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# ON THE BIHARMONIC WAVE EQUATION

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**Under appropriate restrictions of material and motion the equation of motion for a vibrating elastic bar is $(\partial_x^4 + \partial_t^2)u = 0$. Because of its mechanical importance, there is a large literature devoted to the eigenvalue problem for this equation but solutions of boundary value problems for the equation itself seem to have been ignored. It appears that Pini was the first to seek a solution in terms of integrals analogous to thermal potentials. Like Pini, we use a fundamental solution very similar to that of the heat kernel to obtain potential terms which lead to a system of integral equations. While Pini uses Laplace transforms to obtain solutions to the integral equations, we observe that the problem may be reduced to one integral equation of a complex valued function, $f = a + \lambda k * \bar{f}$, effecting a significant simplification.**

**Along the way, we obtain, by reduction to Abel integral equations, a general method of solving semi-infinite problems which can solve boundary value problems not available to Fourier transforms, the technique presently used.**

The first appendix is a justification of the change of order of integration for a key iterated integral; the computation of some important integrals is given in the second appendix.

2. **The fundamental solutions.** By standard Fourier transform techniques, one finds that a fundamental solution for the equation

$$(\partial_x^4 + \partial_t^2)u = 0 \tag{1}$$

is

$$K(x, t) = -\pi^{-1/2}t^{1-1/2}\exp\left(\frac{ix^2}{4t} + i\frac{\pi}{4}\right). \tag{2}$$

We also define

$$C(x, t) = \operatorname{Re} K(x, t)\ , \quad S(x, t) = \operatorname{Im} K(x, t)\ . \tag{3}$$

We obtain by straightforward computation:

$$\partial_x K = \frac{ix}{2t} K\ , \tag{4}$$

$$(\partial_x^2 + i\partial_t)K = 0\ , \tag{5}$$

$$(\partial_x^2 - i\partial_t)\bar{K} = 0\ , \tag{6}$$

and hence

$$(\partial_x^4 + \partial_t^2)\begin{Bmatrix} C \\ S \end{Bmatrix} = 0 . \tag{7}$$

It is convenient to set

$$K_1(x, t) = \int_0^t K(x, s)ds ; \tag{8}$$

thus

$$K_1(x, t) = \frac{ax}{2}\int_h^\infty s^{-3/2}e^{is}ds \tag{9}$$

where

$$a = -\pi^{-1/2}e^{i\pi/4} \quad \text{and} \quad h = \frac{x^2}{4t} .$$

Then

$$\partial_x K_1 = \frac{1}{x}K_1 - \frac{2t}{x}K , \tag{10}$$

$$\partial_x^2 K_1 = -iK , \tag{11}$$

$$(\partial_x^2 + i\partial_t)K_1 = 0 . \tag{12}$$

3. **Semi-infinite bar.** We consider now the problem of the semi-infinite bar; that is, we seek a function $u(x, t)$ on $D = \{0 < x\} \times (0, T)$ such that

$$(\partial_x^4 + \partial_t^2)u = 0 , \tag{13}$$

and in the limit

$$u(x, 0) = \partial_t u(x, 0) = 0 , \tag{14}$$

$$\begin{aligned} u(0+, t) &= a(t) \\ \partial_x u(0+, t) &= b(t) \end{aligned} \tag{15}$$

where the conditions on the functions $a$ and $b$ will be determined presently.

We try a solution in the form

$$u(x, t) = \int_0^t [C(x, t-s)\varphi(s) + S(x, t-s)\psi(s)]ds . \tag{16}$$

To relate $\varphi$ and $\psi$ to $a$ and $b$, we consider

$$U(x, t; \varphi) = \int_0^t K(x, t-s)\varphi(s)ds . \tag{17}$$

The next two theorems are essentially contained in [3] but for completeness we include them here. (We use BV and CBV to mean respectively the classes of functions of bounded variation and continuous and of bounded variation.)

THEOREM 1. *If* $\varphi$ *is* BV *on* $[0, T]$, *then*

$$(\partial_x^2 + i\partial_t)U = 0$$

*on* $D$.

*Proof.* By integration by parts

$$U = \varphi(0)K_1 + \int_0^t K_1(x, t-s)d\varphi(s) \tag{18}$$

so that by (11)

$$\partial_x^2 U = -i\varphi(0)K(x, t) + \int_0^t K(x, t-s)d\varphi(s) \,. \tag{19}$$

Differentiation of (18) with respect to $t$ and comparison with (19) completes the proof. Defining

$$\begin{aligned} u_c(x, t; \varphi) &= \operatorname{Re} U(x, t; \varphi) \\ u_s(x, t; \varphi) &= \operatorname{Im} U \,, \end{aligned} \tag{20}$$

we have

COROLLARY. *If* $\varphi \in C^1$ *and* $\varphi' \in$ BV *on* $[0, T]$, *then* $u_c$ *and* $u_s$ *satisfy* (13).

*Proof.* Since (19) can be written as

$$\partial_x^2 U = -i\varphi(0)K(x, t) - iU(x, t; \varphi') \,,$$

we can apply Theorem 1 again.

THEOREM 2. *If* $\varphi \in \mathrm{CBV}([0, T])$, *then*

$$\lim_{\substack{x\to 0\pm \\ t\to t_0}} U(x, t; \varphi) = -\pi^{-1/2}e^{i\pi/4}\int_0^{t_0}(t_0 - s)^{-1/2}\varphi(s)ds \,, \tag{21}$$

$$\lim_{\substack{x\to 0\pm \\ x\to t_0}} \partial_x U(x, t; \varphi) = \pm\varphi(t_0) \tag{22}$$

*on* $(0, T)$, *and*

$$\lim_{\substack{t\to 0+ \\ x\to x_0}} U = \lim_{\substack{t\to 0+ \\ x\to x_0\neq 0}} \partial_x U = 0 \,. \tag{23}$$

