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ON QUASI-ISOMORPHIC INVARIANTS OF PRIMARY GROUPS

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Two primary groups G and H are quasi-isomorphic if there exist subgroups G^* and H^* of G and H such that G^* and H^* are isomorphic and such that G/G^* and H/H^* are bounded. The paper is concerned with properties, of primary groups, that are invariant under the relation of quasi-isomorphism. In the final section, a condition is given which is necessary and sufficient in order that the primary groups G and H be quasi-isomorphic in case G and H are both direct sums of closed groups.

The main result of the paper is that quasi-isomorphism commutes with direct decomposition for the class of primary groups whose first Ulm factors are direct sums of countable groups.

The connection between the relation of quasi-isomorphism of primary groups and their Ulm invariants was investigated by Beaumont and Pierce in [1] and [2] and by the author in [4]. Cutler [3] has recently studied properties of primary groups which are invariant under the relation of quasi-isomorphism. For example, it was proved in [3] that the property of being a direct sum of cyclic groups is invariant under quasi-isomorphism. Irwin and Richman proved, for primary groups, in [9] that the property of being a direct sum of countable groups is also a quasi-isomorphic invariant. We shall establish a decomposition theorem which contains these results as special cases; specifically, the following theorem is proved. Suppose that G and H are primary groups and suppose that $G = \sum G_\lambda$ where $G_\lambda/p^\omega G_\lambda$ is a direct sum of cyclic groups. If H is quasi-isomorphic to G, then $H = \sum H_\lambda$ where H_λ is quasi-isomorphic to G_λ. There is also a decomposition theorem proved for group pairs (G, H) where G is a direct sum of cyclic groups or a closed group and H is a cobounded subgroup of G. In answer to a question in [3], we show that the property of being a direct sum of closed groups is not a quasi-isomorphic invariant. Even in one of the simplest cases, where $G = \bar{B} + B$, a group H quasi-isomorphic to a direct sum G of closed groups need not be a direct sum of closed groups. However, we show that if G and H are direct sums of closed groups, then G and H are quasi-isomorphic if they satisfy a condition which is obviously necessary; if for some bounded subgroup B there exists an isomorphism from $(G + B)[p]$ onto $(H + B)[p]$ that does not alter heights more than a fixed positive integer k, then G and H are quasi-isomorphic.

Two abelian groups G and H are said to be quasi-isomorphic if there exist isomorphic subgroups H^* and G^* of H and G, respectively, such that G/G^* and H/H^* are bounded. The notation $G \doteq H$ is used to mean that G and H are quasi-isomorphic. We shall call a subgroup A of G cobounded if G/A is bounded.

2. Cobounded subgroups inherit basic subgroups. Suppose that G is a primary group and that H is a subgroup of G such that $H \supseteqq p^n G$ for some positive integer n. Let B be a basic subgroup of G. The question was raised in [3] as to whether there exists a basic subgroup B' of H such that $P^n B \subseteqq B' \subseteqq B$, and some partial results were obtained concerning the problem. Megibben has pointed out to the author that not only does such a B' always exists but that it is unique. In fact, the following theorem can readily be established. And we shall make use of it later on in the paper.

THEOREM 2.1. *Suppose that H is a cobounded subgroup of the primary group G. If B is pure and dense in G, relative to the p-adic topology, then $B \cap H$ is pure and dense in H.*

Proof. There is, of course, no proper cobounded subgroup of a divisible group. Thus $\{B, H\}/B = G/B$ and $\{B, H\} = G$, so $H/B \cap H \cong \{B, H\}/B = G/B$ and $B \cap H$ is dense in H. Since H is cobounded and since $B[p]$ is dense in $G[p]$, it is immediate that $(B \cap H)[p]$ is dense in $H[p]$. In order to prove that $B \cap H$ is pure in H, it suffices, according to [6], to prove that $B \cap H$ is a neat subgroup of H. Suppose that $ph = b$ where $h \in H$ and $b \in B \cap H$. Since B is pure, there is an element $b' \in B$ such that $pb' = b = ph$. Now $(h - b') - b'' \in H$ for some $b'' \in B[p]$ since $B[p]$ is dense in $G[p]$ and since H is cobounded. Observe that $b' + b'' \in B \cap H$ and that $p(b' + b'') = b$. This completes the proof of the theorem.

