

Hence, by Theorem 1.2 we have $qX \leqq qL_n$.

THEOREM 2.21. $qL_n \ll q(L_n \cup J)$, $n \in J$.

Proof. Since L_n is closed in $L_n \cup J$ we have $qL_n \leqq (L_n \cup J)$. As a consequence of Theorem 2.20 we have $q(L_n \cup J) \neq qL_n$ and so $qL_n < q(L_n \cup J)$. Suppose there is a set X such that

$$qL_n < qX < q(L_n \cup J) .$$

Then Lemma 2.3 implies X does not have a component which is an open interval. So Lemma 2.8 can be applied (twice) to get that X has exactly n components, say $K_1, \cdots, K_n$, which are half-open intervals. Let $K = \bigcup_{i=1}^n K_i$. Then $\mathrm{Cl}\,(X - K)$ is not compact (otherwise $qL_n = qX$). Therefore $\mathrm{Cl}\,(X - K)$ contains a closed subset Z homeomorphic to J and so $q(L_n \cup J) \leqq q(K \cup Z) \leqq qX$. This is a contradiction.

The next four theorems have proofs similar to those of Theorems 2.16–2.19 and therefore omitted.

THEOREM 2.22. $qX = q(L_n \cup J)$ $(n \in J)$ *if and only if* (1) X *has exactly* n *components which are half-open intervals, say* $K_1, \cdots, K_n$,

(2) *the closure* (*in* X) *of* $X - \bigcup_{i=1}^{n} K_i$ *is not compact and* (3) $N(X - \bigcup_{i=1}^{n} K_i)$ *is compact.*

THEOREM 2.23. $q(L_n \cup J) \ll q(L_n \cup J^*)$, $n \in J$.

THEOREM 2.24. $qX = q(L_n \cup J^*)$ $(n \in J)$ *if and only if* (1) X *has exactly* n *components which are half-open intervals, say* $K_1, \cdots, K_n$, (2) *no component of* X *is an open interval and* (3) $N(X - \bigcup_{i=1}^{n} K_i)$ *is not compact.*

THEOREM 2.25. $q(L_n \cup J^*) \ll qL_{n+1}$, $n \in J$.

THEOREM 2.26. $qL_n \leqq qX < qL_{n+1}(n \in J)$ *if and only if* X *has exactly* n *components which are half-open interval and none which is an open interval.*

Proof. Assume $qL_n \leqq qX < qL_{n+1}$. Then, by Theorems 2.21, 2.23, and 2.25, we have $qX = qL_n$, $qX = q(L_n \cup J)$ or $qX = q(L_n \cup J^*)$. So, by Theorem 2.20, 2.22 and 2.24, we have X has exactly n components which are half-open intervals and none which is an open interval.

Now assume that X has exactly n components which are half-open intervals and none which is an open interval. By Theorems 2.20, 2.22 and 2.24 we have that $qX = qL_n$, $qX = q(L_n \cup J)$ or $qX = q(L_n \cup J^*)$. Hence, by Theorems 2.21, 2.23 and 2.25 we have $qL_n \leqq qX < qL_{n+1}$.

THEOREM 2.27. $qL^* \ll qL_\infty$.

Proof. Any covering α of L^* can be refined by a covering $\beta = \{B_j \mid j = 0, 1, 2, \cdots\}$ such that:

(1) $B_0 = [0, 1/(2m+1)) \cap L^*$ for some $m \in J$,

(2) $B_0 \cap B_j = \varnothing$ for $B_j \in \beta$ and $j \neq 0$.

Since $B = B_0$ is open and closed in L^* so is $Y = \bigcup_{j=1}^{\infty} B_j$. Moreover, Y and B are disjoint, $L^* = Y \cup B$ and

$$Y = \bigcup \{[1/(i+1), 1/i) \mid i = 1, 3, \cdots, 2m-1\} .$$

We now define a closed β-mapping, f_β, of L^* into $L_{\cdot m}$ (a closed subset of L_∞). Let $f_\beta[B] = -4m$ and $(f_\beta \mid [1/(i+i), 1/i))$ be a homeomorphism of $[1/(i+1), 1/i)$ onto $(-2i - 1/2, -2i]$, for $i = 1, 3, \cdots, 2m-1$. Since $\beta > \alpha$ we have that f_β is a closed α-mapping of L^* into L_∞. Therefore, we may conclude that $qL^* \leqq qL_\infty$.

Now suppose that $qL_\infty \leqq qL^*$. Then, for any covering α of L_∞, there is a closed α-mapping, f_α of L_∞ into L^*. We can write each component of L_∞ as the union of at least two distinct subsets which are open in L_∞. Let α be any covering of L_∞ of this type. Now Lemma 2.8 implies that f_α takes distinct components of L_∞ into distinct components of L^*, for α sufficiently fine. So $f_\alpha[L_\infty]$ must contain a sequence of distinct components of L^*. Since 0 is a limit point of any such sequence and $f_\alpha[L_\infty]$ is closed, it follows that $0 \in f_\alpha[L_\infty]$. Therefore, some component of L_∞ must be mapped into 0. This contradicts the fact that f_α is an α-mapping. Thus we may conclude that $qL^* < qL_\infty$.

Now suppose there is a set X such that $qL^* < qX < qL_\infty$. Then X has denumerably many components, $H_1, H_2, \cdots$, which are half-open intervals and no components which are open intervals, from Lemmas 2.3 and 2.8. moreover, every sequence of distinct H_i's contains a convergent subsequence (see [13, Theorem 7.1, p. 11]) which converges to some point of X (otherwise $qL_\infty \leqq qX$). Furthermore, $S = \limsup \{H_i \mid i \in J\}$ is a totally disconnected subset of X and we will show that S is compact. Suppose S is not compact. Then S has a closed subset Z which is homeomorphic to J. For each point p_j of Z, we have that infinitely many H_i's are in each neighborhood of p_j. Let $N_1, N_2, \cdots$ be a collection of disjoint open intervals such that $\delta(N_j) < 1/j$ and $p_j \in N_j$, $j \in J$. Choose an H_i in each N_j and let H be the union of these H_i. Then H is a closed subset of X which is homeomorphic to L_∞. So $qL_\infty \leqq qH \leqq qX$ which is a contradiction. Therefore, S is compact.