*Proof.* Equation (21) and the first equality in (23) follow easily by standard arguments. That

$$\int_0^t \partial_x K(x, t - s)\varphi(s)ds$$

exists for all $x > 0$ follows immediately from

$$\left|\int_{t_1}^{t_2} \partial_x K\varphi ds\right| \leqq (|\varphi(t_1)| + V(\varphi; t_1, t_2))M(t_1, t_2) ,$$

where

$$M(t_1, t_2) = \sup\left|\int_w^z \partial_x K ds\right| , \quad t_1 \leqq w < z \leqq t_2$$

(see [2, p. 623]) and, since the integral exists, goes to zero as $t_1, t_2 \to t$. (The second part of (23) obtains from $M(0, t) \to 0$ as $t \to 0$.) On the other hand, by integration by parts,

$$\left|\int_w^z \partial_x K ds\right| < -\frac{1}{2\sqrt{\pi}}\frac{1}{|x|}\Big\{2[z^{1/2} - w^{1/2}] + \left|\int_w^z \exp\left(\frac{ix^2}{4s}\right)s^{-1/2}ds\right|\Big\} \leqq \frac{1}{\sqrt{\pi}}\frac{1}{|x|}[z^{1/2} - w^{1/2}]$$

so that the convergence is uniform with respect to $x$ for $|x| \geqq \delta > 0$. Therefore

$$U_x(x, t; \varphi) = \int_0^t K_x(x, t - s)\varphi(s)ds . \tag{24}$$

For $\varphi = 1$,

$$U_x(x, t; 1) = \int_0^t K_x(x, s)\,ds = -\frac{e^{i\pi/4}}{2\sqrt{\pi}}ix\int_0^t s^{-3/2}\exp\left(\frac{ix^2}{4s}\right)ds$$

which through a change of variable becomes

$$U_x(x, t; 1) = -\frac{ie^{i\pi/4}}{\sqrt{\pi}}\operatorname{sgn}(x)\int_h^\infty m^{-1/2}e^{im}dm$$

where $h = x^2/4t$. Since

$$\lim_{\substack{x\to 0\pm \\ t\to t_0}} h = 0 ,$$

$$\lim_{\substack{x\to 0\pm \\ t\to t_0}} U_x(x, t; 1) = \pm 1 . \tag{25}$$

Now, for $x > 0$

$$\begin{aligned}\int_0^t \partial_x K(x, t-s)[\varphi(s) - \varphi(t_0)]ds &= \left(\int_0^{t-\delta} + \int_{t-\delta}^t\right)\partial_x K[\varphi(s) - \varphi(t_0)]ds \\ &= I_1 + I_2 ,\end{aligned}$$

and

$$|I_1| \leqq \frac{|x|}{\sqrt{\pi}} \sup|\varphi(s) - \varphi(t_0)|\,(\delta^{-1/2} - t^{-1/2})$$

which goes to zero as $t \to t_0$ and $x \to 0+$. Also,

$$|I_2| \leqq \left\{\sup_{t-\delta \leqq s \leqq t} |\varphi(s) - \varphi(t_0)| + V(\varphi(s) - \varphi(t_0); t-\delta, t)\right\} M(x)$$

where

$$M(x) = \sup\left|\int_{t_1}^{t_2} \partial_x K(x, t-s)ds\right| ,$$

$t - \delta \leqq t_1 < t_2 \leqq t$. But,

$$\left|\int_{t_2}^{t_2} \partial_x K ds\right| = \frac{1}{\pi}\left|\int_{h(t_1)}^{h(t_2)} m^{-1/2} e^{im} dm\right|$$

where $h(s) = x^2/4(t-s)$, and since the limits of integration are always positive and the integral on $(0, \infty)$ exists, the last integral is uniformly bounded in $x$ and $t$, i.e. $M(x) \leqq M_1$. Therefore,

$$|I_2| \leqq M_1\{\sup|\varphi(s) - \varphi(t_0)| + V(\varphi(s) - \varphi(t_0); (t-\delta, t))\} .$$

Thus, for $t$ sufficiently close to $t_0$ and $\delta$ so small that $t - \delta$ is also close to $t_0$, the continuity of $\varphi$ and of its variation implies that $|I_2|$ is small which completes the proof.

We shall later find it necessary to extend these theorems for an important special case which is not contained in the above hypotheses. We can show that our theorems do not hold without the BV requirement, but as we shall see, BV is not necessary; thus, the present conditions are not the most natural for the kernel in question.

From the preceding we see that the conditions (14) are satisfied and that the boundary conditions (15) must be related to the density functions $\varphi$ and $\psi$ by

$$(26) \qquad \begin{gathered} -\frac{1}{\sqrt{2\pi}}\int_0^t (t-s)^{-1/2}\varphi(s)ds - \frac{1}{\sqrt{2\pi}}\int_0^t (t-s)^{-1/2}\psi(s)ds = a(t) \\ \varphi(t) = b(t) . \end{gathered}$$

We then have

$$\begin{aligned} \varphi(t) &= b(t) \\ \psi(t) &= -\sqrt{2}\,\frac{d}{dt}\int_0^t (t-s)^{-1/2}a(s)ds - b(t)\,. \end{aligned} \tag{27}$$

That is, we have proved that:

THEOREM 3. *If $b(t)$ and $d/dt\int_0^t (t-s)^{-1/2}a(s)ds$ have first derivatives which are* CBV$[0, T]$, *then* (16) *satisfies* (13) *with conditions* (14) *and* (15) *where $\varphi$ and $\psi$ are given by* (27).

**4. The finite bar problem.** We consider next the problem of finding a function $u(x, t)$ on the domain

$$D = \{0 < x < 2\} \times (0, T)$$

which satisfies

$$(\partial_x^4 + \partial_t^2)u = 0 \text{ in } D \tag{28}$$

with

$$u(x, 0+) = \partial_t u(x, 0+) = 0$$

and

$$\begin{aligned} u(0+, t) &= a_1(t)\,, \quad u(2-, t) = a_2(t) \\ \partial_x u(0+, t) &= b_1(t)\,, \quad \partial_x u(2-, t) = b_2(t) \end{aligned} \tag{29}$$

for $t > 0$.

