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ON A THEOREM OF ORLICZ AND PETTIS

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In this paper a direct proof of the following theorem of Orlicz, Pettis, and Grothendieck is given.

THEOREM 1. In a locally convex Hausdorff space each subseries of a series converges with respect to the initial topology of the space if and only if each subseries of the series converges with respect to the weak topology of the space.

In a second theorem each of three additional conditions is shown to be equivalent to subseries convergence in complete locally convex Hausdorff spaces. Two of these equivalence are known for Banach spaces. The third condition, a weak compactness condition on the unordered partial sums of the series, is new even for Banach spaces. It is a consequence of the first theorem that a weak unconditional basis for a weak sequentially complete locally convex Hausdorff space is an unconditional basis.

Theorem 1 was first proved by Orlicz [8, Satz 2] for weakly sequentially complete Banach spaces. Banach [1, p. 240] noted the hypothesis of weak sequential completeness was unnecessary. A proof of Theorem 1 for Banach spaces was given by Pettis [9, Th. 2.32]. Grothendieck [4, Cor. 2, p. 141] obtains Theorem 1 for locally convex spaces as a special case of a theorem on vector valued integrals.

The proof of Lemma 3 was suggested by the referee in place of a longer proof by the author. It uses a result of a paper of James [5] which appeared after this paper was submitted.

For clarity we now state the basic definitions in more detail. If E is a Hausdorff linear topological space with topology $\mathscr{T}$ then a series $\sum_{i=1}^{\infty} x_i$ in E is *subseries convergent* relative to $\mathscr{T}$ if and only if:

(A) Corresponding to each subseries $\sum_{i=1}^{\infty} x_{k_i}$ there is an element $x \in E$ such that $\lim_n \sum_{i=1}^{n} x_{k_i} = x$, the convergence being relative to $\mathscr{T}$.

Let E^* denote the space of $\mathscr{T}$-continuous linear functionals on E. Then $\sum_{i=1}^{\infty} x_i$ is *weak subseries convergent* if and only if:

(B) $\sum_{i=1}^{\infty} x_i$ is subseries convergent relative to the $w(E, E^*)$ topology for E.

For a series $\sum_{i=1}^{\infty} x_i$ in a linear topological space $(E, \mathscr{T})$ let $S = \{\sum_{i \in \sigma} x_i : \sigma \text{ finite}\}$ and consider the following conditions where $\overline{\mathrm{sp}}\{x_i\}$ denotes the closure, relative to $\mathscr{T}$, of the linear span of $\{x_i\}$:

(C) S is totally bounded relative to $\mathscr{T}$.

(D) Whenever $\{f_n\}$ is an equicontinuous sequence in E^* such that $f_n(x) \to 0$ for all $x \in \overline{\mathrm{sp}}\,\{x_i\}$ then $f_n(x) \to 0$ uniformly on S.

(E) The $w(E, E^*)$ closure of S is $w(E, E^*)$ compact.

In this paper we also prove,

THEOREM 2. *For a series in a locally convex, complete, Hausdorff space the conditions* (A), (B), (C), (D), *and* (E) *are equivalent.*

The equivalence of (A), (B), (C), and a variant of (D) for series in Banach spaces is known (see, e.g., [7]).

2. Proof of Theorem 1.

LEMMA 1. *If $\sum_{i=1}^{\infty} x_i$ is a series in a locally convex Hausdorff space E which satisfies the condition* (D) *then it is subseries Cauchy.*

Proof. We first observe that if $M \subset E$, then M is bounded if, whenever $\{f_n\}$ is an equicontinuous sequence in E^* such that $\lim f_n(x) = 0$ for all $x \in E$, it follows that $\lim f_n(x) = 0$ uniformly on M. For if M is not bounded and U is a closed convex circled neighborhood of zero that does not absorb M, then for each integer n there is an $f_n \in E^*$ and an $x_n \in M$ such that $|f_n(x)| \leqq 1/n$ on U and $f_n(x_n) \geqq 1$. The sequence $\{f_n\}$ is equicontinuous and $\lim f_n(x) = 0$ on E but not uniformly on M. We observe further that if $\sum_{i=1}^{\infty} x_i$ satisfies condition (D) then S is bounded from which it follows that $\sum_{i=1}^{\infty} |f(x_i)| < +\infty$ for each $f \in E^*$.

