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ON THE RIGIDITY OF SEMI-DIRECT PRODUCTS OF LIE ALGEBRAS

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Roughly speaking, a Lie algebra L is *rigid* if every Lie algebra near L is isomorphic to L. It is known that L is rigid if the Lie algebra cohomology space $H^2(L, L)$ vanishes. In this paper we give an elementary set of necessary and sufficient conditions, independent of Lie algebra cohomology, for the rigidity of a semi-direct product $L = S +_\rho W$, where ρ is an irreducible representation of a semi-simple Lie algebra S on a vector space W. These conditions lead to a number of new examples of rigid Lie algebras. In particular, we obtain a rigid Lie algebra L with $H^2(L, L) \neq 0$.

It follows from [9] that there is only a finite number of isomorphism classes of rigid Lie algebras with a given underlying vector space. The "rigidity theorem" of [9] shows that L is rigid if $H^2(L, L) = 0$. Thus semi-simple Lie algebras are rigid. In general, however, it is difficult to compute $H^2(L, L)$ and there are few known examples of rigid Lie algebras which are not semi-simple. In considering the rigidity of semi-direct products $L = S +_\rho W$, we avoid the use of Lie algebra cohomology and appeal instead to the "stability theorem" of [10]. Our results essentially reduce the problem of rigidity for such semi-direct products to a classification problem in the theory of semi-simple Lie algebras.

In a series of papers [6] written with an eye towards applications to physics, R. Hermann has obtained results similar to ours in a number of special cases. His method involves a direct computation of $H^2(L, L)$.

1. Preliminaries. Let V be a finite-dimensional real or complex vector space and let $A^2(V)$ denote the vector space of all alternating bilinear maps of $V \times V$ into V. Let $\mathscr{M}$ be the algebraic set in $A^2(V)$ consisting of all Lie algebra multiplications on V. There is a canonical representation of the group $G = GL(V)$ of all vector space automorphisms of V on the vector space $A^2(V)$ defined as follows. If $g \in G$ and $\varphi \in A^2(V)$, then $(g.\varphi)(x, y) = g(\varphi(g^{-1}x, g^{-1}y))$ for all $x, y \in V$. The algebraic set $\mathscr{M}$ is stable under the corresponding action of G on $A^2(V)$. Moreover, the orbits of G on $\mathscr{M}$ correspond precisely to the isomorphism classes of Lie algebra structures on V.

Let $\mu \in \mathscr{M}$ and let $L = (V, \mu)$ be the corresponding Lie algebra. Then L is *rigid* if the orbit $G(\mu)$ is an open subset of $\mathscr{M}$. If V is a complex (resp. real) vector space, then it follows from [9, Prop. 17.1, p. 21] that $G(\mu)$ is in fact a Zariski-open subset of $\mathscr{M}$ (resp. one component of a Zariski-open subset of $\mathscr{M}$). Hence there exists only a finite number of isomorphism classes of rigid Lie algebras with underlying vector space V.

If $\mu, \mu' \in \mathscr{M}$ and if $L = (V, \mu)$ and $L' = (V, \mu')$ are the corresponding Lie algebras, then L is a *contraction* of L' if μ lies in the closure of the orbit $G(\mu')$. If L is rigid and is a contraction of L', then it follows that L is isomorphic L'.

2. Rigidity of semi-direct products. Let S be a semisimple (real or complex) Lie algebra and let ρ be an irreducible representation of S on a finite-dimensional vector space W. We consider W as an abelian Lie algebra and form the corresponding semi-direct product $L = S +_\rho W$. (See [1, pp. 17–20] for the appropriate definitions.)

THEOREM 2.1. *Let $L = S +_\rho W$ be as above. Then L is not rigid if and only if there exists a semi-simple Lie algebra L' which satisfies the following conditions*: (a) *there exists a semi-simple subalgebra S' of L' which is isomorphic to S*; (b) *if we identify S and S' by an isomorphism, then L'/S' is isomorphic as an S-module to W.*

Here the S-module structure of L'/S' is determined by the adjoint representation of S' on L'.