We seek a solution in the form

$$\begin{aligned} u(x, t) = u_c(x, t; \varphi_1) &+ u_s(x, t; \psi_1) \\ &+ u_c(2 - x, t; \varphi_2) + u_2(2 - x, t; \psi_2). \end{aligned} \tag{30}$$

We observe that, by the corollary to Theorem 1, if $\varphi_i$ and $\psi_i$ $(1 = 1, 2)$ have CBV derivatives, then (30) satisfies (28) and, by Theorem 2, that (30) satisfies the initial conditions and finally that there hold the relations

$$\begin{aligned} 2^{-1/2}I^{1/2}\varphi_1 + 2^{-1/2}I^{1/2}\psi_1 &+ \left\{t^{-1/2}\cos\left(\frac{1}{t}\right) + \frac{\pi}{4}\right\}*\varphi_2 \\ &+ \left\{t^{-1/2}\sin\left(\frac{1}{t} + \frac{\pi}{4}\right)\right\}*\psi_2 = -a_1 \end{aligned} \tag{$31_1$}$$

$$\varphi_1 + 2\left\{t^{-3/2}\sin\left(\frac{1}{t} + \frac{\pi}{4}\right)\right\}*\varphi_2 - 2\left\{t^{-3/2}\cos\left(\frac{1}{t} + \frac{\pi}{4}\right)\right\}*\varphi_2 = b_1 \tag{$31_2$}$$

$$\begin{aligned}(31_3)\qquad &\left\{t^{-1/2}\cos\left(\frac{1}{t}+\frac{\pi}{4}\right)\right\}*\varphi_1+\left\{t^{-1/2}\sin\left(\frac{1}{t}+\frac{\pi}{4}\right)*\psi_1\right.\\ &\quad+2^{-1/2}I^{1/2}\varphi_2+2^{-1/2}I^{1/2}\psi_2=-a_2\end{aligned}$$

$$(31_4)\qquad 2\left\{t^{-3/2}\sin\left(\frac{1}{t}+\frac{\pi}{4}\right)\right\}*\varphi_1-2\left\{t^{-3/2}\cos\left(\frac{1}{t}+\frac{\pi}{4}\right)\right\}*\psi_1+\varphi_2=-b_2\,,$$

where

$$f*g=\int_0^t f(t-s)g(s)ds$$

and

$$I^\alpha\varphi(t)=(\Gamma(\alpha))^{-1}\int_0^t(t-s)^{\alpha-1}\varphi(s)ds\,.$$

This system is equivalent to that found by Pini [3, p. 101].

For convenience we define operators

$$(32)\qquad \begin{aligned}E^\alpha\varphi&=\frac{e^{i\pi/4}}{\Gamma(\alpha)}\{t^{\alpha-1}\exp(it^{-1})\}*\varphi\\ \tilde{E}^\alpha\varphi&=\frac{1}{\Gamma(\alpha)}\{t^{\alpha-1}\exp(it^{-1})\}*\varphi\end{aligned}$$

and

$$(33)\qquad C^\alpha+iS^\alpha=E^\alpha\,,\quad \tilde{C}^\alpha+i\tilde{S}^\alpha=\tilde{E}^\alpha\,,$$

so that the system (31) can be written

$$(34)\qquad \begin{aligned}2^{-1/2}I^{1/2}\varphi_1+2^{-1/2}I^{1/2}\psi_1+C^{1/2}\varphi_2-S^{1/2}\psi_2&=-a_1\\ I^0\varphi_1+2S^{-1/2}\varphi_2-2C^{-1/2}\psi_2&=b_1\\ C^{1/2}\varphi_1+S^{1/2}\psi_1+2^{-1/2}I^{1/2}\varphi_2+2^{-1/2}I^{1/2}\psi_2&=-a_2\\ 2S^{-1/2}\varphi_1-2C^{-1/2}\psi_1+\varphi_2&=-b_2\,.\end{aligned}$$

Adding and subtracting the first and third equations and the second and fourth equations respectively and setting

$$(35)\qquad \begin{aligned}\varphi_1+\varphi_2&=f_1 &\quad \psi_1+\psi_2&=g_1\\ \varphi_1-\varphi_2&=f_2 &\quad \psi_1-\psi_2&=g_2\\ a_1+a_2&=A_1 &\quad b_1-b_2&=B_1\\ a_1-a_2&=A_2 &\quad b_1+b_2&=B_2\,,\end{aligned}$$

we obtain two systems of two equations each:

$$(36)\qquad \begin{aligned}-2^{-1/2}I^{1/2}f_1-2^{-1/2}I^{1/2}g_1-C^{1/2}f_1-S^{1/2}g_1&=A_1\\ f_1+2S^{-1/2}f_1-2C^{-1/2}g_1&=B_1\end{aligned}$$

and

$$-2^{-1/2}I^{1/2}f_2 - 2^{-1/2}I^{1/2}g_2 + C^{1/2}f_2 + S^{1/2}g_2 = A_2 \tag{37}$$
$$f_2 - 2S^{-1/2}f_2 + 2C^{-1/2}g_2 = B_2 \,.$$

Defining operator matrices

$$M = \begin{bmatrix} -2^{-1/2}I^{1/2} & -2^{-1/2}I^{1/2} \\ I^0 & 0 \end{bmatrix} \tag{38}$$

and

$$N_1 = \begin{bmatrix} C^{1/2} & S^{1/2} \\ -2C^{-1/2} & 2C^{-1/2} \end{bmatrix}, \tag{39}$$

we can write (26) and (27) respectively as

$$M\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{40}$$

and

$$\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = M^{-1}\begin{bmatrix} a \\ b \end{bmatrix}$$

where

$$M^{-1} = \begin{bmatrix} 0 & I^0 \\ -2^{1/2}I^{-1/2} & I^0 \end{bmatrix}. \tag{41}$$

Hence, (36) can be written as

$$\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} + M^{-1}N_1\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \tag{42}$$

with

$$\begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = M^{-1}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}.$$

