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ON H-EQUIVALENCE OF UNIFORMITIES (THE ISBELL-SMITH PROBLEM)

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I have recently given an example of two different uniformities for the same set X, such that the corresponding Hausdorff uniformities for the set of nonempty subsets of X are topologically equivalent; when this is the case we shall call the original uniformities H-equivalent. The problem posed by Isbell and discussed in a recent paper by D. H. Smith may therefore be reformulated as follows:- (a) Under what conditions are two uniformities H-equivalent? (b) Under what conditions does H-equivalence of uniformities imply identity? The theorems given below supply an answer to (a) and a partial answer to (b). In particular, they show that neither R^n nor Q^n (Q denoting the set of rational numbers with the usual metric) has any other uniformity H-equivalent to its metric uniformity. In a sense, therefore, the example in (1) is the simplest possible one of its kind, though we give in the course of this paper another simple example using transfinite ordinals.

TERMINOLOGY. Let $\mathfrak{U}, \mathfrak{V}$ be two uniformities for the same set X, and let $X_1 \subset X_2 \subset X$. We say that $\mathfrak{U}$ is *uniformly finer than* $\mathfrak{V}$ *on* X_1 *over* X_2 if and only if, given any $V \in \mathfrak{V}, \exists U \in \mathfrak{U}$ such that $U \cap (X_1 \times X_2) \subset V$; usually we take $X_2 = X$. (The use of the different words 'on' and 'over', and the, logically unnecessary, condition $X_1 \subset X_2$, are intended to suggest the motivation and use of the definition.) We say also (a) that $\mathfrak{U}$ is *proximity-finer than* $\mathfrak{V}$ if and only if every pair of sets A, B which are $\mathfrak{V}$-remote (i.e. such that $V(A) \cap B = \phi$ for some $V \in \mathfrak{V}$) are also $\mathfrak{U}$-remote; (b) that $\mathfrak{U}$ is *H-finer than* $\mathfrak{V}$ if and only if the topology of its Hausdorff uniformity $\boldsymbol{\mathfrak{U}}$ is finer than that of the Hausdorff uniformity $\boldsymbol{\mathfrak{V}}$ corresponding to $\mathfrak{V}$; i.e. if and only if given any (nonempty) $E_0 \subset X$ and any $V \in \mathfrak{V}, \exists U \in \mathfrak{U}$ (depending on E_0) such that $E \subset U(E_0)$ and $E_0 \subset U(E)$ together imply $E \subset V(E_0)$ and $E_0 \subset V(E)$[1]. The corresponding phrases with 'coarser than' or 'equivalent to' are defined similarly. Note that we use 'finer' in the wide sense, allowing possible equivalence; also that in discussing subsets of X we shall frequently omit the word 'nonempty' where it is obviously implied. Finally, we say (cf. (1)) that a set E is *V-discrete* ($V \in \mathfrak{V}$) if and only if, for x and y in E, $(x, y) \in V$ implies $x = y$, and $\mathfrak{V}$-*discrete*

[1] While this is the form in which the definition, derived from that of the Hausdorff uniformity, is naturally phrased, it is easily seen that the implications are actually respective rather than joint.

if and only if it is V-discrete for some $V \in \mathfrak{V}$.

We can now state

THEOREM 1. *If* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are two uniformities for the same set* X *then* $\mathfrak{U}$ *is H-finer than* $\mathfrak{V}$ *if and only if it is both* (a) *proximity-finer, and* (b) *on every* $\mathfrak{V}$*-discrete set, uniformly finer over* X. *If it is given that* $\mathfrak{U} \subset \mathfrak{V}$, *then* (b) *may be replaced by the weaker condition* (b′):- *every* $\mathfrak{V}$*-discrete set is also* $\mathfrak{U}$*-discrete.*

Proof. Necessity. Let $\mathfrak{U}$ be H-finer than $\mathfrak{V}$. The proof that $\mathfrak{U}$ is proximity-finer than $\mathfrak{V}$ is essentially contained in the proof of Theorem 1 of (3), and is omitted.