We now prove Lemma 1 by showing that if $\sum_{i=1}^{\infty} x_i$ is not subseries convergent and $\sum_{i=1}^{\infty} |f(x_i)| < +\infty$ for each $f \in E^*$ then condition (D) does not hold. Suppose there exists a subseries $\sum_{i=1}^{\infty} x_{k_i}$ whose sequence of partial sums $\{\sum_{i=1}^{n} x_{k_i}\}$ is not a Cauchy sequence. Thus there exists a closed convex circled neighborhood V of zero and an increasing sequence $\{p_n\}$ of positive integers such that for each n,

$$s_n = \sum_{i=p_n+1}^{p_{n+1}} x_{k_i} \notin V .$$

Then, [6, 14.4, p. 119] for each n there exists an element $f_n \in E^*$ such that $f_n(s_n) = 1$ and $\sup\{|f_n(x)| : x \in V\} < 1$. Thus the sequence $\{f_n\}$ is equicontinuous. Since $\{f_n\}$ is equicontinuous it is pointwise bounded on E. Using this and the diagonal process we select a subsequence $\{f_{n_m}\}$ which has [6, 17.4, p. 155] a w^*-cluster point f_0 with the property that $\lim_m f_{n_m}(x) = f_0(x)$, $x \in \overline{\mathrm{sp}}\,[x_i]$. From the hypothesis (D) we have that $\lim_m f_{n_m}(x) = f_0(x)$ uniformly for $x \in S$. Given $\varepsilon > 0$ there exists N such that

$$\sum_{i=N}^{\infty} |f_0(x_i)| < \varepsilon/2 .$$

Thus, for m sufficiently large, $|f_0(s_{n_m})| < \varepsilon/2$. Now there exists $N' > N$ such that if $m \geqq N'$ then $|f_{n_m}(x) - f_0(x)| < \varepsilon/2$ for all $x \in S$. Thus, for m sufficiently large,

$$|f_{n_m}(s_{n_m})| \leqq |f_{n_m}(s_{n_m}) - f_0(s_{n_m})| + |f_0(s_{n_m})| < \varepsilon$$

but this contradicts $f_{n_m}(s_{n_m}) = 1$ for all m.

It is well known [6, 17.8, p. 156] that each weakly compact subset of a locally convex space is complete. The proof of the following lemma follows the same line of argument as the above mentioned result, so is omitted.

LEMMA 2. *Let $(E, \mathscr{T})$ be a locally convex space. If $\{x_\alpha\}$ is a Cauchy net in E relative to the topology $\mathscr{T}$ and if $x \in E$ is a $w(E, E^*)$-cluster point of $\{x_\alpha\}$ then $x_\alpha \to x$ in the topology $\mathscr{T}$.*

Proof of Theorem 1. It is clear that (A) implies (B). We now assume (B) and show that (A) follows. It is clear that when (B) holds we have $\sum_{i=1}^{\infty} |f(x_i)| < +\infty$ for each $f \in E^*$, i.e., $\{f(x_i)\}_{i=1}^{\infty}$ is an element of (l) for each $f \in E^*$. The space (l) here is either real or complex depending on whether the scalar field of E is real or complex. For either real or complex (l) sequential convergence in the $w((l), (l)^*)$ topology for (l) implies convergence in the norm topology of (l) and elements of the form $\{\varepsilon_i\}$, where $\varepsilon_i = \pm 1$ or 0, are fundamental in $(l)^*$. Let $\{f_n\}$ be an arbitrary sequence in E^* such that $\lim_n f_n(x) = 0$ for all $x \in \overline{\mathrm{sp}}\{x_i\}$. Let $\lambda_n = \{f_n(x_i)\}_{i \in \omega}$. We will show, following Pettis [9], that