To simplify $M^{-1}N_1$, we prove:

LEMMA. *If* $\varphi$ *is* CBV $[0, T]$, *then*

$$I^{-1/2}E^{1/2}\varphi = -2\widetilde{E}^{-1/2}\varphi \,. \tag{43}$$

*Proof.* Consider

$$H(t) = \frac{1}{\pi}\int_0^t\int_0^s \varphi(r) \exp\,(i/(s - r))[(t - s)(s - r)]^{-1/2}drds$$

which is a absolutely integrable and, hence, can be written as

$$H(t) = \frac{1}{\pi}\int_0^t dr\varphi(r)\int_r^t \exp(i/(s-r))[(t-s)(s-r)]^{-1/2}ds$$
$$= \int_0^t \varphi(r)Erfc(e^{-i\pi/4}(t-r)^{-1/2})dr .$$

Observe that

$$\frac{d}{dt}Erfc(e^{-i\pi/4}(t-r)^{-1/2}) = \frac{1}{2}\sqrt{\pi}\,\partial_x K(2, t-r)$$

and define

$$F(t) = \frac{1}{2}\sqrt{\pi}\,\partial_x K(2, t)*\varphi .$$

Then,

$$t^{-1}[H(t+r) - H(t)] - F(t)$$
$$= \int_0^{t-\delta}\varphi(r)\Big\{\tau^{-1}[E((t+\tau-r)^{-1/2}) - E((t-r)^{-1/2})]$$
$$- \frac{1}{2}\pi^{3/2}\partial_x K(2, t-r)\Big\}dr$$
$$+ \tau^{-1}\int_{t-\delta}^{t+\tau}\varphi(r)E((t+\tau-r)^{1/2})dr$$
$$- \tau^{-1}\int_{t-\delta}^{t}\varphi(\tau)E((t-r)^{-1/2})dr$$
$$= I_1 + I_2 - I_3 ,$$

where $E(s) = Erfc(e^{-i\pi/4}s)$ and $\delta$ is to be chosen.

If $\tau > 0$, we write

$$I_2 - I_3 = \int_0^\delta \varphi(t-p)\tau^{-1}[E(p+\tau)^{-1/2}) - E(p^{-1/2})]\,dp$$
$$+ \tau^{-1}\int_0^\tau \varphi(t+\tau-p)E(p^{-1/2})dp = J_1 + J_2 .$$

By the properties of the complimentary error functions,

$$E((p+\tau)^{-1/2}) - E(p^{-1/2}) = 2e^{i\pi/4}\int_p^{p+\tau} e^{-i/n}n^{-3/2}dn ,$$

whence,

$$|J_1| \leqq 2\{\sup(|\varphi|; [0, T]) + V(\varphi; [0, T])\}M(\tau, \delta)$$

where

$$M(\tau, \delta) = \sup_{0\leqq w<z\leqq\delta}\left|\int_w^z\left(\tau^{-1}\int_p^{p+\tau} e^{-i/n}n^{-3/2}dn\right)dp\right| .$$

Integrating the inside integral by parts and then integrating by parts with respect to $p$, we obtain

$$\begin{aligned}\int_w^z(\cdots)dp = &-i[\tau^{-1}(e^{-i/(p+\tau)}(p+\tau)^{5/2} - e^{-i/p}p^{5/2})]_w^z \\ &+ \frac{5}{2}i\int_w^z \tau^{-1}(e^{-i/(p+\tau)}(p+\tau)^{3/2} - e^{-i/p}p^{3/2})dp \\ &+ \frac{1}{2}i\int_w^z\left(\tau^{-1}\int_p^{p+\tau} e^{i/n}n^{-1/2}dn\right)dp\,.\end{aligned}$$

Now then, we are justified in taking the limit as $\tau \longrightarrow 0$ to obtain

$$\begin{aligned}\overline{\lim_{\tau\to 0}}\, M(\tau, \delta) \leqq \sup\bigg| &[e^{-i/p}(ip^{1/2} + 5/2p^{3/2})]_w^z \\ &- 5/2\int_w^z e^{-ip}(ip^{-1/2} + 3/2p^{1/2})dp \\ &- 1/2\int_w^z e^{-i/p}p^{-1/2}dp\bigg| \\ \leqq A\delta^{1/2} &\qquad \text{if } \delta < 1\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}|J_2| &\leqq \sup(|\varphi|; [0, T])\left(\tau^{-1}\int_0^\tau |E(p^{-1/2})|\,dp\right) \\ &\leqq B\tau^{-1}\int_0^\tau [p^{+1/2} + O(p^{3/2})]dp \\ &\leqq B\{\tau^{1/2} + O(\tau^{3/2})\}\end{aligned}$$

which goes to zero with $\tau$. Thus, choosing $\delta$ such that

$$\overline{\lim_{\tau\to 0}}\, |J_1| < \varepsilon$$

and holding it fixed, we have

$$\overline{\lim_{\tau\to 0}}\, |I_1| = 0$$

and hence,

$$\overline{\lim_{\tau\to 0}}\, |\tau^{-1}(H(t+\tau) - H(\tau) - F(t)| < \varepsilon\,.$$

For $\tau < 0$,

$$\begin{aligned}I_2 - I_3 &= \int_0^{\delta-\tau} \varphi(t+\tau-p)\{E(p^{-1/2}) - E((p-\tau)^{-1/2}\}dp \\ &\quad + \tau^{-1}\int_0^\tau \varphi(t-p)E(p^{-1/2})dp \\ &= J_1' + J_2'\,.\end{aligned}$$

The estimates for $J_2'$ are exactly like those of $J_2$; the only change in $J_1'$ occurs in the range, $0 \leqq t_1 < t_2 \leqq \delta - \tau$, over which $M'(\delta, \tau)$ is taken, but if $|\tau| < \delta$, then we strengthen the inequalities if we extend the range to $0 \leqq t < t_2 \leqq 2\delta$ and everything goes as before.