Proof. Let V denote the vector space direct sum $S \oplus W$; V is the underlying vector space of L. We identify S and W with subspaces of V in the usual manner. Let μ be the Lie algebra multiplication on V corresponding to L. Suppose there exists a semi-simple Lie algebra L' satisfying conditions (a) and (b) above. We may assume that V is the underlying vector space of L'. If μ' denotes the Lie algebra multiplication on V corresponding to L', we may assume further that $\mu(s, s') = \mu'(s, s')$ for every $s, s' \in S$ and that $\mu(s, w) = \mu'(s, w)$ for every $s \in S$, $w \in W$. Let F denote either the real field or the complex field. For each $t \in F, t \neq 0$, let $g_t \in GL(V)$ be defined by: $g_t(s) = s$ if $s \in S$ and $g_t(w) = tw$ if $w \in W$. We let μ_t be the Lie algebra multiplication on V given by $\mu_t(x, y) = g_t(\mu'(g_t^{-1}(x), g_t^{-1}(y))$ for $x, x \in V$. Then the Lie algebra $L_t = (V, \mu_t)$ is isomorphic to L'. It is easy to check the following conditions: if $s, s' \in S$, then $\mu(s, s') = \mu_t(s, s')$; if $s \in S, w : W$, then $\mu(s,w) = \mu_t(s, w)$; if $w, w' \in W$, then

$$\mu^t(w, w') = t^{-1}\mu'(w, w').$$

It follows immediately that $\lim_{t\to\infty}\mu_t = \mu$. Thus L is a contraction of L' and hence L is not rigid.

Now for the converse. Let $\mathscr{M}$ denote the set of Lie algebra multiplications on V. It follows from the "stability theorem" of [10] (see, in particular Corollary 11.4) that there exists a neighborhood U of μ in M such that if $\mu_1 \in U$, then the Lie algebra $L_1 = (V, \mu_1)$ is isomorphic to a Lie algebra $L' = (V, \mu')$ which satisfies the following conditions: (1) if $s, s' \in S$, then $\mu(s, s') = \mu'(s, s')$; (2) if $s \in S$ and $w \in W$, then $\mu(s,w) = \mu'(s, w)$. If L is not rigid, we may assume that L' is not isomorphic to L. Let R denote the radical of L' and let $pr_W : V \to W$ denote the projection with kernel S. Since $R \cap S = \{0\}$, it follows that the restriction of pr_W to R is an injection. Since the representation ρ of S on W is irreducible, it follows easily from (1) and (2) that either $R = \{0\}$ or that pr_W maps R isomorphically onto W.

Suppose $R \neq \{0\}$. Then $[R, R] \neq R$ and $[R, R]$ is stable under the adjoint representation of S (considered as a subalgebra of L') on L'. The argument given above shows that $[R, R] = \{0\}$, hence that R is abelian. In this case, it is an easy consequence of the Levi-Whitehead Theorem that L' is isomorphic to L, thus giving a contradiction.

Thus $R = \{0\}$, and consequently the Lie algebra L' is semisimple. It follows immediately from (1) and (2) above that L' satisfies (a) and (b) of Theorem 2.1. This completes the proof.

COROLLARY 2.2. *Let L be as in Theorem 2.1 and let L_1 be a Lie algebra with the same underlying vector space as L such that L is a contraction of L_1. Then either L_1 is semisimple or L_1 is isomorphic to L. Hence there exist only a finite number of isomorphism classes of Lie algebras L_1 such that L is a contraction of L_1.*

This was proved in the course of the proof of Theorem 2.1.

3. **A classification problem.** If a Lie algebra L' satisfying conditions (a) and (b) of Theorem 2.1 exists, it follows easily that S' is a maximal semi-simple subalgebra of L'. Consider now the problem of finding, for each semi-direct product $L = S +_\rho W$, with S semi-simple and ρ irreducible, the set of all (isomorphism classes of) Lie algebras L' such that L is a contraction of L'. It follows from the results of § 2 that this problem reduces to the following classification problem:

Classify to within isomorphism the set of all pairs (L', S'), where L' is a semi-simple Lie algebra and S' is a maximal semi-simple subalgebra of L' such that the adjoint representation of S' on L'/S' is irreducible. For each such pair describe the adjoint representation of S' on L'/S'.

