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SOME TOPOLOGICAL PROPERTIES OF PIERCING POINTS

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Let K be the closure of one of the complementary domains of a 2-sphere S topologically embedded in the 3-sphere, S^3. We give first (Theorem 1) a characterization of those points $p \in S$ with the following property: there exists a homeomorphism $h : K \to S^3$ such that $h(S)$ can be pierced with a tame arc at $h(p)$. The topological property of K which distinguishes such a "piercing point" p is this: $K - p$ is 1-ULC. Using this result, we find (Theorems 2 and 3) that p is a piercing point of K if and only if S is arcwise accessible at p by a tame arc from $S^3 - K$ (note: perhaps S cannot be pierced with a tame arc at p, even if p is a piercing point of K). Thus, the "tamely arcwise accessible" property is independent of the embedding of K in S^3. The corollary to Theorem 2 gives an alternate proof of an as yet unpublished fact, first proven by R. H. Bing: a topological 2-sphere in S^3 is arcwise accessible at each point by a tame arc from at least one of its complementary domains.

In the last section of the paper, we give two applications of the above theorems. First, we show in Theorem 4 that S can be pierced with a tame arc at p if and only if p is a piercing point of both K and the closure of $S^3 - K$. Finally, we remark in Theorem 5 that S can be pierced with a tame arc at each of its points if it is "free" in the sense that for each $\varepsilon > 0$, S can be mapped into each of its complementary domains by a mapping which moves each point less than ε. It is not known whether each 2-sphere S with this last property is tame.

A space homeomorphic to such a set K in S^3 (as described at the beginning of the Introduction) is called a *crumpled cube*. We write $\text{Bd}\, K = S$ and $\text{Int}\, K = K - \text{Bd}\, K$. An arc A in S^3 is said to *pierce* a 2-sphere S in S^3 if $A \cap S$ is an interior point p of A and the two components of $A - p$ lie in different components of $S^3 - S$. The *piercing points of a crumpled cube* are defined as above and were first considered by Martin [10]. It follows from Lemmas 2 and 3 of [10] and [6; Th. 11] that the nonpiercing points of a crumpled cube K form a 0-dimensional F_σ subset of $\text{Bd}\, K$.

If C and D are subsets of a space Y with metric d, and $\varepsilon > 0$, we use $B(C, D; \varepsilon)$ to denote the set of all points $x \in D$ such that for some $y \in C$, $d(x, y) < \varepsilon$. The metric on E^3 and S^3 is always assumed to be the ordinary Euclidean one. Let $\Delta^n (n \geqq 1)$ denote a closed n-

simplex. If Y is a metric space and $A \subset Y$, we say that A is n-LC ($n \geqq 0$) at $p \in \text{Cl}\, A \subset Y$ (Cl A = the closure of A) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each mapping of Bd Δ^{n+1} into $B(p, A; \delta)$ extends to a mapping of Δ^{n+1} into $B(p, A; \varepsilon)$. We say that A is n-ULC ($n \geqq 0$) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each mapping of Bd Δ^{n+1} into a subset of A of diameter less than δ extends to a mapping of Δ^{n+1} into a subset of A of diameter less than ε. We refer to a mapping $f : \text{Bd}\, \Delta^2 \rightarrow Y$ as a *loop*.

By a *null* sequence of subsets of a metric space, we mean one such that the diameters of its elements converge to zero. A *Sierpinski curve* X is (uniquely) defined as any space homeomorphic to $[\text{Bd}\, \Delta^3] - \bigcup \text{Int}\, D_i$, where $D_1, D_2, \cdots,$ is a null sequence of disjoint 2-cells whose union is a dense subset of Bd Δ^3. The *inaccessible part* of X corresponds to $[\text{Bd}\, \Delta^3] - \bigcup D_i$. For a more detailed discussion of Sierpinski curves, see [3].

2. Preliminary lemmas.

The following is Theorem 1 of [12], stated here for the reader's convenience.