Thus $(d/dt)H(t) = F(t)$ and the lemma follows by easy comparison with (43).

The lemma established, we return to (42) where we now see that

$$M^{-1}N_1 = \begin{bmatrix} -2S^{-1/2} & 2C^{-1/2} \\ 2\sqrt{2}\,\tilde{C}^{-1/2} - 2S^{-1/2} & 2\sqrt{2}\,\tilde{S}^{-1/2} + 2C^{-1/2} \end{bmatrix}.$$

But

$$\sqrt{2}\begin{Bmatrix} C \\ S \end{Bmatrix}^{-1/2} = \tilde{C}^{-1/2} \mp \tilde{S}^{-1/2}$$

so that

$$M^{-1}N_1 = 2\begin{bmatrix} -S^{-1/2} & C^{-1/2} \\ C^{-1/2} & S^{-1/2} \end{bmatrix},$$

and (42) becomes

$$\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} + 2\begin{bmatrix} -S^{-1/2} & C^{-1/2} \\ C^{-1/2} & S^{-1/2} \end{bmatrix}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}. \tag{44}$$

Further, if we multiply the second equation in (44) by $i$ and add it to the first with

$$\begin{aligned} f &= f_1 + ig_1 \\ a &= C_1 + iD_i\,, \end{aligned} \tag{45}$$

(44) becomes

$$f = a + 2iE^{-1/2}\bar{f}$$

or

$$f(t) = a(t) - \int_0^t k(t-s)\bar{f}(s)ds \tag{46}$$

where

$$k(t) = i\pi^{-1/2}e^{i\pi/4}t^{-3/2}\exp(i/t)\,. \tag{47}$$

Starting from system (40) we obtain in an analogous fashion

$$g = b - 2iE^{-1/2}\bar{g} \tag{48}$$

where $g = f_2 + ig_2$, $b = C_2 + iD_2$. Or,

$$g(t) = b(t) + \int_0^t k(t-s)\overline{g(s)}ds \tag{49}$$

where $k$ is defined in (47).

These two problems are combined in the equation

$$f(t) = a(t) + \lambda k * \bar{f}(t) . \tag{50}$$

**5. Solution of the integral equation.** We here solve the integral equation (50) collecting our results at the end in the form of a theorem. By formal successive substitution of (50) into itself, we obtain

$$f = a + \lambda k * \bar{a} + |\lambda|^2 k * (\bar{k} * a) + \lambda |\lambda|^2 k * (\bar{k} * (k * \bar{a})) + \cdots . \tag{51}$$

In Appendix A, we show that for functions $a(t) \in \text{CBV}\ [0, T]$ we can interchange the order of integration to obtain

$$k * (\bar{k} * a) = (k * \bar{k}) * a = k_2 * a$$

where by explicit evaluation

$$k_2(t) = (2/\pi)^{1/2} t^{-3/2} \exp(-2/t) .$$

This function has the same form as $k$ (it is even absolutely integrable) so that again

$$k * (\bar{k} * (k * \bar{a})) = k * (\bar{k}_2 * \bar{a}) = (k * \bar{k}_2) * \bar{a} .$$

Thus, we obtain for (51)

$$f = a + \lambda k * \bar{a} + |\lambda|^2 k_2 * a + \lambda |\lambda|^2 k_3 * \bar{a} + \cdots \tag{52}$$

where

$$\begin{aligned} k_{2n} &= (2/\pi)^{1/2} n t^{-3/2} \exp(-2n^2/t) \\ k_{2n+1} &= -\frac{1}{2}(2/\pi)^{1/2}(2n+1-i)t^{-3/2} \exp(-[n(2n+2) - i(2n+1)]/t) \end{aligned}$$

(see Appendix B for the computation).

To see that this series has meaning, one observes that on $[0, T]$

$$|k_2| \leqq M_2 \quad \text{and} \quad |k_3| \leqq M_3$$

whence

$$|k_{2n}| = M_2^n t^{n-1}/(n-1)! , \qquad n > 0$$

and

$$\begin{aligned} |k_{2n+1}| &\leqq \int_0^t |k_3(t-s)|\,|k_{2(n-1)}(s)|\,ds \\ &\leqq M_3 M_2^{n-1} t^{n-1}/(n-1)! , \qquad n > 0 . \end{aligned}$$

Hence, if $a \in \mathrm{CBV}$ and one sets $A = \max(a(t); [0, T])$ then

$$\begin{aligned} |f| &\leqq |a| + |\lambda|\,|k*\bar{a}| \\ &\quad + |\lambda| A(M_2 + |\lambda| M_3) \sum_{n=1}^{\infty} (|\lambda| M_2)^{n-1} T^{n-1}/(n-1)! \end{aligned}$$

which shows that (52) is a uniformly convergent series for all $\lambda$ and

$$f = a + \lambda k*\bar{a} + \Sigma\, |\lambda|^{2n} (k_{2n}*a + \lambda k_{2n+1}*\bar{a})\,. \tag{53}$$

To show that $f$, defined by (53), is continuous, one need only show that $k*\bar{a}$ is continuous; the other terms have very smooth kernels. If

$$g(t) = \int_0^t s^{-3/2} \exp(i/s) a(t-s) ds\,,$$

then for $t_1 > t_2$

$$\begin{aligned} g(t_1) - g(t_2) &= \int_{t_2}^{t_1} s^{-3/2} \exp(i/s) a(t_1 - s) ds \\ &\quad + \int_0^{\delta} s^{-3/2} \exp(i/s) a_2(s) ds \\ &\quad + \int_{\delta}^{t_2} s^{-3/2} \exp(i/s) a_2(s) ds \\ &= I_1 + I_2 + I_3 \end{aligned}$$

where $a_2(s) = a(t_1 - s) - a(t_2 - s)$ and $\delta < (1/2)t_1$ is to be chosen. Now

$$|I_i| \leqq (2 \sup |a| + 2V(a; [0, T]))M_i\,, \qquad i = 1, 2\,.$$

The existence of the integral implies that

$$M_1 = \sup \left| \int_u^v s^{-3/2} \exp(i/s) ds \right|, \qquad t_2 \leqq u < v \leqq t_1$$

goes to zero as $t_2 \to t_1$ and that

$$M_2 = \sup \left| \int_u^v \cdots \right|, \qquad 0 \leqq u < v \leqq \delta\,,$$

is small if $\delta$ is small. The uniform continutity of $a$ makes $I_3$ small as $t_2 \to t_1$.