Now let E_0 be V_0-discrete, $V_0 \in \mathfrak{V}$. Given $V_1 \in \mathfrak{V}$, put $V_2 = V_1 \cap V_0$. We suppose for simplicity (w.l. of g.) that V_0 and V_1 are symmetric. There exists a (symmetric) $U \in \mathfrak{U}$ such that $E \subset U(E_0)$, $E_0 \subset U(E)$ imply that $E \subset V_2(E_0)$ and $E_0 \subset V_2(E)$. Consider in particular the set $\{y\} \cup (E_0 \backslash \{x_0\})$, where $x_0 \in E_0$ and $(x_0, y) \in U$. This set satisfies the conditions just stated, so that $E_0 \subset V_2(E)$; in particular $\exists y' \in E$, such that $(x_0, y') \in V_2 \subset V_0$. Since E_0 is V_0-discrete, y' can only be y. Thus, for $x \in E_0$ and $y \in X$, $(x, y) \in U$ implies $(x, y) \in V_1$. As V_1 is arbitrary this proves statement (b). (We remark that it follows that every $\mathfrak{V}$-discrete set is also $\mathfrak{U}$-discrete.)

Sufficiency. Suppose the conditions satisfied; let $E_0 \subset X$ and $V_0 \in \mathfrak{V}$ be given arbitrarily. Since $X \backslash V_0(E_0)$ is $\mathfrak{V}$-remote, hence also $\mathfrak{U}$-remote, from E_0, $\exists U_0 \in \mathfrak{U}$ such that $U_0(E_0) \subset V_0(E_0)$. Now take $V_1 \in \mathfrak{V}$, symmetric, such that $V_1^2 \subset V_0$, and let E_1 be a maximal V_1-discrete subset of E_0, so that $E_0 \subset V_1(E_1)$. By (b) we can take a symmetric $U_1 \in \mathfrak{U}$ such that $U_1 \cap (E_1 \times X) \subset V_1$, and also $U_1 \subset U_0$. If E is a set such that $E_0 \subset U_1(E)$, then in particular (as $E_1 \subset E_0$) for any $x \in E_1$, $\exists y \in E$ such that $(x, y) \in U_1$ and so also $(x, y) \in V_1$. That is, $E_1 \subset V_1(E)$ and so $E_0 \subset V_1^2(E) \subset V_0(E)$.

We have thus shown that $E_0 \subset U_1(E)$ implies $E_0 \subset V_0(E)$ and also, since $U_1 \subset U_0$, that $E \subset U_1(E_0)$ implies that $E \subset V_0(E_0)$. Since E_0 and V_0 were arbitrary, $\mathfrak{U}$ is H-finer than $\mathfrak{V}$.

In the case when $\mathfrak{U} \subset \mathfrak{V}$ and (a), (b′) are given, let E_0 be any $\mathfrak{V}$-discrete set. Then $\exists$ symmetric $U_0 \in \mathfrak{U}$ with E_0 U_0^2-discrete. Now let $V_1 \in \mathfrak{V}$ be given arbitrarily; as $\mathfrak{U} \subset \mathfrak{V}$, $\exists$ symmetric $V_2 \in \mathfrak{V}$, $V_2 \subset U_0 \cap V_1$. Since $\mathfrak{U}$ is proximity-finer than $\mathfrak{V}$ we see as before that $\exists U_1 \in \mathfrak{U}$ such that $U_1(E_0) \subset V_2(E_0)$: we may take $U_1 \subset U_0$ and symmetric. We say that $U_1 \subset (E_0 \times X) \subset V_2 \subset V_1$. For let $x_0 \in E_0$, $(x_0, x) \in U_1$; so $x \in U_1(E_0) \subset V_2(E_0)$. That is, $\exists x' \in E_0$, $(x', x) \in V_2$; then as U_1 and V_2 are both contained in U_0 we have $(x_0, x') \in U_0^2$ and by definition of U_0 this implies $x' = x_0$, $(x_0, x) \in V_2$. Thus we have proved that (b) is satisfied: the result follows.

We now turn to the second question raised in the introduction; we can give an answer only when X has a certain homogeneity of structure.