Recall that the final rank of a basic subgroup is called the critical number of a primary group. Since a cobounded subgroup has the same final rank as the group, an immediate consequence of Theorem 2.1 is the following corollary.

COROLLARY 2.2. *The critical number of a primary group is a quasi-isomorphic invariant.*

Theorem 2.1 also yields a refinement of Theorem 5.1 in [3].

COROLLARY 2.3. *If H is a cobounded subgroup of the primary group G and N is high in G, then $M = H \cap N$ is high in H.*

Proof. Since N is high in G, we have the relation $G[p] = N[p] + G^1[p]$ where $G^1 = \bigcap_{n<\omega} p^n G$. Thus $H[p] = M[p] + H^1[p]$ since $H^1 = G^1$. Now the purity of M in H implies that M is a high subgroup of H.

The question now arises as to whether for each pure and dense subgroup B of a cobounded subgroup H of the primary group G there exists a pure and dense subgroup A of G such that $B = A \cap H$.

THEOREM 2.4. *If H is a cobounded subgroup of the primary group G and if B is pure and dense in H, then there exists a pure and dense subgroup A of G such that $B = A \cap H$.*

Proof. Choose A maximal in G with respect to $A \cap H = B$. Then A is neat in G. We show that the socle of A is dense in $G[p]$. Let $x \in G[p]$. There is an element $a \in A$ such that $x = a + h$ where $h \in H$. Since $ph \in A \cap H = B$ and since B is pure, there exists $b \in B$ such that $ph = pb$. Thus

$$x = (a + b) + (h - b) \in \{A[p], H[p]\} \subseteqq \{A[p], \overline{B[p]}\} \subseteqq \overline{A[p]}.$$

By Theorem 1 in [6], A is pure and dense in G.

Observe that $B = A \cap H$ is cobounded in A since H is cobounded in G. It follows that B is a direct sum of cyclic groups if and only if A is a direct sum of cyclic groups. Hence we have the following corollary.

COROLLARY 2.5. *Let H be a cobounded subgroup of the primary group G. The correspondence $B \to B \cap H$ is a function from the basic subgroups of G onto the basic subgroups of H.*

3. The decompositions theorems.

THEOREM 3.1. *Suppose that the primary group G is a direct sum of cyclic groups. If H is a cobounded subgroup of G, then there exist a nonnegative integer k and a decomposition $G = \sum_{i=0}^{k} G_i$ such that $H \cong \sum_{i=0}^{k} p^i G_i$.*

Proof. Let $G[p] = P + H[p]$. Since P is a discrete subsocle [7] of G, it supports a pure subgroup A of G; indeed, P supports a p^k-bounded direct summand A of G if $p^k(G/H) = 0$. Let $G = A + G'$ and let H' be the image of H under the natural projection of G onto G'. Since $H \cap A = 0$, H' is isomorphic to H. Furthermore, $H'[p] = G'[p]$; hence it suffices to prove the theorem for the case $H[p] = G[p]$. Let $S = H[p] = G[p]$. Since H is a p^k-cobounded subgroup of G, the Ulm invariants of H and G are related by the inequalities:

$$\sum_n^{n+r} f_H(j) \leqq \sum_n^{n+r+k} f_G(j) \text{ and } \sum_{n+k}^{n+r+k} f_G(j) \leqq \sum_k^{n+r+k} f_H(j)$$

for all n, $r \geqq 0$, where f is the Ulm function. It follows from an obvious modification of Lemma 1 in [4] that there exist an automorphism π of S and decompositions $S = \sum P_n = \sum Q_n$ such that the nonzero elements of P_n and Q_n have height n in H and G, respectively, and such that for each $x_n \in P_n$ the relation $\pi(x_n) \in Q_{n+i}$ holds for some nonnegative $i \leqq k$. Hence there exist decompositions $G = \sum_{i=0}^{k} G_i$ and $H = \sum_{i=0}^{k} H_i$ of G and H such that $H_i \cong p^i G_i$ for $0 \leqq i \leqq k$. Thus $H \cong \sum_{i=0}^{k} p^i G_i$.