LEMMA 1. *Let C be a q-cell ($q = 1, 2,$ or 3) topologically embedded in E^3, and let $D \subset \text{Bd}\ C$ be a $(q-1)$-cell. Let $A_1, A_2, \cdots, A_k$ be a finite disjoint collection of tame arcs in $E^3 - D$ with each* Bd *$A_i \subset E^3 - C$. Then, there exists a compact set $E \subset C - D$ such that, for each $\varepsilon > 0$, there is a homeomorphism $h : E^3 \rightarrow E^3$ with each $h(A_i) \subset E^3 - C$ and h is the identity outside the ε-neighborhood of E.*

We shall also need the following [5; Th. 2].

LEMMA 2. *Let B be a closed subset of Δ^2; let A be a subset of the separable metric space Y and suppose that A is* 0-LC *and* 1-LC *at each point of Y. Let $\varepsilon > 0$ and a mapping $f : \Delta^2 \rightarrow \text{Cl}\ A$ be given. Then, There is a mapping $f^* : \Delta^2 \rightarrow \text{Cl}\ A$ such that*

$$f^*(\Delta^2 - B) \subset A, f^* \mid B = f \mid B, \text{ and } d(f^*(x), f(x)) < \varepsilon$$

for each $x \in \Delta^2$, where d is the metric for Y.

Let X be a topological space, and Y a closed subset of X. A loop $f : \text{Bd}\, \Delta^2 \rightarrow X$ will be said to be *contractible in X (mod Y)* if there exists a connected open set N in Δ^2 such that Bd $\Delta^2 \subset N$, and a mapping $F : \text{Cl}\, N \rightarrow X$ such that $F \mid \text{Bd}\, \Delta^2 = f$, and F maps the (point-set) boundary of N (in Δ^2) into Y.

LEMMA 3. *Let K be a crumpled cube in S^3, and let U be an* *open subset of K such that $U \cap \mathrm{Bd}\, K$ is an open 2-cell T. Let A be a compact subset of K such that $A \cap \mathrm{Bd}\, K$ consists of a single point p in T, where $K^* - p$ is 1-LC at p and K^* is the crumpled cube $S^3 - \mathrm{Int}\, K$. Then, if a loop in $U - A$ is contractible in $U - A$ (mod $T - p$), it is contractible to a point in $(U - A) \cup (W - A)$, where W is any open set in S^3 containing p.*

Proof. Let N be a connected open set in Δ^2 containing $\mathrm{Bd}\, \Delta^2$, let W be an open set in S^3 containing p, and let

$$F : \mathrm{Cl}\, N \rightarrow U - A$$

be a mapping which takes the boundary B of N in Δ^2 into $T - p$. By the homotopy extension theorem, $F \mid B : B \rightarrow T$ extends to a mapping $G : \Delta^2 \rightarrow T$. Hence, by Lemma 2, and the fact that $K^* - p$ is 1-LC at p, $F \mid B$ extends to

$$G^* : \Delta^2 \rightarrow [T - p] \cup [(W \cap K^*) - p] .$$

Finally, define $H : \Delta^2 \rightarrow (U - A) \cup (W - A)$ by $H \mid \mathrm{Cl}\, N = F \mid \mathrm{Cl}\, N$ and $H \mid \Delta^2 - N = G^* \mid \Delta^2 - N$. Then H is the required contraction of $F \mid \mathrm{Bd}\, \Delta^2$.

REMARK. Given the notation of the lemma, and a loop $f : \mathrm{Bd}\, \Delta^2 \rightarrow U - A$, a necessary condition for f to be contractible to a point in $(U - A) \cup (W - A)$, where W is a small neighborhood of p in S^3, is that f be contractible in $U - A$ (mod $T - p$).

3. Characterizations of piercing points.

THEOREM 1. *Let K be a crumpled cube and p a point of* Bd K. *Then p is a piercing point of K if and only if $K - p$ is* 1-LC *at p.*

Proof. We may assume, by [8] and [9], that K is embedded in S^3 in such a manner that there exists a homeomorphism h of C, the closure of $S^3 - K$, onto the closed unit ball in E^3. Let A be the inverse image under h of the straight line segment in E^3 from the origin to $h(p)$. Then A is an arc which is locally tame in S^3 except possibly at p, and according to Martin [10], p is a piercing point of K if and only if A is tame. By [11, Lemma 5], A is tame if and only if $S^3 - A$ is 1-LC at p. Hence the problem is reduced to showing that $S^3 - A$ is 1-LC at p if and only if $K - p$ is 1-LC at p.