(F) $\lim_n \sum_{i=1}^{\infty} |f_n(x_i)| = 0$, i.e., $\lambda_n \to 0$ in the norm topology of (l), by showing that

$$\lim_n \sum_{i=1}^{\infty} \varepsilon_i f_n(x_i) = 0$$

for each sequence $\{\varepsilon_i\}$, where $\varepsilon_i = \pm 1$ or 0. For such a sequence $\{\varepsilon_i\}$ let $\sigma + = \{i : \varepsilon_i \geqq 0\}$ and $\sigma - = \{i : \varepsilon_i < 0\}$. By (B) there exist $x_{\sigma+}$ and $x_{\sigma-}$ such that $f(x_{\sigma+}) = \sum_{i \in \sigma+} f(x_i)$ and $f(x_{\sigma-}) = \sum_{i \in \sigma-} f(x_i)$ for all $f \in E^*$. Now $x_{\sigma+}$ and $x_{\sigma-}$ are elements of $\overline{\mathrm{sp}}[x_i]$. Suppose $x_{\sigma+} \notin \overline{\mathrm{sp}}[x_i]$. Then there exists $f \in E^*$ such that $f(x_{\sigma+}) \neq 0$ and $f(x) = 0$ for $x \in \overline{\mathrm{sp}}[x_i]$. This, however, implies the contradiction $f(x_{\sigma+}) = 0$ since $f(x_{\sigma+}) = \sum_{i \in \sigma+} f(x_i)$ where $x_i \in \overline{\mathrm{sp}}[x_i]$. Hence, it follows that

$$\lim_n \sum_{i=1}^{\infty} \varepsilon_i f_n(x_i) = \lim_n f_n(x_{\sigma+} - x_{\sigma-}) = 0 .$$

Thus we have shown that (F) holds, from which it is evident that $\lim_n f_n(x) = 0$ uniformly for $x \in S$. It now follows, by Lemma 1, that $\sum_{i=1}^\infty x_i$ is subseries Cauchy. Then from (B) and Lemma 2, $\sum_{i=1}^\infty x_i$ is subseries convergent.

3. Proof of Theorem 2.

LEMMA 3. *Let E be a complete Hausdorff locally convex space and let E_0 be a closed separable subspace of E. Let M be a subset of E_0 such that whenever $\{f_n\}$ is an equicontinuous sequence in E^* such that $\lim_n f_n(x) = 0$ for all $x \in E_0$ it follows that $\lim_n f_n(x) = 0$ uniformly for $x \in M$. Then the weak closure of M is $w(E, E^*)$ compact.*

Proof. If the weak closure of M is not weakly compact then, by the result of James [5, Condition (9), p. 104], there is a positive number ε, an equi-continuous sequence $\{f_n\}$, and a sequence $\{z_n\}$ from the weak closure of M such that

$$|f_n(z_k)| > \varepsilon \quad \text{if} \quad n \leqq k \quad \text{and} \quad f_n(z_k) = 0$$

if $n > k$. Let $\{x_n\}$ denote a sequence in E_0 which is dense in E_0. Using the diagonal technique we select a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ which has a w^*-cluster point f_0 such that $\lim_j f_{n_j}(x_m) = f_0(x_m)$, $m \in w$. Hence, $\lim_j f_{n_j}(x) = f_0(x)$ for all $x \in E_0$. We then have from the uniformity hypothesis that $\lim_j f_{n_j}(x) = f_0(x)$ uniformly for $x \in M$ and hence uniformly for x in the weak closure of M. Since z_k is in the weak closure of M this contradicts the fact that $|f_{n_j}(z_k)| > \varepsilon$ if $n_j \leqq k$ and $f_{n_j}(z_k) = 0$ if $n_j < k$.

LEMMA 4. *For a series $\sum_{i=1}^\infty x_i$ in a locally convex Hausdorff space E,* (A) *implies* (C) *implies* (D) *and* (E) *implies* (B).

Proof. (A) implies (C): Subseries convergence implies unordered convergence [2, p. 59]. Let U be a neighborhood of 0. Then there exists a finite set σ_0 of positive integers such that if σ is a finite subset of positive integers and $\sigma \supset \sigma_0$ then

$$\sum_{i \in \sigma} x_i - \sum_{i \in \sigma_0} x_i \in U\,.$$

Let $B = \{\sum_{i \in \sigma} x_i : \sigma \subset \sigma_0\} \cup \{0\}$. Observe that B is a finite set and for an arbitrary finite subset σ of positive integers