Let γ be any covering of X. We proceed as in the proof of Theorem 2.10, except for taking $\varDelta$ to be a denumerable dense subset of $R^1 - X$ which contains the excluded end points of the half-open intervals which are components of X. With this modification we obtain as in the proof of Theorem 2.10:

(1) a covering β of X which refines γ, and

(2) a collection $V = \{K_i\}$ covering X whose elements are open, closed, disjoint and $K_i = (a_i, b_i) \cap X$ where $a_i, b_i \in \varDelta$.

Since S is compact and the elements of V are disjoint, finitely many elements of V meet S, say $K_1, \cdots, K_r$. Let $K = \bigcup_{i=1}^r K_i$ and $Y = X - K$. Now K contains all but a finite number of the components of X which are half-open intervals. Therefore, Y contains $m(<\infty)$ components of X which are half-open intervals. Applying Theorem 2.26 we get that $qY \leqq qL_{m+1}$ and, since L_{m+1} is homeomorphic to $\varDelta_{m+1} = \bigcup_{i=1}^{m+1} \{[1/2i, 1/2i-1)\}$, $qY \leqq q\,\varDelta_{m+1}$.

Now we will show that $qK \leqq q(L^* - \varDelta_{m+1})$. For $i = 1, \cdots, r$, each $K_i = (a_i, b_i) \cap X$, $a_i, b_i \in \varDelta$, contains some $C_{j(i)}$, a half-open interval which is a component of X. Let $u_i = 1/(2m + 2i + 2)$ and $v_i = 1/(2m + 2i + 1)$ for $i = 1, \cdots, r$. Then we define a closed β-mapping f_β of K_i onto $[u_i, v_i)$ as follows: $(f_\beta \mid C_{j(i)})$ is a homeomorphism of

$C_{j(i)}$ onto $[u_i, v_i)$ and $f_\beta[K_i - C_{j(i)}] = u_i$, for $i = 1, \cdots, r$. The proof that f_β is a closed β-mapping is analogous to the proof given for the same purpose in Theorem 2.10. But $\beta > \gamma$ so f_β is also a γ-mapping. Therefore, $qK \leqq q(L^* - \Lambda_{m+1})$ and this together with the previously obtained relation, $qY \leqq q\Lambda_{m+1}$, imply that $qX \leqq qL^*$ (by Theorem 1.2). This is a contradiction of our original assumption that $qL^* < qX < qL_\infty$, so we may conclude that $qL^* \ll qL_\infty$.

THEOREM 2.28. *$qX = qL^*$ if and only if* (1) *no component of X is an open interval,* (2) *precisely denumerable many components of X are half-open intervals, say $H_1, H_2, \cdots$ and* (3) *every sequence of distinct H_i's contains a subsequence which converges to some point of X.*

Proof. Assume $qX = qL^*$. Then, by Lemma 2.3, X does not have a component which is an open interval. By Lemma 2.8 (used twice), X has precisely denumerably many components which are half-open intervals. Now suppose some sequence of distinct H_i's does not contain a subsequence which converges to a point of X. Then X contains a closed subset homeomorphic to L_∞. Therefore, $qL_\infty \leqq qX = qL^*$ which contradicts Theorem 2.27. Hence every sequence of distinct H_i's contains a subsequence which converges to a point of X.

Now assume that conditions (1), (2) and (3) hold. Let $Y = H \cup \{p\}$ where H is the union of a sequence of distinct H_i's which converges to p. Then Y is homeomorphic to L^*. Hence $qL^* \leqq qX$.

Now let γ be any covering of X. Since $S = \limsup \{H_i \mid i \in J\}$ is compact (by (3)), totally disconnected subset of X we can proceed as in the latter part of the proof of Theorem 2.27 to obtain the relation $qX \leqq qL^*$. This together with the previously obtained relation $qL^* \leqq qX$ imply that $qX = qL^*$.

THEOREM 2.29. *$qX = qL_\infty$ if and only if* (1) *no component of X is an open interval,* (2) *precisely denumerably many components of X are half-open intervals, say $H_1, H_2, \cdots$, and* (3) *some sequence of distinct H_i's does not contain a subsequence which converges to a point of X.*

Proof. Assume $qX = qL_\infty$. Then conditions (1) and (2) hold just as in the proof of Theorem 2.28. If condition (3) is not true, then by Theorem 2.28 and we have $qX = qL^*$. But this contradicts Theorem 2.27 and so condition (3) must hold.

Now assume that conditions (1), (2) and (3) hold. Condition (3) implies that X contains a closed subset homeomorphic to L_∞. Therefore,

$qL_\infty \leqq qX$. Let γ be any covering of X. We proceed as in the proof of Theorem 2.10, except for taking $\varDelta$ to be a denumerable dense subset of $R^1 - X$ which contains the excluded end points of the half-open intervals which are components of X. With this modification we obtain as in the proof of Theorem 2.10: (1) a covering β of X which refines γ, (2) a collection $V = \{K_i\}$ covering X whose elements are open, closed, disjoint and $K_i = (a_i, b_i) \cap X$ where $a_i, b_i \in \varDelta$.

We index the elements of V as follows: even positive integers are used to index the totally disconnected elements of V and odd positive integers are used to index the elements of V which are not totally disconnected. We define a closed β-mapping, f_β, of X into L_∞ as follows:

(1) if i is even, $f_\beta[K_i] = 2i$,

(2) if i is odd and K_i contains a half-open interval C_j, say $C_j = [c_j, d_j)$ which is a component of X,

$$(f_\beta \mid C_j) \text{ is a homeomorphism of } C_j \text{ onto } (-2i - 1/2, -2i]$$

and $f_\beta[K_i - C_j] = -2i$,

(3) if i is odd and K_i contains no such half-open interval, then K_i contains a closed interval $C_j = [c_j, d_j]$, which is a component of X and we define $(f_\beta \mid K_i)$ as an order preserving homeomorphism of C_j onto

$$[-2i - 1/4, -2i];\ f_\beta[(a_i, c_j) \cap X] = -2i - 1/4,\ f_\beta[d_j, b_i) \cap X] = -2i\ .$$

The proof that f_β is a closed β-mapping is analogous to the proof given for the same purpose in Theorem 2.10. But $\beta > \gamma$ so f_β is also γ-mapping of X into L_∞. Therefore, $qX \leqq qL_\infty$ and so $qX = qL_\infty$.