REMARK. The isomorphism between H and $\sum p^i G_i$ in Theorem 3.1 cannot be replaced by set theoretic equality. This can be demonstrated by very simple examples.

COROLLARY 3.2. *If H is a cobounded subgroup of the closed group G, then there exist a nonnegative integer k and a decomposition $G = \sum_{i=0}^{k} G_i$ such that $H \cong \sum_{i=0}^{k} P^i G_i$.*

Proof. Let B be a basic subgroup of G. It follows from Theorem 2.1 that $B \cap H$ is a basic subgroup of the cobounded subgroup H of G. Since $B \cap H$ is a cobounded subgroup of B, Theorem 3.1 implies that there exist a nonnegative integer k and a decomposition $B = \sum_{i=0}^{k} B_i$ such that $B \cap H \cong \sum_{i=0}^{k} p^i B_i$. Thus $G = \bar{B} = \sum_{i=0}^{k} \bar{B}_i$ and $H \cong \sum_{i=0}^{k} p^i \bar{B}_i$ since G and H are closed.

Our main decomposition theorem concerning quasi-isomorphism is the following.

THEOREM 3.3. *Suppose for the primary group G that $G = \sum_{\lambda \in \Lambda} G_\lambda$ where $G_\lambda / p^\omega G_\lambda$ is a direct sum of countable groups for each $\lambda \in \Lambda$. If H is quasi-isomorphic to G, then $H = \sum_{\lambda \in \Lambda} H_\lambda$ where H_λ is quasi-isomorphic to G_λ for each $\lambda \in \Lambda$.*

Proof. Since decompositions lift, for arbitrary G, from $p^n G$ to G, it suffices to prove the theorem for the case that H is a cobounded subgroup of G. Suppose that $p^k G \subseteqq H \subseteqq G$. In case G is a direct sum of cyclic groups, the theorem follows from Theorem 3.1 and the isomorphic refinement theorem for direct sums of cyclic groups. In fact, since H is p^k-cobounded in G, we can write $H = \sum_{\lambda \in \Lambda} H_\lambda$ where H_λ is isomorphic to a p^k-cobounded subgroup of G_λ. Thus, in the general case, we can write $H/p^\omega H = \sum_{\lambda \in \Lambda} H_\lambda^*$ where H_λ^* is isomorphic to a p^k-cobounded subgroup of $G_\lambda / p^\omega G_\lambda$. For each $\lambda \in \Lambda$, let G_λ^* be a cobounded subgroup of G_λ such that $G_\lambda^* / p^\omega G_\lambda \cong H_\lambda^*$. Set $G^* = \sum_{\lambda \in \Lambda} G_\lambda^*$. Now $p^\omega G^* = p^\omega G = p^\omega H$, and $G^*/p^\omega G^* = G^*/p^\omega G = \sum_{\lambda \in \Lambda}(G_\lambda^*/p^\omega G_\lambda) = \sum_{\lambda \in \Lambda} H_\lambda^* = H/p^\omega H$. It follows that $H \cong G^*$ by the uniqueness theorem [8] of Hill and Megibben, and the theorem is proved.

DEFINITION 3.4. A primary group G is said to be a pillared group if $G/p^{\omega}G$ is a direct sum of cyclic groups.

Since the property of being a direct sum of cyclic groups is a quasi-isomorphic invariant, the property of being a pillared group is a quasi-isomorphic invariant. Our Theorem 3.3 shows that for pillared groups quasi-isomorphism is compatible with direct decompositions. We conclude this section with the following consequence of Theorem 3.3.