THEOREM 2. *Let* $(X, \mathfrak{V})$ *be a uniform space such that there exist a set* $K \subset X$, *compact in the topology induced by* $\mathfrak{V}$, *and a family of functions* $f_i, K \to X$ $(i \in I)$, *satisfying*:-

(i) $\bigcup f_i(K) = X$;

(ii) *the set* $E_x = \{f_i(x); i \in I\}$ *is* $\mathfrak{V}$*-discrete for every* $x \in K$;

(iii) *the functions* f_i, $i \in I$, *are equi-uniformly continuous: that is, given* $V \in \mathfrak{V}$, $\exists V_1 \in \mathfrak{V}$ *such that* $(x, x') \in V_1$ *implies* $[f_i(x), f_i(x')] \in V$, *for all* x, x' *in* K *and all* $i \in I$.

Then there is no other uniformity for X H*-equivalent to* $\mathfrak{V}$.

Proof. Let $\mathfrak{U}$ be H-equivalent to $\mathfrak{V}$. Given $U_0 \in \mathfrak{U}$, take a symmetric $U_1 \in \mathfrak{U}$ with $U_1^2 \subset U_0$. By Theorem 1, every $\mathfrak{V}$-discrete set is $\mathfrak{U}$ discrete, and conversely; thus condition (ii) and Theorem 1 imply that, for any $k \in K, \exists V_k \in \mathfrak{V}$ such that $V_k^2 \cap (E_k \times X) \subset U_1$. By condition (iii), there then exists $W_k \in \mathfrak{V}$ such that, for all k' in K, i in $I, (k, k') \in W_k$ implies $(f_i(k), f_i(k')) \in V_k$. The compact set K may be covered by a finite number of sets of the form $W_{k(r)}(k(r)), r = 1, 2, \cdots, n$ say; let $V \in \mathfrak{V}$ be the intersection of the corresponding $V_{k(r)}$. Then if $(x, x') \in V$ we can put $x = f_i(k)$, some $i \in I, k \in K$. For some $r, (k(r), k) \in W_{k(r)}$ and so $(f_i(k(r)), x) \in V_{k(r)}$; since $V \subset V_{k(r)}$ this gives $(f_i(kr)), x') \in V_{k(r)}^2$. Thus $(f_i(k(r)), x)$ and $(f_i(k(r)), x')$ are both in U_1, so as U_1 is symmetric we have $(x, x') \in U_1^2 \subset U_0$. We have thus proved that $V \subset U_0$; it follows that $\mathfrak{V} \supset \mathfrak{U}$.

Since $\mathfrak{U}, \mathfrak{V}$ (being H-equivalent) induce identical topologies, they must induce the same uniformity over the compact subspace K. As we have now shown that $\mathfrak{V} \supset \mathfrak{U}$, condition (iii) must be satisfied with $\mathfrak{V}$ replaced by $\mathfrak{U}$. A proof exactly similar to the above then shows that $\mathfrak{V} \subset \mathfrak{U}$, so that $\mathfrak{U}$ and $\mathfrak{V}$ are identical.

We now apply Theorem 2 to the case of a topological group.

THEOREM 3. *If* $(G, \mathscr{Y})$ *is a locally compact topological group there is no other uniformity for* G H*-equivalent to* $\mathfrak{L}$, *the left-invariant uniformity associated with* $\mathscr{Y}$.

We recall that $\mathfrak{L}$ is the uniformity with a base consisting of sets of the form $L_N = \{(x, x'); x^{-1}x' \in N\}$, where N is any $\mathscr{Y}$-neighbourhood of the identity e.

Let K be a symmetrical compact neighbourhood of e. Take a set

$Y \subset G$ which is maximal subject to the condition that for y, y' in Y and $y \neq y'$ we have $y^{-1}y'$ not in K; then $YK = G$, for given $g \in G$, $\exists y \in Y$ such that one (and hence both) of $y^{-1}g, g^{-1}y$ is in K. Write $f_y, y \in Y$, for the function such that $f_y(x) = yx, x \in K$. This family of functions clearly satisfies conditions (i) and (iii) of Theorem 2, with G for X, Y for I, and $\mathfrak{L}$ for $\mathfrak{V}$; we shall now prove that condition (ii) is satisfied; the result then follows at once.