THEOREM 2.30. *If X does not have a component which is an open interval, then $qX \leqq qL_\infty$.*

Proof. If X has exactly $n(<\infty)$ components which are half-open intervals, then by Theorem 2.26 we have $qX \leqq qL_{n+1}$. If X has precisely denumerably many components which are half-open intervals, then either Theorem 2.28 or Theorem 2.29 applies. Therefore, we have $qX \leqq qL^*$ or $qX \leqq qL_\infty$. But from any one of these three inequalities involving qX it follows that $qX \leqq qL_\infty$.

DEFINITION 2.2. *Then n-chains $\mathscr{S}_n$.* We call the following ordering of quasi dimension types the 0-chain $\mathscr{S}_0$ (since there are no open intervals which are components of any of the sets in it). The sets inside the brackets are not comparable. (We call the spaces occuring here the *standard* 0-*spaces*.)

$$\mathscr{S}_0\colon\ qJ_0 \ll qJ_1 \ll qJ_2 \ll \cdots \ll qJ_i \ll qJ_{i+1} \ll \cdots < qC \ll \begin{Bmatrix} qI \\ qJ \end{Bmatrix}$$
$$\ll q(I \cup J) \ll qJ^* \ll qL_1 \ll q(L_1 \cup J) \ll q(L_1 \cup J^*) \ll qL_2$$
$$\ll q(L_2 \cup J) \ll q(L_2 \cup J^*) \ll qL_3 \ll \cdots \ll qL_i \ll q(L_i \cup J)$$
$$\ll q(L_i \cup J^*) \ll qL_{i+1} \ll \cdots < qL^* \ll qL_\infty .$$

The n-chain $\mathscr{S}_n$, for $n = 1, 2, 3, \cdots$, is obtained from $\mathscr{S}_0$ by replacing each standard 0-space by its union with M_n. (We call these the *standard n-spaces.*) So we have the following chain for $\mathscr{S}_n$.

$$\mathscr{S}_n\colon\ qM_n \ll q(M_n \cup J_1) \ll q(M_n \cup J_2) \ll \cdots \ll q(M_n \cup J_i)$$
$$\ll q(M_n \cup J_{i+1}) \ll \cdots < q(M_n \cup C) \ll \begin{Bmatrix} q(M_n \cup I) \\ q(M_n \cup J) \end{Bmatrix}$$
$$\ll q(M_n \cup I \cup J) \ll q(M_n \cup J^*) \ll q(M_n \cup L_1)$$
$$\ll q(M_n \cup L_1 \cup J) \ll q(M_n \cup L_1 \cup J^*) \ll q(M_n \cup L_2)$$
$$\ll q(M_n \cup L_2 \cup J) \ll q(M_n \cup L_2 \cup J^*) \ll q(M_n \cup L_3)$$
$$\ll \cdots \ll q(M_n \cup L_i) \ll q(M_n \cup L_i \cup J) \ll q(M_n \cup L_i \cup J^*)$$
$$\ll q(M_n \cup L_{i+1}) \ll \cdots < q(M_n \cup L^*) \ll q(M_n \cup L_\infty) .$$

Finally we define $\mathscr{S}_\infty : qM^* \ll qM_\infty$. (We call M^*, M_∞ and the standard n-spaces, $n = 0, 1, 2, \cdots$, the *standard spaces.*)

REMARK. Note that it follows from the previous theorems that every linear set with no components which are open intervals has the same quasi dimension type as some standard 0-space.

THEOREM 2.3. *The n-chains, for $n = 0, 1, 2, \cdots$, have the ordering indicated in Definition 2.2 (where $q(M_n \cup C)$ and $q(M_n \cup L^*)$ are each the least upper bounds of their predecessors).*

Proof. The previous results of § 2 are proof of the case $n = 0$. So we may consider $n > 0$. Since M_n is open and closed in its union with any standard 0-space we can use Theorem 1.2 to get: if X, Y are standard 0-spaces and $qX < qY$, then $q(M_n \cup X) \leqq q(M_n \cup Y)$. We wish to show that this last inequality is, in fact, a strict inequality. Suppose not, i.e., $q(M_n \cup Y) \leqq q(M_n \cup X)$. Then, for each covering α of $M_n \cup Y$, there is a closed α-mapping, f_α, of $M_n \cup Y$ into $M_n \cup X$. By the proof of Lemma 2.3 we have $f_\alpha[M_n] \supset M_n$, for α sufficiently fine. Therefore, $f_\alpha[Y] \subset X - M_n$, for α sufficiently fine, from which it follows that $qY \leqq q(X - M_n)$. Since $X - M_n$ is a closed subset of X we may conclude that $qY \leqq qX$. This, however, is a contradiction and so we have $q(M_n \cup X) < q(M_n \cup Y)$. A similar argument shows that: if X, Y are standard 0-spaces which are not comparable, then

$M_n \cup X$ and $M_n \cup Y$ are not comparable.

Now we wish to show the following: if X, Y are standard 0-spaces and $qX \ll qY$, then $q(M_n \cup X) \ll q(M_n \cup Y)$. By the previous paragraph we need only show that there is no set Z such that $q(M_n \cup X) < qZ < q(M_n \cup Y)$. Suppose not i.e., there is such a set Z. Then, by Lemma 2.3, Z contains exactly n components $W_1, \cdots, W_n$ which are open intervals. Hence $Z = \bigcup_{i=1}^{n} W_i \cup Z'$ where $qZ' = qW$ for some standard 0-space T and so $qZ = q(M_n \cup T)$. Now since $qX \neq qT \neq qY$ we have $qT < qX$, $qY < qT$ or T is not comparable to X or Y (but not both by the construction of $\mathscr{S}_0$). All there alternatives clearly lead to contradictions. Hence the n-chain $\mathscr{S}_n$ has the ordering indication in Definition 2.2.