One observes that

$$\begin{aligned} \lambda k*\bar{f} &= \lambda k*\bar{a} + |\lambda|^2 k*\overline{(k*\bar{a})} \\ &\quad + \lambda k*\overline{[\Sigma\, |\lambda|^{2n}(k_{2n}*a + \lambda k_{2n+1}*\bar{a})]} \\ &= -a + f\,, \end{aligned}$$

and we have proved:

THEOREM 4. *If* $a \in \mathrm{CBV}[0, T]$, *then* $f$, *defined by* (53), *is a continuous solution of the equation*

$$f = a + \lambda k * \bar{f}$$

*for all* $\lambda$.

**6. Solution for the finite problem.** The function $f$, determined by (53), may not be BV; for example set $a(t) = 1$. Thus, the solution of the system (34) may not satisfy the conditions of Theorems 1 and 2. However, we can establish the validity of the results directly.

PROPOSITION. If a is CBV, then

$$U(x, t; k * a) = -U(x + 2, t; a) . \tag{54}$$

The proof of this proposition is easily effected by an inversion of the order of integration and evaluation of the interior integral. Justification of the interchange can be obtained by adapting the argument of Appendix A.

From (53) $f - \lambda k * \bar{a}$ is CBV, $a$ is CBV by assumption and the remaining sum has a bounded first derivative because of the form of $k_m$, $m \geqq 2$. Therefore,

$$U(x, t; f) = U(x, t; f - \lambda k * \bar{a}) - \lambda U(x + 2, t; \bar{a}) ,$$

and from Theorem 2

$$\lim_{\substack{x \to 0\pm \\ t \to t_0}} U(x, t; f) = -e^{i\pi/4} I^{1/2}[f - \lambda k * \bar{a}] + \lambda E^{1/2} \bar{a} .$$

We observe that $k * \bar{a} = -2iE^{-1/2}\bar{a}$ and $-2E^{-1/2} = iI^{-1/2}\widetilde{E}^{1/2}$ (the lemma) so that

$$U(x, t; f) \to -e^{i\pi/4} I^{1/2} f .$$

Also, from Theorem 2

$$\lim_{\substack{x \to 0\pm \\ t \to t_0}} U_x = \pm(f - \lambda k * \bar{a}) - \lambda U_x(2, t; \bar{a}) .$$

Since $U_x(2, t; \bar{a}) = -k * \bar{a}$, we see that $U_x(x, t; f) \to f(t)$ as $x \to 0+$ which is the only case of interest.

We thus conclude that if $I^{-1/2}a_i$ and $b_i$ $(i = 1, 2)$ have first derivatives which are CBV then (33) provides a solution to the equation $(\partial_x^4 + \partial_t^2)u = 0$ with homogeneous initial conditions and boundary values given by (32).

**Appendix A.** To solve the integral equation above, it was necessary to interchange the order of integration for a very singular integrand. We here prove that the interchange is valid.

*If $f$ is* CBV$[0, T]$ *and* $|a| > 0$, $|b| > 0$, $\operatorname{Re} a \geqq 0$, $\operatorname{Re} b \geqq 0$, *then for* $0 \leqq t \leqq T$

$$\text{(A.1)}\quad \begin{aligned} J_1 &= \int_0^t \int_0^s [(t-s)(s-u)]^{-3/2} \exp(-a(t-s)^{-1} - b(s-u)^{-1}) \\ &\quad \cdot x f(u) du ds \\ &= \int_0^t f(u) \int_u^t [(t-s)(s-u)^{-3/2} \exp(-a(t-s)^{-1} - b(s-u)^{-1}) \\ &\quad \cdot x ds du = J_2 \,. \end{aligned}$$

The method of proof will be to show that $J_2$ exists, to restrict the domain to that on which interchange is easily justified, and then to show that the neglected terms vanish in the limit which will establish the existence of $J_1$ and the equality $J_1 = J_2$.

The first integral of $J_2$ is evaluated explicitly in Appendix B as

$$\sqrt{\pi}\left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{a}}\right)(t-u)^{-3/2} \exp\left[-\frac{(\sqrt{a}+\sqrt{b})^2}{t}\right].$$

Thus

$$J_2 = C_1 \int_0^t (t-u)^{-3/2} \exp(-c(t-u)^{-1}) f(u) du$$

which is equivalent to the first integration of $J_1$.

Now then,

$$\text{(A.2)}\quad \begin{aligned} I(s) &= \int_0^s (s-u)^{-3/2} \exp(-b(s-u)^{-1}) f(u) du \\ &= \int_0^s u^{-3/2} \exp(-b/u) f(s-u) du \end{aligned}$$

will exist if we can show that

$$I(m, n) = \int_m^n u^{-3/2} \exp(-b/u) f(s-u) du \longrightarrow 0 \quad \text{as } n, m \longrightarrow 0\,.$$

But

$$I(m, n) \leqq (B + V(f, [0, T])) M(m, n)$$

where $B = \sup|f(t)|$, $0 \leqq t \leqq T$, and

$$M(m, n) = \sup\left|\int_w^z u^{-3/2} \exp(-b/u) du\right|, \quad m \leqq z \leqq n\,.$$

The existence of the integral implies that $M(m, n) \longrightarrow 0$ as $m, n \longrightarrow 0$; hence, $I(s)$ exists.