The maximal semi-simple subalgebras S' of a complex semisimple Lie algebra L' have been classified by Dynkin [3, 4]. There remains the problem of finding those pairs (L', S') for which the adjoint representation of S' on L'/S' is irreducible and, for each such pair, finding the highest weight of the representation of S' on L'/S'. In the case of real Lie algebras the problem becomes considerably more complicated.

3. Some examples. (1) Let $\mathfrak{o}_n$ denote the Lie algebra of all skew symmetric n by n matrices with real entries. Let ρ denote the identity representation of $\mathfrak{o}_n$ on $\boldsymbol{R}^n$ and let $\mathfrak{m}_n = \mathfrak{o}_n +_\rho \boldsymbol{R}^n$; $\mathfrak{m}_n$ is the Lie algebra of the Lie group of all rigid motions of $\boldsymbol{R}^n$. We may imbed $\mathfrak{o}_n$ as a subalgebra of $\mathfrak{o}_{n+1}$ in an obvious manner. We consider $\mathfrak{o}_{n+1}$ as an $\mathfrak{o}_n$-module via the adjoint representation. Then $\mathfrak{o}_{n+1}$ splits, as an $\mathfrak{o}_n$-module, into a direct sum of $\mathfrak{o}_n$ and an $\mathfrak{o}_n$-submodule which is isomorphic to $\boldsymbol{R}^n$. It follows from Theorem 2.1 that $\mathfrak{m}_n$ is a contraction of $\mathfrak{o}_{n+1}$; hence $\mathfrak{o}_{n+1}$ is not rigid.

(2) Let S denote the unique simple Lie algebra of dimension three over the field $\boldsymbol{C}$ of complex numbers. By a half-integer we mean an element of the set $\{1/2, 1, 3/2, \cdots\}$. For each half-integer k let ρ_k denote the irreducible representation of weight k of S on $\boldsymbol{C}^{2k+1}$. Every irreducible representation of S is equivalent to some ρ_k. We denote by L_k the semidirect product $S +_{\rho_k} \boldsymbol{C}^{2k+1}$. If S is embedded as a subalgebra of a semisimple Lie algebra L of rank r, then it is shown in [8, p. 996, Th. 5.2] that the number of irreducible components occuring in the complete reduction of the adjoint representation of S on L is at least r. Moreover there always exists a three-dimensional simple subalgebra of L (the principal three-dimensional subalgebra) such that exactly r irreducible components occur. Combining this result with Theorem 2.1 it follows that L_k is not rigid if and only if there exists a semisimple Lie algebra of rank 2 and of dimension $2k + 4$. From the classification of simple Lie algebras over $\boldsymbol{C}$, it follows easily that L_k is rigid unless $k = 1, 2,3$ or 5. If L_k is not rigid, there is precisely one semisimple Lie algebra L (to within isomorphism) such that L_k is a contraction of L.

4. Remarks on Lie algebra cohomology. A representation ρ of a Lie algebra L on a vector space X defines on X the structure of an L-module. If $a \in L$ and $x \in X$ we denote $\rho(a).x$ simply by $a.x$. An element $x \in X$ is an *invariant* of L if $a.x = 0$ for every $a \in L$. The set of invariants of L forms an L-submodule of X which we denote by X^L. If $\varphi : X \to Y$ is a homomorphism of L-modules, then $\varphi(X^L) \subset Y^L$. Let S be a semi-simple Lie algebra and let $X \to Y \to Z$ be an exact sequence of finite-dimensional S-modules (and S-module homomorphisms). It follows easily from the fact that every finite-dimensional S-module is semi-simple that the corresponding sequence $X^S \to Y^S \to Z^S$ of S-modules is again exact.