THEOREM 2.32. *$\mathscr{S}_\infty$ has the ordering indicated in Definition* 2.2 (i.e., $qM^* \ll qM_\infty$).

Proof. Any covering α of M^* can be refined by a covering $\beta = \{B_j \mid j = 0, 1, 2, \cdots\}$ such that:

(2) $B_0 = [0, 1/(2m+1)) \cap M^*$ for some $m \in J$,

(2) $B_0 \cap B_j = \varnothing$ for $B_j \in \beta$ and $j \neq 0$.

Since $B = B_0$ is open and closed in M^* so is $Y = \bigcup_{j=1}^{\infty} B_j$. Moreover, Y and B are disjoint, $M^* = Y \cup B$ and

$$Y = \bigcup \{(1/(i+1), 1/i) \mid i = 1, 3, \cdots, 2m-1\} .$$

We now define a closed β-mapping, f_β, of M^* into M_{2m} (a closed subset of M_∞). Let $f_\beta[B] = -4m + 3/4$ and $(f_\beta \mid (1/(i+1), 1/i))$ be a homeomorphism of $(1/(i+1, 1/i)$ onto $(-2i + 1/2, -2i + 1)$, for $i = 1, 3, \cdots, 2m-1$. Since $\beta > \alpha$ we have that f_β is a closed α-mapping of M^* into M_∞. Therefore, we may conclude that $qM^* \leqq qM_\infty$.

Now suppose that $qM_\infty \leqq qM^*$. Then, for any covering α of M_∞, there is a closed α-mapping, f_α, of M_∞ into M^*. Each component of M_∞ con be written as the union of at least two distinct subsets which are open in M_∞. Let α be any covering of M_∞ of this type. Now f_α takes distinct components of M_∞ into distinct components of M^*, for α sufficiently fine (this follows from the proof of Lemma 2.3). So $f_\alpha[M_\infty]$ must contain a sequence of distinct components of M^*. Since 0 is a limit point of any such sequence and $f_\alpha[M_\infty]$ is closed, it follows that $0 \in f_\alpha[M_\infty]$. Therefore, some component of M_∞ must be mapped into 0. This contradicts the fact that f_α is an α-mapping. Thus we may conclude that $qM^* < qM_\infty$.

Now suppose there is a set X such that $qM^* < qX < qM_\infty$. Then X has denumerably many components, $W_1, W_2, \cdots$, which are open intervals. Moreover, every sequence of distinct W_i's contains a convergent subsequence which converges to some point of X (otherwise $qM_\infty \leqq qX$). Furthermore, $S = \limsup \{W_i \mid i \in J\}$ is a totally disconnected subset of X which is compact (see the proof of Theorem 2.27).

Let γ be any covering of X. We proceed as in the proof of Theorem 2.10, except for taking Δ to be a denumerable dense subset of $R^1 - X$ which contains the excluded end points of the open or half-open intervals which are components of X. With this modification we obtain as in the proof of Theorem 2.10:

(1) a covering β of X which refines γ, and

(2) a collection $V = \{K_i\}$ covering X whose elements are open, closed, disjoint and $K_i = (a_i, b_i) \cap X$ where $a_i, b_i \in \Delta$.

Since S is compact and the elements of V are disjoint, finitely many elements of V meet S, say $K_1, \cdots, K_r$. Let $K = \bigcap_{i=1}^r K_i$ and $Y = X - K$. Now K contains all but a finite number of the components of X which are open intervals. So Y has only $m(<\infty)$ components which are open intervals. Applying Theorem 2.31 we get that $qY \leqq q(M_m \cup L_\infty)$ and since

$$q(M_m \cup L_\infty) \leqq q(M^* - \Omega_r)$$

(where $\Omega_r = \bigcup_{i=1}^r \{(1/2i, 1/2i-1)\}$) we have $qY \leqq q(M^* - \Omega_r)$.

Now we will show that $qK \leqq q\Omega_r$ by defining a closed β-mapping, f_β, of K into Ω_r as follows:

(1) if K_i contains a half-open interval C_j, say $C_j = [c_j, d_j)$, which is a component of X, then $(f_\beta \mid K_i)$ is a homeomorphism of C_j onto $[2/(4i-1), 1/(2i-1))$ and $f_\beta[K_i - C_j] = 2/(4i-1)$,

(2) if K_i contains no such half-open interval, then K_i contains a closed interval $C_j = [c_j, d_j]$ which is a component of X and we define f_β to be an order preserving homeomorphism of C_j onto

$$[2/(4i-1), 4/(8i-1)];\ f_\beta[X \cap (a_i, c_j)] = 2/(4i-1),\ f_\beta[X \cap (d_j, b_i)] = 4/(8i-1)\ .$$

The proof that f_β is a closed β-mapping is analogous to the proof given for that purpose in Theorem 2.10. But $\beta > \gamma$ so f_β is also a γ-mapping of K into Ω_r and, therefore, $qK \leqq q\Omega_r$. This together with the relation $qY \leqq q(M^* - \Omega_r)$ imply $qX \leqq qM^*$. However, this is a contradiction so we may conclude that $qM^* \ll qM_\infty$.

THEOREM 2.33. *$qX = qM^*$ if and only if (1) X has precisely denumerably many components which are open intervals say W_1, $W_2, \cdots$ and (2) every sequence of distinct W_i's contains a subsequence which converges to some point of X.*

Proof. First assume that $qX = qM^*$. Then Lemma 2.3 implies that X has precisely denumerably many components which are open intervals, say $W_1, W_2, \cdots$. Now suppose some sequence of distinct W_i's does not contain a subsequence which converges to a point of X. Then X contains a closed subset homeomorphic to M_∞. Therefore, $qM_\infty \leqq qX = qM^*$ which contradicts the previous theorem. Hence (2) holds.