COROLLARY 3.5. *Suppose that* $G = \sum_{\lambda \in \Lambda} G_{\lambda}$ *is a pillared group. If* H *is quasi-isomorphic to* G, *then* $H = \sum_{\lambda \in \Lambda} H_{\lambda}$ *where* H_{λ} *is isomorphic to a direct summand of* $G_{\lambda} + C_{\lambda}$ *and* C_{λ} *is a direct sum of cyclic groups.*

4. Some quasi-isomorphic variants.

As we have mentioned, it was established in [3] and [9] that the property of being a direct sum of cyclic groups and the property of being a direct sum of countable groups are invariant under the relation of quasi-isomorphism of primary groups; this is also an immediate consequence of Corollary 3.5. Cutler observed in [3] that the property of being a closed group is a quasi-isomorphic invariant, but he left open the following question. If G is a direct sum of closed groups and if H is quasi-isomorphic to G, does H have to be a direct sum of closed groups? The next theorem shows that the answer is negative.

Let $A = \sum \{a_n\}$ and $B = \sum \{b_n\}$ be copies of the standard basic subgroup, that is, let $\{a_n\}$ and $\{b_n\}$ denote cyclic groups of order p^n. Denote by $\bar{A}$ the closed group $\sum_T^* \{a_n\}$, the torsion completion of A. We want to consider the group $G = \bar{A} + B$ and a certain cobounded subgroup of G.

THEOREM 4.1. *The group* $G = \bar{A} + B$, *where* A *and* B *are copies of the standard basic subgroup, has a cobounded subgroup* H *with the following properties.*

(1) *H is not pure-complete.*

(2) *H is not semi-complete.*

(3) *H is not a direct sum of closed groups.*

Proof. Define $H = \{pG, a_1, a_{n+1} + b_n\}_{n<\omega}$. Let $S = \bar{A}[p]$, and observe that $S \subseteqq H$. We show that S does not support a pure subgroup of H. Assume that S does support a pure subgroup K of H. Since each element of S has the same height in H as in G, it follows that K is pure in G. Since $K[p] = \bar{A}[p]$, K is a closed group; hence K is a direct summand of G. In fact, we have the decompositions $G = K + B$ and $H = K + (H \cap B)$. It is easily verified that $H \cap B = pB$, so we have the equation $H = K + pB$. The defining equation for H and the above decompositions imply that $\{pK + pB, a_1, a_{n+1} + b_n\}_{n<\omega} = K + pB$. Thus, for each positive integer n,

$$p^n(a_{n+2} + b_{n+1}) = p^n k_{(n)} + p^{n+1} b_{(n)}$$

where $k_{(n)} \in K$ and $b_{(n)} \in B$.

Define $s(1) = 1$ and suppose that a positive integer $s(i)$ has been chosen for $i \leqq n$ such that $s(1) < s(2) < \cdots < s(n)$. Choose $s(n+1) > s(n)$ such that $b(n) \in \sum_{i<s(n+1)}\{b_i\}$. Since $p^2(p^n k_{(n)}) = 0$ for each n and since K is closed, $\sum p^{s(n)} k_{(s(n))}$ must converge in K. Since

$$p^n k_{(n)} = p^n a_{n+2} + (p^n b_{n+1} - p^{n+1} b_{(n)})$$

and since $G = \bar{A} + B$, it follows that $\sum (p^{s(n)} b_{s(n)+1} - p^{s(n)+1} b_{(s(n))})$ must converge in B. However, this is impossible since, for each positive integer n, the projection of the limit onto $\{b_{s(n)+1}\}$ is $p^{s(n)} b_{s(n)+1} \neq 0$. Thus S does not support a pure subgroup of H, and we have verified (1).

Assume that H is a direct sum of closed groups. Then H is a direct sum of a countable number of closed groups since G, and therefore H by Theorem 2.1, has a countable basic subgroup. It follows from Theorem 5.6 in [7] that a direct sum of a countable number of closed groups is pure-complete. Since we have already verified (1), we conclude that H is not a direct sum of closed groups. Furthermore, every semi-complete group [10] is a direct sum of closed groups, so the theorem is proved.