Moreover, similar estimates give

$$|I| \leqq (B + V(f, (0, s)))M(0, s)\,, \tag{A.3}$$

where by integration by parts

$$M(0, s) \leqq |b|^{-1} |\exp(-b/z)z^{1/2} - \exp(-b/w)w^{1/2}| + \frac{1}{2}|b|^{-1}\left|\int_w^z u^{-1/2} du\right|,$$

and thus $M(0, s) \leqq 3|b|^{-1}s^{-1/2}$. Hence, $I(s)$ is well behaved near the origin so that the outside integral of $J_1$ is not improper at the origin and its existence will be established if

$$J_1' = \int_\delta^{t-\delta} (t - s)^{-3/2} \exp(-a(t - s)^{-1})I(s)ds$$

has a limit as $\delta \to 0$. However,

$$J_1' = \int_\delta^{t-\delta} s^{-3/2} \exp(-a/s)I(t - \delta - s)ds + \int_\delta^{t-\delta} s^{-3/2} \exp(-a/s)[I(t - s) - I(t - \delta - s)]ds$$

where the first integral is proper and we may interchange the order of integration to obtain

$$\int_0^{t-2\delta}\left(\int_u^t - \int_u^{u+\delta} - \int_{t-\delta}^t\right) \cdots dsdu = J_2' - I_1 - I_2\,.$$

Substitution into $J_1'$ gives $J_1' = J_2' - I_1 - I_2 + I_3$, where $I_3$ is the second integral in $J_1'$ above. Since $J_2$ exists, $\lim_{\delta\to 0} J_2' = J_2$. We have only to show that $I_i \to 0$ ($i = 1, 2, 3$) with $\delta$ to have $J_1 = J_2$.

We integrate the first integral of $I_3$ by parts to obtain

$$I(t - s) - I(t - \delta - s) = b^{-1} \exp(-b/\delta)\delta^{1/2} f(s - \delta) - \frac{1}{2} b^{-1} \int_0^s u^{-1/2} \exp(-b/u) f(s - u)du + b^{-1} \int_{s-\delta}^s (s - u)^{1/2} \exp(-b(s - u)^{-1})df(u)$$

which substituted into $I_3$ gives

$$I_3 = b^{-1}(I_{31} - I_{32} + I_{33})\,.$$

If we extend $f$ to the negative reals by $f(t) = f(0)$ for $t < 0$, then

$$I_{31} = b^{-1}\delta^{1/2} \exp(-b/\delta) \int_\delta^{t-\delta} s^{-3/2} \exp(-a/s) f(s - \delta)ds\,.$$

The integral converges to $I(s)$ as $\delta \to 0$ so that $I_{31} \to 0$ as $\delta \to 0$.

Observe that

$$\left|\int_0^\delta u^{-1/2} \exp(-b/u) f(s-u) du\right| \leqq [B + V(f)] M_1(0, \delta)$$

where

$$M_1(0, \delta) = \sup\left|\int_w^z u^{-1/2} \exp(-b/u) du\right|, \quad 0 \leqq w < z \leqq \delta .$$

By the same analysis as used to estimate $M(0, s)$ we see that $M_1(0, \delta) \leqq$ const. $\delta^{3/2}$ and hence

$$| I_{32} | \leqq K\delta^{3/2} \int_\delta^{t-\delta} s^{-3/2} ds = K\delta^{3/2}(\delta^{-1/2} + (t-\delta)^{-1/2}) .$$

Finally we have

$$\left|\int_{s-\delta}^s (s-u)^{1/2} \exp(-u(s-u)^{-1}) df(u)\right| \leqq K\delta^{1/2} V(f; (s-\delta, s)) .$$

Since $f$ is uniformly continuous on $[0, T]$ so is $V(f; (0, s))$; thus, $V(f; (s-\delta, s)) = o(1)$ as $\delta \to 0$ uniformly in $s$, and

$$| I_{33} | \leqq K\delta^{1/2} o(1) \int_\delta^{t-\delta} s^{-3/2} ds = o(1) .$$

Integrals $I_1$ and $I_2$ are essentially the same as can be seen by substituting $s = u + v$ into $I_1$ and $s = t - v$ into $I_2$. We thus consider only $I_1$ which we write as

$$I_1 = \int_0^{t-2\delta} f(u) \int_0^t [(t-u-v)v]^{-3/2} \exp(-a(t-u-v)^{-1} - b/v) dv du .$$

Integrating the first integral by parts (integrating $\exp(-b/v)$), we obtain

$$\begin{aligned} I_1 = & \; b^{-1}\delta^{-1/2} \exp(-b/\delta) \int_\delta^{t-\delta} u^{-3/2} \exp(-a/u) f(t-u) du \\ & - b^{-1} \int_0^{t-2\delta} du f(u) \int_0^\delta \exp(-b/v - a(t-u-v)^{-1}) \\ & \times \left[\frac{1}{2} v^{-1/2}(t-u-v)^{-3/2} + 3/2 v^{1/2}(t-u-v)^{-5/2} \right. \\ & \left. - a v^{1/2}(t-u-v)^{-7/2}\right] dv , \end{aligned}$$

$I_1 = b^{-1}(I_{11} - I_{12} - I_{13} + I_{14})$. Since $I_{11}$ is just $I_{31}$, $I_{11} \to 0$ as $\delta \to 0$

Since $t - u - v \geqq \delta$, hence $(t-u-v)^{-\alpha} \leqq \delta^{-\alpha}$ $(\alpha > 0)$, $I_{1j}$ $(j = 2, 3, 4)$ are absolute integrable as double integrals so that Fubini's theorem is applicable. Reversing the order of integration in $I_{12}$ we obtain

$$I_{12} = b^{-1}\int_0^{\delta} \frac{1}{2} v^{-1/2} \exp(-b/v) \int_0^{t-2\delta} (t-u-v)^{3/2} \exp(-a(t-u-v)^{-1}) \cdot xf(u)dudv .$$

The inside integral can be estimated by $[B + V(f)]M_3(\delta, v)$ where

$$M_3(\delta, v) = \sup\left|\int_w^z (t-u-v)^{-3/2} \exp(-a(t-u-v)^{-1}du\right| ,$$

$0 \leqq w < z \leqq t - 2\delta$. By integration by parts we see that $M_3$ satisfies

$$M_3(\delta, 0) \leqq 3 \mid a \mid^{-1} (t-v)^{1/2} + \mid a \mid^{-1} (2\delta - v)^{1/2} ;$$

thus, $I_{12} \longrightarrow 0$ as $\delta \longrightarrow 0$. Similarly the inside integral $I_{13}$, after reversing the order of integration, can be estimated by $K[(2\delta - v)^{-1/2} + (t-v)^{-1/2}]$ so that $I_{13} \longrightarrow 0$ as $\delta \longrightarrow 0$.