We shall give the details of the "if" part of the above assertion. The converse is merely a rearrangement of the same ideas. Suppose

$K - p$ is 1-LC at p, and let ε be a positive number. We must find a $\delta > 0$ such that each loop in $B(p, S^3 - A; \delta)$ is contractible in $B(p, S^3 - A; \varepsilon)$. We assume that ε is less than the distance from p to $h^{-1}((0, 0, 0))$. Since $K - p$ is 1-LC at p, there exists $\rho > 0$ such that each loop in $B(p, K - p; \rho)$ is contractible in $B(p, K - p; \varepsilon)$. Let U be an open subset of S^3 such that $p \in U \subset B(p, S^3; \rho)$ and such that there is a homeomorphism of $U \cap C$ onto the set of points in E^3 having nonnegative z-coordinates which takes $U \cap A$ into the z-axis. Finally, choose $\delta > 0$ so that $B(p, S^3; \delta) \subset U$.

Now, a given loop in $B(p, S^3 - A; \delta)$ is homotopic in $U - A$ to a loop in

$$(U \cap K) - p \subset B(p, K - p; \rho) ,$$

and this loop in turn is contractible to a point in $B(p, K - p; \varepsilon)$, as required.

REMARK. Since K is compact and locally contractible, the condition "$K - p$ is 1-LC at p" is equivalent to "$K - p$ is 1-ULC".

COROLLARY. *Let K be a crumpled cube, and p a point of $S = \operatorname{Bd} K$. Then p is a piercing point of K if and only if the following condition holds: For each $\varepsilon > 0$, there is a $\delta > 0$ such that each simple closed curve in $B(p, S - p; \delta)$ is contractible in $B(p, K - p; \varepsilon)$.*

Proof. The condition is necessary by the preceding theorem. To show sufficiency, assume the notation of the preceding proof and let $\varepsilon > 0$ be given as before. Let $\delta > 0$ be chosen to satisfy the above condition and so that only the component of $A - B(p, S^3; \delta)$ which contains $h^{-1}((0, 0, 0))$ fails to lie in $B(p, S^3; \varepsilon)$. We also assume that A is locally polyhedral at each point of $A - p$. Then, each piecewise-linear homeomorphism

$$f : \operatorname{Bd} \Delta^2 \to B(p, S^3 - A; \delta)$$

extends to a piecewise-linear mapping F of Δ^2 into $B(p, S^3 - p; \delta)$ such that F is in general position relative to A. Hence $F^{-1}(A)$ is finite. If $x \in F^{-1}(A)$, then F restricted to a sufficiently small curve enclosing x represents a loop in $B(p, S^3 - A; \delta)$ which is homotopic in $B(p, S^3 - A; \varepsilon)$ to a loop in $B(p, S - p; \delta)$, and hence is contractible in $B(p, K - p; \varepsilon)$. This permits us to redefine F in a small neighborhood of each $x \in F^{-1}(A)$, and thus obtain an extension of f mapping Δ^2 into $B(p, S^3 - A; \varepsilon)$. Hence $S^3 - A$ is 1-LC at p and the result follows.