The function $\phi(x, z) = xz^{-1}$ is continuous, hence $\mathfrak{L}$-uniformly continuous, on the set $KK \times K$, for KK is compact as it is a continuous map of $K \times K$. Thus there exists a (symmetrical) neighbourhood $N \subset K$ of e such that for z in K and g in N (so that $zg \in L_N(z) = zN \subset KK$) we have $zgz^{-1} \in L_K(zz^{-1}) = K$. We now say that for any $z \in K$ the set $\{yz; y \in Y\}$ is L_N-discrete. For if $(yz)^{-1}(y'z) = g \in N$ then $z^{-1}y^{-1}y'z = g$ or $y^{-1}y' = zgz^{-1} \in K$; for y, y' in Y this implies that $y = y'$.

COROLLARY. *If (G, d) is a locally compact metric group with the property that, given $\varepsilon > 0, \exists\delta > 0$ such that, for all x, y, z of G, $d(x, y) < \delta$ implies $d(zx, zy) > \varepsilon$, then there is no other uniformity for G H-equivalent to that defined by d.*

For the left-invariant uniformity coincides in this case with the metric uniformity defined by d. (In fact, it is known that there exists a left-invariant metric uniformly equivalent to d.)

In view of the close relation between a space and its completion, one might hope to be able to replace 'compact' by 'precompact Hausdorff' in Theorem 2. If certain additional conditions are imposed this can in fact be done (Theorems 4 and 5 below). It seems possible that somewhat weaker conditions might suffice, particularly in the case of a group, where the algebraic structure imposes homogeneity on the space. However, the immediate analogue of Theorem 2 is certainly not true in general, even if the set of functions $\{f_i\}$ is enumerable, as we shall now show by an example.

Let Z be the set of positive integers, $\mathfrak{D}$ its standard (metric) uniformity; sot hat Z is $\mathfrak{D}$-discrete. Let Ω_1 be the set of ordinals $<\omega_1$ and $\Omega_1^* = \Omega_1 \cup \{\omega_1\}$. With its natural order-topology, Ω_1^* is a compact Hausdorff space and hence has a unique natural uniformity $\mathfrak{W}^*$ say. Now write $X^* = Z \times \Omega_1^*$, $X = Z \times \Omega_1$, $\mathfrak{V}^*$ for the product-uniformity defined by $\mathfrak{D}$, $\mathfrak{W}^*$, and $\mathfrak{V}$ for the uniformity induced by $\mathfrak{V}^*$ on X (as a subset of X^*) by restriction.

Let $B \subset X$ be the set $\{(1, \alpha); \alpha < \omega_1\}$; B is clearly $\mathfrak{V}$-precompact as its $\mathfrak{V}^*$-closure is the compact set $\{1\} \times \Omega_1^*$. Define $f_n: B \to X$ by

$f_n(1, \alpha) = (n, \alpha)$; then the set of functions $\{f_n; n \in Z\}$ satisfies the conditions (i) to (iii) of Theorem 2 (with B for K).

Now let $\mathscr{C}$ be any finite partition of Z, and α_0 any ordinal $<\omega_1$. Write $U^*(\mathscr{C}, \alpha_0)$ for the set of all pairs $(m, \alpha), (m', \alpha')$ such that α and α' are both $\geqq \alpha_0$ (and $\leqq \omega_1$), while m and m' are in the same set of $\mathscr{C}$.

Define $\mathfrak{U}^*$ as the set of those members of $\mathfrak{V}^*$ which contain some $U^*(\mathscr{C}, \alpha_0)$, $\mathscr{C}$ and α_0 being arbitrary. It is easily verified that $\mathfrak{U}^*$ is a uniformity for X^*, and that the uniformity $\mathfrak{U}$ which it induces on X is strictly contained in (i.e. strictly coarser than) $\mathfrak{V}$. We shall show that $\mathfrak{U}$ and $\mathfrak{V}$ are H-equivalent.