Now assume (1) and (2) hold. Let $Y = W \cup \{p\}$ where W is the union of a sequence of distinct W_i's which converges to $p \in X$. Then Y is homeomorphic to M^*. Hence $qM^* \leqq qX$.

Now let γ be any covering of X. Since $S = \limsup \{W_i \mid i \in J\}$ is compact (by (2)), totally disconnected subset of X we can proceed as in the latter part of the proof of Theorem 2.32 to obtain the relation, $qX \leqq qM^*$. This together with the previously obtained relation, $qM^* \leqq qX$ imply $qX = qM^*$.

THEOREM 2.34. *$qX = qM_\infty$ if and only if* (1) *X has precisely denumerably many components which are open intervals, say W_1, $W_2, \cdots$ and* (2) *some sequence of distinct W_i's does not contain a subsequence which converges to a point of X.*

Proof. Assume $qX = qM_\infty$. The Lemma 2.3 implies that X has precisely denumerably many components which are open intervals, say $W_1, W_2, \cdots$. Now suppose condition (2) does not hold. Then by the previous theorem we have $qX = qM^*$. But this contradicts Theorem 2.32 and so condition (2) must hold.

Now assume that conditions (1) and (2) hold. Condition (2) implies that X contains a closed subset homeomorphic to M_∞. Therefore, $qM_\infty \leqq qX$.

Now let γ be any covering of X. We proceed as in the proof of Theorem 2.10, except for taking Δ to be a denumerable dense subset of $R^1 - X$ which contains the excluded end points of the half-open or open intervals which are components of X. With this modification we obtain as in the proof of Theorem 2.10:

(1) a covering β which refines γ,

(2) a collection $V = \{K_i\}$ covering X whose elements are open, closed, disjoint and $K_i = (a_i, b_i) \cap X$ where $a_i, b_i \in \Delta$.

We index the elements V as follows even positive integers are used to index the totally disconnected elements of V and odd positive integers are used to index the elements of V which are not totally disconnected. We now define a closed β-mapping, f_β, of X into M_∞ as follows:

(1) if i is even, $f_\beta[K_i] = -2i + 3/4$,

(2) if i is odd and K_i contains a half-open interval C_j (say

$C_j = [c_j, d_j))$ which is a component of X, $(f_\beta | C_j)$ is a homeomorphism onto $[-2i + 3/4, -2i + 1)$ and $f_\beta[K_i - C_j] = -2i + 3/4$,

(3) if i is odd and K_i contains a closed interval $C_j = [c_j, d_j]$ which is a component of X, but no half-open interval which is a component of X, then we define $(f_\beta | K_i)$ as an order preserving homeomorphism of C_j onto $[-2i + 3/8, -2i + 5/8]$; $f_\beta[(a_i, c_j) \cap X] = -2i + 3/8$, $f_\beta[(d_j, b_i) \cap X] = -2i + 5/8$,

(4) is i is odd and K_i contains an open interval $C_j = (c_j, d_j)$ which is a component of X, but no half-open or closed interval which is a component of X, then we define $(f_\beta | K_i)$ as an order preserving homeomorphism of C_j onto $(-2i + 1/2, -2i + 1)$; if $c_j \neq a_i$, then $f_\beta[(a_i, c_j) \cap X]$ is a point of $f_\beta[B_1 \cap C_j]$ where B_1 is a link of β meeting C_j with end point a_i; if $d_j \neq b_i$, then $f_\beta[(d_j, b_i) \cap X]$ is a point of $f_\beta[B_2 \cap C_j]$ where B_2 is a link of β meeting C_j with end point b_i.

The proof that f_β is a closed β-mapping is analogous to the proof given for the same purpose in Theorem 2.10. Since $\beta > \gamma$ we have that f_β is also a γ-mapping of X into M_∞. Therefore, $qX \leqq qM_\infty$ and so $qX = qM_\infty$.

We next prove a theorem which determines all the order relationships between the n-chains. Together with the above theorem this determines completely the partial ordering of the quasi dimension types of subsets of R^1.

LEMMA 2.9. *If $qY \leqq qX$ and X has exactly n components which are open intervals, then the number of components of Y which are half-open intervals is at most $2n$ more than the corresponding number for X. (This generalizes Lemma 2.8.)*

Proof. First note that $qL_3 \not\leqq qM_1$ since, for any sufficiently fine cover α of L_3, the images of the components of L_3 under a closed α-map would be mutually disjoint (each being a subinterval of M_1 of the form $(-3/2, y]$ or $[y, -1)$). Continuing in this manner we can show that $qL_{2n+1} \not\leqq qM_n$ for any $n \in J$. Let i be the number of components of Y which are half-open intervals and let k be the corresponding number for X.

Suppose the lemma is false, i.e., $i \geqq k + 2n + 1$. Now for any covering α of Y there is a closed α-map $f_\alpha: Y \to X$. Then at most $2n$ of the components of Y which are half-open intervals map under f_α into W (the union of the components of X which are open intervals) for α sufficiently fine. So the other components of Y which are half-open intervals (of which there is at least $k+1$) map under f_α into $X - W$ for α sufficiently fine. It follows that $qL_{k+1} \leqq q(X - W) \leqq q(L_k \cup J^*)$ which is a contradiction of Theorem 2.25.

LEMMA 2.10. *If* (1) *X and M_n are separated,* (2) *Y and M_n* and *separated and* (3) *$q(M_n \cup Y) \leqq q(M_n \cup X)$, then $qY \leqq qX$.*

Proof. By (3) there is a closed α-map $f_\alpha: (M_n \cup Y) \to (M_n \cup X)$ for every covering α of $M_n \cup Y$. As in the proof of Lemma 2.3 we have $f_\alpha[M_n] \supset M_n$, for α sufficiently fine. Therefore, $f_\alpha[Y] \subset X$, for α sufficiently fine. Since X is a closed subset of $M_n \cup X$ and Y is a closed subset of $M_n \cup Y$, it follows that $qY \leqq qX$.