Estimates like those above are too coarse for $I_{14}$; we must take another approach. If we reverse the order of integration and then integrate the inner integral by parts

$$\begin{aligned} aI_{14} &= f(0)\int_0^{\delta} v^{1/2}(t-v)^{-3/2} \exp(-b/v - a(t-v)^{-1})dv \\ &\quad - f(t-2\delta)\int_0^{\delta} v^{1/2}(2\delta - v)^{-3/2} \exp(-b/v - a(t-v)^{-1}dv \\ &\quad + \int_0^{\delta} v^{1/2} \exp(-b/v) \int_0^{t-2\delta} \exp(-a(t-u-v)^{-1}) \\ &\quad \times [-3/2(t-u-v)^{-5/2}f(u)du + (t-u-v)^{-3/2}df(u)]dv ; \end{aligned}$$

that is

$$aI_{.4} = f(0)I_{141} - f(t-2\delta)I_{142} + I_{143} + I_{144} .$$

Now, $I_{143}$ is just $I_{13}$ and

$$\mid I_{141} \mid \leqq (t-\delta)^{-3/2} \int_0^{\delta} \sqrt{v}\, dv$$

so that $I_{141}$ and $I_{143} \longrightarrow 0$ as $\delta \longrightarrow 0$.

For $I_{144}$ we have immediately that

$$\mid I_{144} \mid \leqq \int_0^{\delta} v^{1/2} \int_0^{t-2\delta} (t-u-v)^{-3/2} dV(f; (0, u))dv .$$

If $\delta < 1$ and $0 < p < 1$, $y = \delta^p > \delta$ and

$$\int_0^{t-2\delta}(t-u-v)^{-3/2}dV(f;0,u)) = \left(\int_0^{t-2y}+\int_{t-y2}^{t-\delta 2}\right)(t-u-v)^{-3/2}dV(f)$$
$$\leqq (2y-v)^{-3/2}V(f;(0,t)) + (2y-v)^{-3/2}V(f;(t-2y,\xi))$$
$$+ (2\delta-v)^{-3/2}V(f;(\xi,t-2\delta))$$

by the second mean value theorem for Riemann-Stieltjes integrals where $t-2y \leqq \xi \leqq t-2\delta$. Therefore,

$$|I_{144}| \leqq 2\delta^{3/2}(2y-\delta)^{-3/2}V(f;(0,t)) + \frac{2}{3}V(f;(\xi,t-2\delta))\,.$$

Since $V(f,(0,u)$ is continuous in $u$, we have

$$|I_{144}| \leqq 2\delta^{3(1-p)/2}(2-\delta^{1-p})^{-3/2}V(f;(0,t)) + o(1)$$

as $\delta \longrightarrow 0$.

To complete the proof we have only to show that $I_{142} \longrightarrow 0$ as $\delta \longrightarrow 0$. The only question arises if $a$ and $b$ are both imaginary for which we write $b = i\beta$, $\alpha = ih\beta$. If we set $v = 2\delta/(w+1)$, than

$$I_{142} = \left(\int_1^N + \int_N^\infty\right)(w+1)^{-1}w^{-3/2}\exp\left[-i\beta(w+1)(hw+1)(2\delta w)^{-1}\right]dw\,.$$

Given $\varepsilon > 0$, we can choose $N$ such that

$$\int_N^\infty \cdots dw \leqq \int_n^\infty (w+1)^{-1}w^{-3/2}dw \leqq 2(N+1)^{-1}N^{-1/2} < \varepsilon\,.$$

The remaining integral goes to zero with $\delta$ by a stationary phase argument.

**Appendix B.** We here evaluate certain integrals which we have used. From [1] p. 146(28)

$$L\left\{t^{-3/2}\exp\left(-\frac{1}{4}a/t\right)\right\} = 2\sqrt{\pi}\,\alpha^{-1/2}e^{-\sqrt{\alpha}\,s}$$

for $\operatorname{Re}(\alpha) > 0$. One can easily show that the formula is valid if $\operatorname{Re}(\alpha) \leqq 0$, $|\alpha| \neq 0$. Thus, using this formula we have

$$\int_0^t [(t-u)u]^{-3/2}\exp(-a(t-u)^{-1}-b/u)du$$
$$= \sqrt{\pi}(a^{-1/2}+b^{-1/2})t^{-3/2}\exp\left[-(a^{1/2}+b^{1/2})^2t^{-1}\right],$$
$$|\arg a| \leqq \pi/2\,, \quad |\arg b| \leqq \pi/2\,.$$

Thus for $k*\bar{k}$, $a = e^{i\pi/2}$ $b = e^{-i\pi/2}$ and $k_2(t) = (2/\pi)^{1/2}t^{-3}/^2\exp(-2/t)$.

$$k_3(t) = k*\bar{k}_2 = -\frac{1}{2}\sqrt{(2/\pi)}(3-i)\exp\left[-(4-3i)t^{-1}\right].$$

Finally, by induction

$$k_{2n}(t) = (2/\pi)^{1/2} n t^{-3/2} \exp(-2n^2/t)$$

and hence

$$\begin{aligned} k_{2n+1}(t) &= k * \bar{k}_{2n} \\ &= -\frac{1}{2}(2/\pi)^{1/2}(2n+1-i)t^{-3/2} \exp(-[n(2n+2) - i(2n+1)]t^{-1}) . \end{aligned}$$

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