LEMMA 4. *Let K be a crumpled cube in S^3, and p a piercing* *point of the crumpled cube* $K^* = S^3 - \operatorname{Int} K$. *Suppose* A *is an arc in* K *having* p *as an end-point, such that* $A \cap S = p$, *where* $S = \operatorname{Bd} K$. *If there exists a homeomorphism* $h : K \to S^3$ *such that* $h(A)$ *is tame, then* A *is tame.*

Proof. Since $h(A)$ is tame, A is locally tame in S^3 except possibly at p. Hence, by [11; Lemma 5], it suffices to show that $S^3 - A$ is 1-LC at p. Suppose $\varepsilon > 0$. Let U be an open set in S^3 such that $p \in U \subset B(p, S^3; \varepsilon)$ and $U \cap S$ is an open 2-cell T. Since h is a homeomorphism, and since $S^3 - h(A)$ is 1-LC at $h(p)$, there exists $\rho > 0$ such that each loop in $B(p, K - A; \rho)$ is contractible in $(U \cap K) - A$ (mod $T - p$). Choose $\mu > 0$ so that each loop in $B(p, K^*; \mu)$ is contractible in $B(p, K^*; \rho)$. Finally, let $\delta > 0$ be such that each pair of points in $B(p, S; \delta)$ can be joined by an arc in $B(p, S; \mu)$.

Now let a loop in $B(p, S^3 - A; \delta)$ be given. We give here an outline of the proof that this loop is contractible in $B(p, S^3 - A; \varepsilon)$. The details are left to the reader. There are three steps:

1. After performing a small homotopy in $B(p, S^3 - A; \delta)$, we assume that this loop is a simple closed curve J such that $J \cap K^*$ consists of a finite number of disjoint arcs $L_1, L_2, \cdots, L_k$, with $L_i \cap S = \operatorname{Bd} L_i$, for each i.

2. For each i, let Z_i be an arc in $B(p, S; \mu) - p$ joining the end-points of L_i. Then L_i is homotopic in $B(p, K^*; \rho)$, with end-points fixed, to Z_i. Since $K^* - p$ is 1-LC at p, Lemma 2 allows us to adjust this homotopy to give one in $B(p, K^*; \rho) - p$ between L_i and Z_i. Hence, by piecing together these homotopies, we see that the given loop is homotopic in $B(p, S^3 - A; \rho)$ to the loop

$$[J - \bigcup \operatorname{Int} L_i] \cup \bigcup Z_i$$

in $B(p, K - A; \rho)$.

3. This last loop is contractible in $(U \cap K) - A$ (mod $T - p$). Hence, by Lemma 3, it is contractible to a point in $B(p, S^3 - A; \varepsilon)$. This completes the proof.

REMARK. Using the same techniques, and Lemma 3, we could prove this lemma with "tame" replaced consistently by "cellular" or "has a simply-connected complement in S^3" everywhere in its statement. In these two alternate formulations, we could permit A to be any compact absolute retract, and p any point of A.

THEOREM 2. *Let* K *be a crumpled cube in* S^3, *and* p *a point of* $S = \operatorname{Bd} K$. *If* p *is a piercing point of* K, *then there is a tame arc* A *in* $K^* = S^3 - \operatorname{Int} K$ *having* p *as an end-point such that* $A \cap S = p$.

Proof. By Lemma 4, it suffices to show that there is an arc A in K^* having p as an end-point such that $A \cap S = p$, and such that for some embedding $h: K^* \to S^3$, $h(A)$ is tame. We choose h so that the closure of $S^3 - h(K^*)$ is a 3-cell ([8] and [9]). Hence, the theorem will follow as stated above if we can prove it in the special case when K is a closed 3-cell. We make this assumption to simplify the notation.

Let f be a homeomorphism of the closed unit ball B in E^3 onto K, with $f((0, 0, 1)) = p$. Let $T_i\,(i = 1, 2, \cdots)$ be the 2-cell which is the f-image of the intersection of B with the plane $z = 1 - 1/i$. Let the 3-cell $C_i\,(i = 1, 2, \cdots)$ be defined inductively as follows: C_1 is the closure of the component of $K - T_1$ not containing p; $C_i\,(i \geqq 2)$ is the closure of the component of

$$K - T_i - \bigcup_{j<i} C_j$$

not containing p. Finally, let A^* be a tame arc in S^3 having p as one end-point and the other end-point not in K. We assume that $A^* \cap C_1 = \phi$.