LEMMA 2.11. *The spaces L_∞ and M^* are not comparable.*

Proof. By the proof of Lemma 2.3 we have that $qM^* \nleqq qL_\infty$. Suppose that $qL_\infty \leqq qM^*$. Then, for any covering α of L_∞, there is a closed α-mapping, f_α, of L_∞ into M^*. Now, since $qL_3 \nleqq qM_1$ (Lemma 2.9), at most two of the components of L_∞ go into a single component of M^* under f_α, for α sufficiently fine. Therefore, $f_\alpha[L_\infty]$ contains denumerably many components of M^* and so 0 is a limit point of $f_\alpha[L_\infty]$. Since $f_\alpha[L_\infty]$ is closed (in M^*) we have $0 \in f_\alpha[L_\infty]$. This implies that a component of L_∞ is mapped into 0 which is a contraction of the fact that f_α is an α-mapping (for α sufficiently fine). Therefore, we have $qL_\infty \nleqq qM^*$ and so L_∞ and M^* are not comparable.

THEOREM 2.35. *Every linear set has the quasi dimension type of some* (*unique*) *standard space.*

Proof. Suppose we are given a linear set X and exactly $n(<\infty)$ of its components are open intervals $W_1, \cdots, W_n$ and $(<\infty)$ are half-open intervals $H_1, \cdots, H_m$. Let $W = \bigcup_{i=1}^n W_i$ and $H = \bigcup_{i=1}^m H_i$ where if the upper limit is zero the union is understood to be the empty set. If $m \geqq 1$, then $Z = X - W$ is such that $qZ = qY$ where

$$Y = \begin{Bmatrix} L_m \cup J^*, \text{ if } \mathrm{Cl}\,(Z-H) \text{ and } N(Z-H) \text{ are not compact} \\ L_m \cup J, \text{ if } \mathrm{Cl}\,(Z-H) \text{ is not compact and } N(Z-H) \text{ is compact} \\ L_m, \text{ if } \mathrm{Cl}\,(Z-H) \text{ is compact} \end{Bmatrix}.$$

If $m = 0$, then $qZ = qY$ where

$$Y = \begin{Bmatrix} J^*, \text{ if } Z \text{ and } N(Z) \text{ are not compact} \\ I \cup J, \text{ if } Z \text{ is noncompact, not totally disconnected, and } N(Z) \text{ is compact} \\ J, \text{ if } Z \text{ is noncompact and totally disconnected} \\ I, \text{ if } Z \text{ is compact and not totally disconnected} \\ C, \text{ if } Z \text{ is compact, totally disconnected and infinite} \\ J_i, \text{ if } Z \text{ is finite and card } Z = i \end{Bmatrix}.$$

So $qX = q(M_n \cup Y)$ where Y is given above.

Now we consider the cases when m or n is infinite. If $n < \infty$ and $m = \infty$ (i.e., X has infinitely many components, $H_1, H_2, \cdots$, which are half-open intervals), then $qX = q(M_n \cup Y)$ where

$$Y = \begin{cases} L^*, & \text{if every sequence of distinct } H_i\text{'s contains a} \\ & \text{subsequence which converges to some point of } X \\ L_\infty, & \text{if some sequence of distinct } H_i\text{'s does not contain} \\ & \text{a subsequence which converges to a point of } X \end{cases} .$$

If $n = \infty$ (i.e., X has infinitely many components, $W_1, W_2, \cdots$, which are open intervals), then

$$qX = \begin{cases} qM^*, & \text{if every sequence of distinct } W_i\text{'s contains a} \\ & \text{subsequence which converges to some point of } X \\ qM_\infty, & \text{if some sequence of distinct } W_i\text{'s does not contain} \\ & \text{a subsequence which converges to a point of } X \end{cases} .$$

REMARK. It follows from Theorem 2.35 that for any linear set X, $qX \leqq qM_\infty$.

THEOREM 2.36. *The only order relations between sets in different* $\mathscr{S}_n$*'s are those implied by the following. For* $i = 2, 3, 4, \cdots$ *and* $j = [i/2]$ (*the largest integer* $\leqq i/2$), *we have*

(I) $qL_i \ll q(M_1 \cup L_{i-2}) \ll q(M_2 \cup L_{i-4}) \ll \cdots \ll q(M_j \cup L_{i-2j})$,

(II) $q(L_i \cup J) \ll q(M_1 \cup L_{i-2} \cup J) \ll q(M_2 \cup L_{i-4} \cup J)$
$\ll \cdots \ll q(M_j \cup L_{i-2j} \cup J)$,

(III) $q(L_i \cup J^*) \ll q(M_1 \cup L_{i-2} \cup J^*) \ll q(M_2 \cup L_{i-4} \cup J^*)$
$\ll \cdots \ll q(M_j \cup L_{i-2j} \cup J^*)$,

(IV) $qL^* \ll q(M_1 \cup L^*) \ll q(M_2 \cup L^*) \ll \cdots \ll q(M_n \cup L^*)$
$\ll q(M_{n+1} \cup L^*) \ll \cdots < qM^*$,

(V) $qL_\infty \ll q(M_1 \cup L_\infty) \ll q(M_2 \cup L_\infty) \ll \cdots \ll q(M_n \cup L_\infty)$
$\ll q(M_{n+1} \cup L_\infty) \ll \cdots < qM_\infty$.

Proof. In each case we have at least "$\leqq$". Applying Lemma 2.3 we have "$<$" in each case. Consider the first inequality in (I). Suppose there is a set X such that $qL_i < qX < q(M_1 \cup L_{i-2})$. Then, by Lemma 2.3, X has at most one component which is an open interval and so $qX \in \mathscr{S}_1$ or $qX \in \mathscr{S}_0$. The first alternative and Theorem 2.31 imply that $qX \leqq q(M_1 \cup L_{i-3} \cup J^*)$. By assumption, $qL_i < qX$ so we have $qL_i \leqq q(M_1 \cup L_{i-3} \cup J^*)$. But this contradicts Lemma 2.9. The second alternative and Theorem 2.31 imply that $q(L_i \cup J^*) \leqq qX$. By assumption $qX < q(M_1 \cup L_{i-2})$ so we have $q(L_i \cup J^*) \leqq q(M_1 \cup L_{i-2})$.