$$\sum_{i \in \sigma} x_i \in B + U\,.$$

(C) implies (D): In general totally bounded sets are bounded. Also [6, 8.17, p. 76] an equicontinuous family of linear functionals on E which converges pointwise on a totally bounded set to an element $f \in E^*$ converges uniformly to f on that set. (E) implies (B): If $\sum_{i=1}^{\infty} x_{k_i}$ is any subseries and y is a $w(E, E^*)$ cluster point of $\{\sum_{i=1}^{n} x_{k_i}\}$, then y is a $w(E, E^*)$ sum of $\sum_{i=1}^{\infty} x_{k_i}$, since if

$$\left| f\left(\sum_{i=1}^{N} x_{k_i}\right) - f(y) \right| < \frac{1}{2}\varepsilon$$

and

$$\sum_{i=k_N}^{\infty} | f(x_i) | < \frac{1}{2}\varepsilon ,$$

then

$$\left| f\left(\sum_{i=1}^{n} x_{k_i}\right) - f(y) \right| < \varepsilon$$

if $n > N$.

Proof of Theorem 2. By Lemma 3, (D) implies (E). By Lemma 4, (E) implies (B). Conditions (A) and (B) are equivalent by Theorem 1. By Lemma 4, (A) implies (C) and (C) implies (D).

4. **Applications.** Suppose that A is a set, that $\sum$ is a σ-field of subsets of A, and m is an additive set function defined on $\sum$ with values in a locally convex Hausdorff space E. Then m is *weakly countably additive* if and only if

$$\sum_{i=1}^{\infty} fm(A_i) = fm\left(\bigcup_{i=1}^{\infty} A_i\right)$$

for each $f \in E^*$ and each sequence of disjoint sets A_i in $\sum$. As an immediate consequence of Theorem 1, we obtain

COROLLARY 1. *A weakly countable additive set function m defined on a σ-field Σ with values in a locally convex Hausdorff space is countably additive.*

Corollary 1 is a generalization of a theorem of Pettis [9, Th. 2.4; 3, Th. 1, p. 318] for Banach space valued set functions.

A sequence $\{x_i\}$ in a Hausdorff linear topological space $(E, \mathscr{T})$ is a *basis* if and only if corresponding to each $x \in E$ there is a unique sequence of scalars $\{a_i\}$ such that $x = \lim_n \sum_{i=1}^{n} a_i x_i$, the convergence relative to $\mathscr{T}$. A basis is *unconditional* if for each $x \in E$ every rearrangement of the basis expansion for x converges. It is known that a sequence $\{x_i\}$ in a locally convex, metrizable, complete space $(E, \mathscr{T})$ is a basis relative to $\mathscr{T}$ providing it is a basis relative to the $w(E, E^*)$ topology for E. As a corollary to Theorem 1 we obtain the following weak basis theorem which applies to a class of linear topological spaces which includes certain nonmetrizable spaces.

COROLLARY 2. *If* $(E, \mathscr{T})$ *is a locally convex Hausdorff space which is sequentially complete in its* $w(E, E^*)$ *topology, then a sequence* $\{x_i\}$ *in* E *is an unconditional basis for* $(E, \mathscr{T})$ *provided it is an unconditional basis for* E *with its* $w(E, E^*)$ *topology. Thus, a* $w(E, E^*)$ *unconditional basis in a semireflexive space* $(E, \mathscr{T})$, e.g., *a Montel space, is an unconditional basis relative to* $\mathscr{T}$.

Proof. Suppose $\{x_i\}$ is a $w(E, E^*)$ unconditional basis. For $x \in E$ let $\{a_i\}$ denote the unique sequence of scalars such that for all $f \in E^*$ $f(x) = \sum_{i=1}^{\infty} a_i f(x_i)$ where the convergence of the series is unconditional. Unconditional convergence of a series of real or complex numbers implies subseries convergence for that series. It follows, using the hypothesis of $w(E, E^*)$ sequential completeness, that $\sum_{i=1}^{\infty} a_i x_i$ is weak subseries convergent so by Theorem 1, $x = \lim_n \sum_{i=1}^{n} a_i x_i$, convergence relative to $\mathscr{T}$. If x also has the expansion $x = \lim_n \sum_{i=1}^{n} b_i x_i$, unconditional convergence relative to $\mathscr{T}$, then $f(x) = \sum_{i=1}^{\infty} b_i f(x_i)$, $f \in E^*$ where the convergence is unconditional in the scalar field so $b_i = a_i$, $i \in \omega$, because of the assumed uniqueness of $\{a_i\}$.

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