According to Lemma 1, there is for each $i > 1$, a homeomorphism $g_i: S^3 \to S^3$ which is the identity outside a small neighborhood U_i of T_i and which is such that $g_i(A^*) \cap T_i = \phi$. In particular, the U_i's may be chosen to form a null sequence of disjoint sets. Let g be the homeomorphism of S^3 onto itself which agrees with g_i on U_i, for each i, and otherwise is the identity. Then $g(A^*) \cap T_i = \phi$, for each i, and $g(p) = p$.

Again using Lemma 1, there is, for each $i > 1$, a compact set $E_i \subset C_i - (T_i \cup T_{i-1})$ (by the previous paragraph, there is a 2-cell in $\text{Bd}\, C_i$ containing $T_i \cup T_{i-1}$ and missing $g(A^*)$) and a homeomorphism $k_i: S^3 \to S^3$ which is the identity outside an arbitrarily small neighborhood V_i of E_i and which is such that $k_i g(A^*) \cap C_i = \phi$, for each i. We choose V_i so close to E_i that the V_i's form a null sequence of disjoint sets, and so that V_i misses the closure of $K - C_i$. Let k be the homeomorphism of S^3 onto itself which agrees with k_i on V_i, for each i, and reduces to the identity otherwise. Then $A = kg(A^*)$ is the required arc.

COROLLARY (*Bing*). *A topological 2-sphere in S^3 is arcwise accessible at each point by a tame arc from at least one of its complementary domains.*

Proof. Let K and K^* be the two crumpled cubes into which the 2-sphere S decomposes S^3. If $p \in S$, then either p is a piercing point of K, or p is a piercing point of K^*([10; Theorem]). The result then follows from the preceding theorem.

THEOREM 3. *Let K be a crumpled cube in S^3, and p a point of $S = \text{Bd}\, K$. If there is a tame arc A in $K^* = S^3 - \text{Int}\, K$ having p as an end-point and such that $A \cap S = p$, then p is a piercing point of K.*

Proof. It suffices to establish the condition given in the corollary to Theorem 1. Thus, take $\varepsilon > 0$. We assume that ε is less than the distance between p and q, where q is the other end-point of A. Choose $\delta > 0$ so that $B(p, S; \delta)$ lies interior to a closed 2-cell $D \subset B(p, S; \varepsilon)$.

Since A is locally tame at p, there is a tame 2-sphere

$$Z^* \subset B(p, S^3; \delta)$$

which separates p from q in S^3 and which meets A at precisely one point $r \in \text{Int}\, A$, at which A pierces Z^*. Let T be a small closed 2-cell in Z^* missing K and such that $r \in \text{Int}\, T$. Note that, by linking considerations, Bd T is not contractible in $B(p, K^*; \varepsilon) - A$.

Appealing to [2; Th. 1] and [4; Th. 1], we obtain, for each $\rho > 0$, a tame Sierpinski curve $X \subset S$ such that each component U_i $(i = 1, 2, \cdots)$ of $S - X$ has diameter less than ρ, and a homeomorphism $h : S^3 \to S^3$ which moves each point of S^3 less than ρ, which is the identity outside $B(Z^* \cap S, S^3; \rho)$, and which is such that $h(Z^*) \cap X$ consists of a finite disjoint collection of simple closed curves each in the inaccessible part of X. Let $Z = h(Z^*)$. By choosing ρ sufficiently small, we may ensure that h is the identity on T and that Z retains all the properties originally required of Z^*. A final requirement on ρ is that $\rho < \varepsilon - \delta$ and that the component of $S - X$ containing p should not meet Z (if $p \in X$, then S can be pierced with a tame arc at p, by [6; Th. 6]).

We assert that there is at least one component of $Z \cap S$ separating p from Bd D in D (this component is necessarily a simple closed curve). If not, then $Z \cap X$ consists of a finite number of simple closed curves each of which is contractible in $D - p$, and $Z \cap (S - X)$ can be covered by the null sequence of disjoint open 2-cells of diameter less than ρ in $S : U_1, U_2, \cdots$. Note that $U_i \cap Z$ is compact. It is now easy, using the homotopy extension theorem on each of the inclusions $U_i \cap Z \to U_i$ as in the proof of Lemma 3, to construct a mapping contracting Bd T in

$$[K^* \cap (Z - \text{Int}\, T)] \cup [B(p, S - p; \varepsilon)] \subset B(p, K^*; \varepsilon) - A \,,$$

a contradiction.