COROLLARY 4.2. *The property of being pure-complete is not a quasi-isomorphic invariant.*

COROLLARY 4.3. *The property of being semi-complete is not a quasi-isomorphic invariant.*

COROLLARY 4.4. *The property of being a direct sum of closed groups is not a quasi-isomorphic invariant.*

In view of Corollary 4.4, a natural question is: what are the groups that are quasi-isomorphic to direct sums of closed groups ? In this connection, we make the following observation.

PROPOSITION 4.5. If the primary group G is quasi-isomorphic to a direct sum of closed groups, then $G[p] = \sum S_\lambda$ where (1) S_λ is complete and (2) there exists a fixed positive integer k such that height $(x_1 + x_2 + \cdots + x_n) \leqq$ height $(x_i) + k$ if $x_i \in S_{\lambda_i}$ for distinct $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Proof. Suppose that G is quasi-isomorphic to a direct sum of closed groups. Then there are closed groups H_λ such that $G \doteq \sum_{\lambda \in \Lambda} H_\lambda$ and such that there exists an isomorphism π from $G[p]$ onto $\sum_{\lambda\in\Lambda} H_\lambda[p]$ that does not alter heights (computed in G and $H = \sum H_\lambda$) more than a fixed positive integer k. Defining S_λ by the equation $\pi(S_\lambda) = H_\lambda[p]$, we have S_λ's which satisfy the conditions (1) and (2).

5. **Quasi-isomorphism of direct sums of closed groups.** Although two primary groups G and H can be quasi-isomorphic with one a direct sum of closed groups and the other not, there is a particularly simple criterion which determines whether G and H are quasi-isomorphic in case both G and H are direct sums of closed groups.

THEOREM 5.1. *Suppose that the primary groups G and H are direct sums of closed groups. Then G and H are quasi-isomorphic if and only if there is a bounded group B and an isomorphism between $(G + B)[p]$ and $(H + B)[p]$ that does not alter heights more than a fixed positive integer k.*

Proof. If $G \overset{\bullet}{\cong} H$, there exist p^k-cobounded subgroups G^* and H^* of G and H, respectively, such that $G^* \cong H^*$. Define π from $G^*[p]$ onto $H^*[p]$ as the restriction of some isomorphism φ from G^* onto H^*. Let $G[p] = P + G^*[p]$ and $H[p] = Q + H^*[p]$. The height of a nonzero element in P or Q does not exceed k. For a sufficiently large p-bounded group $B, |P + B| = |Q + B|$ and π can be extended to an isomorphism from $(G + B)[p]$ onto $(H + B)[p]$ having the desired property that heights are not altered more than k.

For the proof of the nontrivial half of the theorem, we may assume that $B = 0$ since the relation of quasi-isomorphism is transitive. Thus suppose that $G = \sum_{\lambda\in\Lambda} G_\lambda$ and $H = \sum_{\mu \in M} H_\mu$ are direct sums of closed groups and that π is an isomorphism from $G[p]$ onto $H[p]$ that does not alter heights more than k. We wish to show that $G \overset{\bullet}{\cong} H$. Since G_λ or H_μ can be zero, there is no loss of generality in assuming that $\Lambda = M$. Thus we shall make this assumption. If Λ is countable, then $G + H$ is pure-complete and the proof that $G \overset{\bullet}{\cong} H$ is essentially the same as the proof of Corollary 1 in [4]. Let K be a pure subgroup of $G + H$ such that $K[p] = \{(x, \pi(x)) : \ x \in G[p]\}$. Then K is a subdirect sum of isomorphic cobounded subgroups G^* and H^* of G and H, respectively.