Then, for any cover α of $L_i \cup J^*$, there is a closed α-map

$$f_\alpha \colon (L_i \cup J^*) \to (M_1 \cup L_{i-2})$$

of $L_i \cup J^*$ into $M_1 \cup L_{i-2}$. Since $qL_{i-1} \not\leqq qL_{i-2}$, $f_\alpha^{-1}[M_1]$ contains at least two components of L_i for α sufficiently fine. Since $qL_3 \not\leqq qM_1$, $f_\alpha^{-1}[M_1]$ contains at most two components of L_i for α sufficiently fine. So, for α sufficiently fine, $f_\alpha^{-1}[M_1]$ contains exactly two components of L_i. It follows that $q(L_{i-2} \cup J^*) \leqq qL_{i-2}$ which contradicts Theorem 2.31. Hence $qL_i \ll q(M_1 \cup L_{i-2})$. The other inequalities of (I)—(III) can be obtained in a similar manner.

Consider the first inequality of (IV). Suppose there is a set X such that $qL^* < qX < q(M_1 \cup L^*)$. Now if X does not contain a component which is an open interval, then by Theorem 2.30 $qX \leqq qL_\infty$ and $qX = qL_\infty$. However, it follows from Lemma 2.11 that $qL_\infty \not\leqq q(M_1 \cup L^*)$ and so X contains at least one component which is an open interval. Lemma 2.3 implies X has exactly one such component. Since $qL^* \not\leqq q(M_1 \cup L_n)$ for any $n \in J$, we have that X must contain denumerably many components which are half-open intervals. Hence, $q(M_1 \cup L^*) \leqq qX$ which is a contradiction. Therefore, we may conclude that $qL^* \ll q(M_1 \cup L^*)$. The other inequalities of (IV), except the last one, can be obtained in a similar manner. We now treat the last inequality of (IV). Suppose there is a set X such that

$$q(M_n \cup L^*) < qX < qM^*$$

for all $n \in J$. Then X must have denumerably many components which are open intervals by Lemma 2.3. Therefore, either Theorem 2.33 or Theorem 2.34 applies and so $qM^* \leqq qX$. This is a contradiction and so the last inequality of (IV) is valid.

Now consider the first inequality of (V). Suppose there is a set X such that $qL_\infty < qX < q(M_1 \cup L_\infty)$. Theorem 2.30 and Lemma 2.3 imply that X has exactly one component, W, which is an open interval. Since $qL_3 \not\leqq qW$, it follows that $qL_\infty \leqq q(X - W)$. The Theorem 1.2 applies and we have $q(M_1 \cup L_\infty) \leqq qX$. This is a contradiction and so we may conclude that $qL_\infty \ll q(M_1 \cup L_\infty)$. The other inequalities of (V), except the last one, can be obtained in a similar manner.

We now treat the last inequality of (V). Suppose there is a set X such that $q(M_n \cup L_\infty) < qX < qM_\infty$ for all $n \in J$. Then X has denumerably many components which are open intervals. Therefore, $qX = qM^*$ or $qX = qM_\infty$. The latter is contrary to our assumption so we have $qX = qM^*$. However, it follows from Lemma 2.11 that $q(M_n \cup L_\infty) \not\leqq qM^*$, and so we have a contradiction. Thus we may conclude that the last inequality of (V) is valid.

We now wish to show that: if $qY < qX$, $qX \in \mathscr{S}_n$, and $qY \in \mathscr{S}_m$,

$0 \leqq m < n \leqq \infty$, then $qY < qX$ can be obtained from (I)–(V), together with the orderings of the individual $\mathscr{S}_i$'s. If $qY < q(M_m \cup L_1)$, then using (I) we get

$$qY < q(M_m \cup L_{2(n-m)}) \ll q(M_{m+1} \cup L_{2(n-m)-2}) \ll q(M_{m+2} \cup L_{2(n-m)-4}) \ll \cdots \ll qM_n \leqq qX .$$

Now assume that $q(M_m \cup L_1) \leqq qY$. Then, either

(1) $qX = q(M_n \cup L_k \cup X_1)$ where $n, k < \infty$; $X_1 = J_0$, J, or J^*,

(2) $qX = q(M_n \cup L^*)$ where $n < \infty$,

(3) $qX = q(M_n \cup L_\infty)$ where $n < \infty$,

(4) $qX = qM^*$,

or

(5) $qX = qM_\infty$;

and either

(6) $qY = q(M_m \cup L_i \cup Y_1)$ where $m < \infty$, $1 \leqq i < \infty$; $Y_1 = J_0$, J or J^*,

(7) $qY = q(M_m \cup L^*)$ where $m < \infty$,

(8) $qY = q(M_m \cup L_\infty)$ where $m < \infty$,

or

(9) $qY = qM^*$.

Since $qY < qX$ none of the following pairs can hold together:
(1)–(7), (1)–(8), (1)–(9),
(2)–(8), (2)–(9),
(3)–(9),
(4)–(8), (4)–(9).

The pair (5)–(9) does not apply since qM^* and qM_∞ are in the same chain: Let $r = i - 2(n - m)$ and let $j = [i/2]$. Then we proceed to consider the remaining cases.

Case 1. (1) and (6) hold. Then since $qY < qX$, it follows from Lemmas 2.9 and 2.10 that $r \leqq k$. We have the following five subcases.