We assume familiarity with Lie algebra cohomology. For details we refer the reader to [7]. If X is an L-module, we denote by $C(L, X) = \bigoplus_n C^n(L, X)$ the cochain complex used to compute the cohomology of L with coefficients in X. We shall denote by

$$H(L, X) = \bigoplus_n H^n(L, X)$$

the corresponding cohomology group. If I is an ideal of L, then there is a natural L-module structure on $C(I, X)$ and this induces an L-module structure on $H(I, X)$. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of L-modules. Then the corresponding exact sequence

$$0 \to C(I, X) \to C(I, X) \to C(I, Z) \to 0$$

of cochain complexes is also an exact sequence of L-modules. Consequently, the corresponding cohomology exact sequence

$$\cdots \to H^{n-1}(I, Z) \to H^n(I, X) \to H^n(I, Y) \to H^n(I, Z) \to \cdots$$

is an exact sequence of L-modules. Suppose now that there is a semi-simple subalgebra S of L which is supplementary (as a vector subspace of L) to I. Then, by restriction, we can consider each $H^n(I, X)$ (resp. $H^n(H, Y)$, $H^n(I, Z)$) as an S-module. Hence the cohomology exact sequence above gives rise to an exact sequence

$$\cdots \to H^{n-1}(I, Z)^S \to H^n(I, X)^S \to H^n(I, Y)^S \to H^n(I, Z)^S \to \cdots .$$

5. **A rigid Lie algebra with $H^2(L, L) \neq 0$.** Let S be the simple 3-dimensional Lie algebra over $\boldsymbol{C}$, let n be a positive integer, let $W = \boldsymbol{C}^{2n+1}$, and let ρ be the irreducible representation of weight n of S on W. Let $L = L_n$ denote the semi-direct product $S +_\rho W$. Then W is an abelian ideal in L and S is supplementary to W in L. We consider L as an L-module via the adjoint representation. If we consider $\boldsymbol{C}$ as a trivial S-module, then $H^1(S, \boldsymbol{C}) = 0 = H^2(S, \boldsymbol{C})$ (see [2. p. 113]). It follows from the Hochschild-Serre spectral sequence [7, p. 603, Th. 13] that $H^2(L, L) = H^2(W, L)^L$. But $H^2(W, L)$ is a trivial W-module. Hence $H^2(L, L) = H^2(W, L)^S$.

Consider the exact sequence $0 \to W \to L \to L/W \to 0$ of L-modules. It follows from the results of § 4 that there is a corresponding cohomology exact sequence

$$\cdots \to H^1(W, L/W)^S \to H^2(W, W)^S \to H^2(W, L)^S \to \cdots .$$

Since W is an abelian Lie algebra and W and L/W are trivial W-modules, it follows that $H^n(W, W) = C^n(W, W)$ and $H^n(W, L/W) =$

$C^n(W, L/W)$. Assume now that $n > 1$. Then it is easy to see that $C^1(W\,L/W)^S = 0$ and hence that $H^1(W, L/W)^S = 0$. Thus we have an exact sequence $0 \longrightarrow H^2(W, W)^S(W, L)^S$.

It follows from the Clebsch-Gordan formula [5, p. 251] that the tensor product representation of S on $W \otimes_C W$ decomposes into a direct sum of representations of weight $2n, 2n-1, \cdots, 1,0$. Let T denote the S-submodule of $W \otimes_C W$ consisting of all skew-symmetric tensors. Then the representation of S on T decomposes into a direct sum of representations of odd weights $2n-1, 2n-3, \cdots, 1$. In particular, if n is odd, the representation of weight n occurs in the complete reduction of T as a direct sum of irreducible S-modules. In this case, it follows immediately that $H^2(W, W)^S = C^2(W, W)^S$ is 1-dimensional. Hence $H^2(L,L) = H^2(W, L)^S \neq 0$. Combining this with the results of (2) of § 3, we obtain:

PROPOSITION 5.1. *For every odd integer $n > 5$, the Lie algebra L_n is a rigid Lie algebra with $H^2(L_n, L_n) \neq 0$.*

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