We show first that $\mathfrak{U}^*$ and $\mathfrak{V}^*$ are proximity-equivalent (which certainly implies the same for $\mathfrak{U}$ and $\mathfrak{V}$).

If E_1, E_2 are $\mathfrak{V}^*$-remote, then for any $m, (m, \omega_1)$ is in at most one of $\bar{E}_1, \bar{E}_2$ (closures relative to $\mathfrak{V}^*$, of course). Thus we can find $\alpha_m < \omega_1$, each $m \in Z$, such that the set $\{(m, \alpha): \alpha \geqq \alpha_m\}$ meets at most one of E_1, E_2. If $\alpha_0 = \sup \alpha_m$ then $\alpha_0 < \omega_1$ and we can clearly find a two-set partition $\mathscr{C}$ such that $E_1 \times E_2 \cap U^*(\mathscr{C}, \alpha_0) = \phi$. It easily follows that E_1, E_2 are $\mathfrak{U}^*$-remote.

Now let $E_0 \subset X$ be any $\mathfrak{V}$-discrete set; as such, it is clearly closed in X^*. Hence (considering the point (m, ω_1)) we see that for each $m \exists \alpha_m < \omega_1$ such that $\alpha < \alpha_m$ for all α with $(m, \alpha) \in E_0$. Put $\alpha_0 = \sup \alpha_m < \omega_1$; then, for every $\mathscr{C}, U^*(\mathscr{C}, \alpha_0) \cap E_0 \times E_0 = \phi$; it follows that E_0 is $\mathfrak{U}$-discrete. By Theorem 1, $\mathfrak{V}$ and $\mathfrak{U}$ are H-equivalent; contrary to the assertion of the supposed analogue of Theorem 2.

We note that $\mathfrak{U}^*$ and $\mathfrak{V}^*$ are certainly not H-equivalent (on X^*): this follows either from application of Theorem 2 (since $\mathfrak{U}^* \neq \mathfrak{V}^*$) or directly from Theorem I on observing that the set $\{(m, \omega_1): m \in Z\}$ is $\mathfrak{V}^*$-discrete but not $\mathfrak{U}^*$-discrete. Thus our example shows also that two uniformities may be proximity-equivalent (hence topologically equivalent) on a space X^* and H-equivalent on an everywhere dense subset X of X^* without being H-equivalent on X^*.

We now state and prove the positive results referred to earlier: it will be seen that the essential step in the proof is the demonstration that, under the conditions we impose, the situation just illustrated does not arise.

THEOREM 4. *Let (X, d) be a metric space, $\mathfrak{V}$ the uniformity defined by d. Suppose there exist a set B, precompact with respect to $\mathfrak{V}$, and a sequence of functions $\{f_r; r = 1, 2, \cdots\}$ satisfying conditions* (i) *to* (iii) *of Theorem* 2 *with K replaced by B (and with $\{f_r; r = 1, 2, \cdots\}$ for $\{f_i; i \in I\}$). Then there is no other uniformity* *for X H-equivalent to* $\mathfrak{V}$.[2]

Proof. Consider X as imbedded in its metric completion $X^\dagger$; let K (compact) be the closure of B in $X^\dagger$. The uniformly continuous functions $f_r, B \to X$, have (unique) continuous, in fact uniformly continuous, extensions $f_r^*, K \to X^\dagger$. We write $X^* = \bigcup_r f_r^*(K)$, and for the rest of the proof work in the space X^* with metric d^* (the extension of d), of which X is a dense subset. (It is not difficult to prove that in fact $X^* = X^\dagger$, but we do not need this result.) We denote by $\mathfrak{V}^*$ the uniformity for X^* defined by d^*; its restriction to $X \times X$ is obviously $\mathfrak{V}$. It is clear, by extension from X, that the set of functions $\{f_r^*; r = 1, 2, \cdots\}$ satisfies conditions (i) to (iii) of Theorem 2 for $X^*, K, \mathfrak{V}^*$.