We now assume that Λ is uncountable and proceed by induction on the cardinality of Λ. According to the next theorem, there are decompositions $G = \sum_{\lambda\in\Lambda} G^*_\lambda$ and $H = \sum_{\lambda\in\Lambda} H^*_\lambda$ of G and H into closed groups such that, for each $\lambda \in \Lambda$, there exists a countable subset M_λ

of Λ having the property that

$$\varphi(G_\lambda[p]) \subseteqq \sum_{M_\lambda} H_\mu \text{ and } \varphi^{-1}(H_\lambda[p] \subseteqq \sum_{M_\lambda} G_\mu$$

for some isomorphism φ between $G[p]$ and $H[p]$ that alters heights no more than k. It follows that there is an ascending chain

$$\Lambda_0 \subseteqq \Lambda_1 \subseteqq \cdots \subseteqq \Lambda_\alpha \subseteqq \cdots, \Lambda_\alpha \subseteqq \Lambda ,$$

that leads up to Λ such that $\varphi(\sum_{\Lambda_\alpha} G_\lambda[P]) = \sum_{\Lambda_\alpha} H_\lambda[p]$ and $|\Lambda_\alpha| < |\Lambda|$. We conclude that there exist decompositions

$$G = \sum_\alpha \sum_{I_\alpha} G_i^* \text{ and } H = \sum_\alpha \sum_{I_\alpha} H_i^*$$

such that there is an isomorphism from $\sum_{I_\alpha} G_i^*[p]$ onto $\sum_{I_\alpha} H_i^*$ that does not alter heights more than k and such that $|I_\alpha| < |\Lambda|$. The proof of the theorem is finished by an application of the induction hypothesis; however, we owe a proof of the following theorem.

THEOREM 5.2. *For the primary groups G and H, suppose that $G = \sum_{\lambda \in \Lambda} G_\lambda$ and $H = \sum_{\lambda \in \Lambda} H_\lambda$ where G_λ and H_λ are closed groups. If π is an isomorphism from $G[p]$ onto $H[p]$ that alters heights no more than k, then there are decompositions $G = \sum_{\lambda \in \Lambda} G_\lambda^*$ and $H = \sum_{\lambda \in \Lambda} H_\lambda^*$ of G and H into closed groups and an isomorphism φ from $G[p]$ onto $H[p]$ that alters heights no more than k such that, for each $\lambda \in \Lambda$, there exists a countable subset M_λ of Λ such that*

$$\varphi(G_\lambda^*[p]) \subseteqq \sum_{M_\lambda} H_\lambda^* \text{ and } \varphi^{-1}(H_\lambda^*[p]) \subseteqq \sum_{M_\lambda} G_\lambda .$$

Proof. The proof is similar to the proof of Theorem 2 in [5], except in the present case most of the details are simplier. Here, we include only an outline of the proof. The following lemma is essential.

LEMMA 5.3. *Suppose that $G = B + \sum_{\lambda \in \Lambda} G_\lambda$ where B is a direct sum of cyclic groups and G_λ is closed for each $\lambda \in \Lambda$. If H is a closed group and if π is an isomorphism from $H[p]$ into $G[p]$ that does not decrease heights more than a fixed positive integer k, then there exists a positive integer n such that $\pi(p^n H \cap H[p])$ is contained in a finite number of the groups G_λ.*

Applying Lemma 5.3 to π and π^{-1} and working back and forth between G_λ and H_λ, we obtain decompositions $G = A + \sum_{\lambda \in \Lambda} G'_\lambda$ and $H = B + \sum_{\lambda \in \Lambda} H'_\lambda$ such that (i) A and B are direct sums of cyclic groups, (ii) $\pi(\sum_{\lambda \in \Lambda} G'_\lambda[\mathrm{p}]) = \sum_{\lambda \in \Lambda} H'_\lambda[\mathrm{p}]$, (iii) For each λ, there exists a finite subset M_λ of Λ such that

$$\pi(G'_\lambda[p]) \subseteqq \sum_{M_\lambda} H'_\mu \text{ and } \pi^{-1}(H'_\lambda[p]) \subseteqq \sum_{M_\lambda} G'_\mu ,$$

(iv) there is an isomorphism φ from $A[p]$ onto $B[p]$ that alters heights no more than k. This essentially finishes the proof of Theorem 5.2.

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