Case 1.1. $Y_1 = J_0$. If $r < 0$, then $m + j < n$ so using (I) we have

$$qY = q(M_m \cup L_i) \ll q(M_{m+1} \cup L_{i-2}) \ll q(M_{m+2} \cup L_{i-4}) \ll \cdots \ll q(M_{m+j} \cup L_{i-2j}) \leqq q(M_{m+j} \cup L_1) \leqq qM_n \leqq qX .$$

On the other hand, if $r \geqq 0$, then using (I) we have

$$qY = q(M_m \cup L_i) \ll q(M_{m+1} \cup L_{i-}) \ll q(M_{m+2} \cup L_{i-4}) \ll \cdots \ll q(M_n \cup L_r) \leqq q(M_n \cup L_k) \leqq qX .$$

Case 1.2. $Y_1 = J$ and $X_1 = J_0$. If $r < 0$, then $m + j < n$ so using

(II) and the fact that $q(L_1 \cup J) < qM_1$ we have

$$qY = q(M_m \cup L_i \cup J) \ll \cdots \ll q(M_{m+j} \cup L_{i-2j} \cup J)$$
$$\leqq q(M_{m+j} \cup L_1 \cup J) \leqq qM_n \,.$$

On the otherhand, if $r > 0$, then $r + 1 \leqq k$. Otherwise, since $r \leqq k$ we would have $r = k$ and so $q(M_m \cup L_i \cup J) < q(M_n \cup L_r)$. This by Lemma 2.10 implies $q(L_i \cup J) \leqq q(M_{n-m} \cup L_r)$ from which it follows that $qL_{i+1} \leqq q(M_{n-m} \cup L_r)$. Applying Lemma 2.9 to the last relation we get $i + 1 \leqq r + 2(n - m)$, so that $i + 1 \leqq i$. This is impossible so we have $r + 1 \leqq k$. Then, using (II) we have

$$qY = q(M_m \cup L_i \cup J) \ll q(M_{m+1} \cup L_{i-2} \cup J)$$
$$\ll \cdots \ll q(M_n \cup L_r \cup J)$$
$$< q(M_n \cup L_{r+1}) \leqq q(M_n \cup L_k) = qX \,.$$

Case 1.3. $Y_1 = J$ and $X_1 = J$ or J^*. If $r < 0$ we can proceed just as in the first part of Case 1.2. If $r \geqq 0$, then using (II) and the fact that $r \leqq k$ we have

$$qY = q(M_m \cup L_i \cup J) \ll q(M_{m+1} \cup L_{i-2} \cup J)$$
$$\ll \cdots \ll q(M_n \cup L_r \cup J)$$
$$\leqq q(M_n \cup L_k \cup J) \leqq qX \,.$$

Case 1.4. $Y_1 = J^*$ and $X_1 = J_0$ or J. This case is analogous to Case 1.2 except that (III) is used instead of (II).

Case 1.5. $Y_1 = J^*$ and $X_1 = J^*$. If $r < 0$ we can proceed just as in the first part of Case 1.2. If $r \geqq 0$ then using (III) and the fact that $r \leqq k$ we have

$$qY = q(M_m \cup L_i \cup J^*) \ll q(M_{m+1} \cup L_{i-2} \cup J^*)$$
$$\ll \cdots \ll q(M_n \cup L_r \cup J^*)$$
$$\leqq q(M_n \cup L_k \cup J^*) = qX \,.$$

Therefore, if (1) and (6) hold, we have that $qY < qX$ is implied by (I)-(III).

Case 2. (2) and (6) hold. If $r < 0$, then we can proceed just as in the first part of Case 1.2 except that (III) is used instead of (II). If $r \geqq 0$ then using (III) we have

$$qY \leqq q(M_m \cup L_i \cup J^*) \ll q(M_{m+1} \cup L_{i-2} \cup J^*) \ll \cdots \ll q(M_n \cup L_r)$$
$$< q(M_n \cup L^*) = qX \,.$$

Case 3. (2) and (7) hold. Then, using (IV) we have

$$qY = q(M_m \cup L^*) \ll q(M_{m+1} \cup L^*) \ll \cdots \ll q(M_n \cup L^*) = qX .$$

Case 4. (3) and (6) hold. If $r < 0$, then we can proceed as in the first part of Case 1.2 except that (III) is used instead of (II). If $r \geqq 0$, then using (III) we have

$$qY \leqq q(M_m \cup L_i \cup J^*) \ll q(M_{m+1} \cup L_{i-2} \cup J^*) \ll \cdots \ll q(M_n \cup L_r \cup J^*) < q(M_n \cup L_\infty) = qX .$$

Case 5. (3) and (7) hold. Then, using (IV) we have

$$qY = q(M_m \cup L^*) \ll q(M_{m+1} \cup L^*) \ll \cdots \ll q(M_n \cup L^*) < q(M_n \cup L_\infty) = qX .$$

Case 6. (3) and (8) hold. Then, using (V) we have

$$qY = q(M_m \cup L_\infty) \ll q(M_{m+1} \cup L_\infty) \ll \cdots \ll q(M_n \cup L_\infty) = qX .$$

Case 7. (4) and (6) hold. Proceed as in Case 4 and use (IV).

Case 8. (4) and (7) hold. Then, using (IV) we have

$$qY = q(M_m \cup L^*) < qM^* = qX .$$

Case 9. (5) and (6) hold. Proceed as in Case 4 and use (V).

Case 10. (5) and (7) hold. Then, using (IV) we have

$$qY \leqq q(M_m \cup L^*) < qM^* < qM_\infty = qX .$$

Case 11. (5) and (8) hold. Then, using (V) we have

$$qY = q(M_m \cup L_\infty) < qM_\infty = qX .$$

Therefore, in all cases, $qY < qX$ is implied by (I)-(V).

REMARK. We can characterize the class of linear sets with quasi dimension type $q(M_n \cup Y)$, for $n = 0, 1, 2, \cdots$, where Y is a standard 0-space. For, if X is a member of this class exactly n of the components of X are open intervals. Let W denote the union of these n components. Then, since $q(X - W) = qY \in \mathscr{S}_0$ and we have already characterized such sets, we can obtain our desired topological characterization of linear sets of given quasi dimension type. Furthermore, if a linear set has quasi dimension type qM^* or qM_∞, then Theorem 2.33 and 2.34 give its topological characterization.

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