Suppose now that $\mathfrak{U}$ is a uniformity for X, H-equivalent and hence proximity-equivalent to $\mathfrak{V}$; as $\mathfrak{V}$ is metric we must have $\mathfrak{U} \subset \mathfrak{V}$. We can define $\mathfrak{U}$ by a family $\{\rho; \rho \in P\}$ of pseudometrics for X; we can suppose further that for $\rho_1, \rho_2 \in P$ we have $\max(\rho_1, \rho_2) \in P$. Since $\mathfrak{U} \subset \mathfrak{V}$, each $\rho \in P$ is d-uniformly continuous (on $X \times X$), so it has a d^*-uniformly continuous extension ρ^* to $X^* \times X^*$, which is clearly a pseudo-metric for X^*. The family P^* of all such ρ^* defines a uniformity $\mathfrak{U}^*$ for X^* whose restriction to $X \times X$ is $\mathfrak{U}$. It is clear that P^*, like P, is closed under the taking of finite maxima; also that $\mathfrak{U}^* \subset \mathfrak{V}^*$. We shall show that $\mathfrak{U}^*$ is H-equivalent to $\mathfrak{V}^*$ on X^*; by Theorem 2 this gives $\mathfrak{U}^* = \mathfrak{V}^*$ and hence $\mathfrak{U} = \mathfrak{V}$.

We first show $\mathfrak{U}^*$ proximity-finer than $\mathfrak{V}^*$. Let A^*, B^* be subsets of X^* such that $d^*(\alpha, \beta) \geqq 3\varepsilon > 0$ for all $\alpha \in A^*, \beta \in B^*$. Put $A = X \cap \{\xi; d^*(\xi, A^*) < \varepsilon\}$ and define B similarly. A and B are clearly $\mathfrak{V}$-remote (i.e. d-remote) in X; by the data and Theorem 1 they are also $\mathfrak{U}$-remote: that is, $\exists \rho \in P, \delta > 0$ such that $\rho(a, b) \geqq \delta$, all $a \in A, b \in B$. Since $A^* \subset \bar{A}$ and $B^* \subset \bar{B}$ we have $\rho^*(\alpha, \beta) \geqq \delta$ for all $\alpha \in A^*, \beta \in B^*$, so that A^*, B^* are $\mathfrak{U}^*$-remote as required. Since $\mathfrak{U}^* \subset \mathfrak{V}^*$ the two uniformities are therefore proximity-equivalent, *a fortiori* topologically equivalent. By Theorem I we need now prove only that any $\mathfrak{V}^*$-discrete set E_0^* in X^* is also $\mathfrak{U}^*$-discrete. Since E_0^* can have only a finite number of points in each of the compact sets $f_n^*(K)$ it must be

[2] Since submitting this paper I have extended the proof of Theorem 4 to the case of a nonenumerable family of functions, provided only that the cardinal of the family is nonmeasurable. Conversely, if there exists a measurable cardinal then there is a counter-example in which both the set of elements and the family of functions have this cardinal. (The usual definition of a measurable cardinal is equivalent to the following form, convenient for topological applications:- a cardinal is *measurable* if and only if it is the cardinal of a set over which there exists a nontrivial ultrafilter with the enumerable intersection property. It is not known whether measurable cardinals actually exist but it is known that any such cardinal must be extremely large.)