By the preceding paragraph, we may let L be an innermost (in $Z - T$) one of the components of $S \cap Z$ which separates p from Bd D in D. Let L bound the 2-cell $F \subset Z - T$. Note that L is not contractible in $B(p, K^*; \varepsilon) - A$ and that no component of $S \cap \text{Int } F$ separates p from Bd D in D. Hence, by the argument of the preceding paragraph, the "large" component of $F - S$ lies in Int K, and L is contractible in

$$[K \cap F] \cup [B(p, S - p; \varepsilon)] \subset B(p, K - p; \varepsilon) .$$

Since each simple closed curve in $B(p, S - p; \delta)$ is homotopic in $D - p$ to L, the proof is complete.

4. Some applications.

THEOREM 4. *Let S be a 2-sphere topologically embedded in S^3, and let K and K^* be the two crumpled cubes into which S divides S^3. Then S can be pierced with a tame arc at a point $p \in S$ if and only if p is a piercing point of K and a piercing point of K^*.*

Proof. The "only if" part of the theorem follows from Theorem 3. For the converse, suppose that p is a piercing point of each of K and K^*, and let A be an arc in S such that A is locally tame except possibly at the end-point p. By [6; Th. 6], S can be pierced with a tame arc at p if A is tame.

To show that A is tame, we proceed in essentially the same manner as in the proof of [6; Lemma 6.1]. That is, let S' be a 2-sphere in S^3 which contains A and is locally tame at each point of $S' - A$, and which is homeomorphically so close to S that p is a piercing point of each of the crumpled cubes L and L^* into which S' divides S^3 (use Theorems 2 and 3). It suffices to show that S' is tame.

Exactly as in [6], S' is locally tame at each point of $A - p$. Hence, S' is locally tame except possibly at p. It follows easily, since $L - p$ and $L^* - p$ are each 1-LC at p, that $S^3 - S'$ is 1-LC at each point of S' and hence that S' is tame by [1; Th. 6]. This completes the proof.

In [7], Hempel studied the properties of a surface S (=Bd K) which is *free* relative to one of its complementary domains (Int K) in S^3 (i.e., S satisfies the mapping condition stated in the following theorem). It is not known whether the crumpled cube of this theorem is necessarily a 3-cell.

THEOREM 5. *Let K be a crumpled cube, and let $S =$ Bd K. Suppose that for each $\varepsilon > 0$ there exists a mapping $f: S \rightarrow$ Int K which* *moves each point of S less than ε. Then each point of S is a piercing point of K.*

Proof. We shall verify the condition given in the corollary to Theorem 1. Suppose $p \in S$ and $\varepsilon > 0$. Choose $\delta > 0$ so that there is a closed 2-cell $D \subset S$ such that

$$B(p, S; \delta) \subset D \subset B(p, S; \varepsilon) .$$

Then, if J is a simple closed curve in $B(p, S - p; \delta)$ bounding a 2-cell $D^* \subset D$, there is a $\rho > 0$ such that ρ is less than the distance from D to the complement of $B(p, K; \varepsilon)$ and such that each mapping of J into K which moves each point of J less than ρ is homotopic in $B(p, K - p; \delta)$ to the inclusion of J into $B(p, K - p; \delta)$.

Suppose $f: S \rightarrow \text{Int } K$ is a mapping which moves each point of S less than ρ. Then J is homotopic in $B(p, K - p; \delta)$ to $f(J)$, and $f(J)$ bounds the singular 2-cell

$$f(D^*) \subset B(p, K; \varepsilon) - S .$$

This completes the proof.

REMARK. If $S \subset S^3$ is a topological 2-sphere which is free relative to *each* of its complementary domains, then it follows from the foregoing theorems that S can be pierced with a tame arc at each of its points.

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