enumerable, say as $(\xi_n; n = 1, 2, \cdots)$. Let ε_0 be such that $d^*(\xi_m, \xi_n) \geqq \varepsilon_0 > 0$ whenever $m \neq n$, and let $\{\varepsilon_n; n = 1, 2, \cdots\}$ be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$ and $3\varepsilon_n < \varepsilon_0$ for all $n \geqq 1$. Choose $x_n \in X$ such that $d^*(x_n, \xi_n) < \varepsilon_n (n \geqq 1)$; thus $d(x_m, x_n) \geqq (1/3)\varepsilon_0$ if $m \neq n$. Put $E_0 = \{x_n; n = 1, 2, \cdots\}$; by Theorem 1, E_0 being $\mathfrak{V}$-discrete is also $\mathfrak{U}$-discrete: thus there exist $\rho \in P$, $\delta > 0$ such that $\rho(x_m, x_n) \geqq 3\delta$ whenever $m \neq n$ (we need take only a single ρ, by the closure condition we imposed on the family P). Since $\mathfrak{U}^* \subset \mathfrak{V}^*$, $\exists \varepsilon > 0$ such that $d^*(\xi, \xi') < \varepsilon$ implies $\rho^*(\xi, \xi') < \delta$. As $\varepsilon_n \to 0$, $\exists n_0$ such that $d^*(x_n, \xi_n) < \varepsilon$ whenever $n \geqq n_0$: it easily follows that $\rho^*(\xi_m, \xi_n) \geqq \delta$ if $m \neq n$ and m, n both exceed n_0. Since each ξ_n is d^*-isolated in E_0^* and $\mathfrak{U}^*$, $\mathfrak{V}^*$ are topologically equivalent, $\exists \rho_1 \in P$, $\delta_1 > 0$ such that, for $m = 1, 2, \cdots, n_0$, $\rho_1^*(\xi_m, \xi_n) \geqq \delta_1$ for all $n \neq m$. Combining these results we see that E_0^* is $\mathfrak{U}^*$-discrete, as required.

To apply Theorem 4 to a topological group we need an additional condition (effectively, that the left—and right—invariant uniformities should be uniformly equivalent on B) which in the locally compact case was proved in the course of the work.

THEOREM 5. *Let (G, d) be a metric group with the (ε, δ) property of* Theorem 3, Corollary. *Suppose there exist a precompact neighbourhood B of the identity such that the function ϕ defined by $\phi(x, z) = xz^{-1}$ is d-uniformly continuous on $B \times B$, and an enumerable set $E_0 \subset G$ such that $E_0 B = G$. Then there is no other uniformity for G H-equivalent to the uniformity $\mathfrak{V}$ induced by d.*

Proof. With the notation of Theorem 3, the (ε, δ) condition implies that $\mathfrak{L} = \mathfrak{V}$. We can take symmetric neighbourhoods N_1, B_1 of the identity e, such that $N_1 N_1 = B_1$, $B_1 B_1 \subset B$. As B is $\mathfrak{L}$-precompact, there is a finite set $\{z_1, \cdots, z_p\} \subset B$ such that $\bigcup_{r \leqq p} L_{N_1}(z_r) = \bigcup (z_r N_1) \supset B$, hence $\bigcup_{r \leqq p} E_0 z_r N_1 = G$. Select from the enumerable set $\bigcup_r E_0 z_r$ a maximal L_{N_1}-discrete set $Y = \{y_s; s = 1, 2, \cdots\}$; then $\bigcup_s (y_s N_1) = \bigcup_s L_{N_1}(y_s) \supset \bigcup_{r \leqq p} E_0 z_r$ and hence $\bigcup_s (y_s B_1) = \bigcup_s (y_s N_1 N_1) \supset \bigcup_{r \leqq p} E_0 z_r N_1 = G$. As in Theorem 3, since ϕ is uniformly continuous on $B \times B$ and $B_1 N_1 \subset B$, $\exists$ symmetric neighbourhood $N_2 \subset N_1$ of e such that for all z in B_1 and g in N_2 we have $zgz^{-1} \in N_1$, and this implies that, for $r \neq s$ and z in B_1, $(y_r z)^{-1}(y_s z)$ is not in N_2. Thus Yz is L_{N_2}-discrete for every z in B_1, so that the set of functions f_s, where $f_s(z) = y_s z$, and the set B_1 satisfy the conditions of Theorem 4.

It is immediately obvious that both R^n and Q^n, considered as metric groups under translation, satisfy the conditions of Theorem 5; R^n (but not Q^n) also satisfies the conditions of Theorem 3, Corollary.

References

1. A. J. Ward, *A counter-example in uniformity theory*, Proc. Camb. Phil. Soc. **62** (1966), 207-208.
2. J. R. Isbell, *Uniform spaces*, (Math. Surveys 12), Amer. Math. Soc. 1964.
3. D. H. Smith *Hyperspaces of a uniformizable space*, Proc. Camb. Phil. Soc. **62** (1966), 25-28.

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