PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 2, 1967

WEIERSTRASS POINTS OF PLANE DOMAINS

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In this paper examples will be given of plane domains which have interior points as Weierstrass points.

The notion of a Weierstrass point of a Riemann surface (or an algebraic curve) is an old one, having been introduced more than a century ago. Surprisingly enough, some of the simplest and most immediate questions concerning Weierstrass points remain unanswered. A beautiful account of the fundamental theory of Weierstrass points can be found in [2]. Nevertheless, we shall, for the sake of convenience, recapitulate some of the basic facts concerning Weierstrass points.

Let $\mathscr{R}$ be a closed Riemann surface of genus $p > 1$ and let $u_1, \cdots, u_p$ be a basis of linearly independent Abelian integrals of the first kind on $\mathscr{R}$. Consider

$$W_a = \begin{vmatrix} \dfrac{du_1}{dz_\alpha}, \cdots, \dfrac{du_p}{dz_\alpha} \\ \vdots \qquad\qquad \vdots \\ \dfrac{d^p u_1}{dz_\alpha^p}, \cdots, \dfrac{d^p u_p}{dz_\alpha^p} \end{vmatrix},$$

defined in the neighborhood U_α of the local uniformizer z_α. If $U_\alpha \cap U_\beta$ is not empty and z_β is the uniformizer of U_β, then in the neighborhood $U_\alpha \cap U_\beta$ we have

$$W_\alpha = \left(\frac{dz_\beta}{dz_\alpha}\right)^{\frac{p(p+1)}{2}} W_\beta .$$

Thus $\{W_\alpha\}$ defines an everywhere finite differential of dimension

$$\frac{p(p+1)}{2}$$

on $\mathscr{R}$. (Some authors refer to the degree of a differential rather than its dimension, others speak of its order. See [4].) Such a differential will have

$$(2p-2)\cdot\frac{p(p+1)}{2} = (p-1)p(p+1)$$

zeros (counted with proper multiplicity). These zeros are the Weierstrass points of $\mathscr{R}$. They do not depend on the particular basis

$u_1, \cdots, u_p$ chosen but are a fixed set of points of $\mathscr{R}$,

$$(p-1)p(p+1)$$

in number.

In order to introduce another basic (and defining) property of Weierstrass points, we consider the following problem: for a given point $q \in \mathscr{R}$ let us try to construct on $\mathscr{R}$ a function which is complex analytic and regular everywhere on $\mathscr{R}$ except at q and at q has a pole of order n. For all but a finite number of points, viz, all but the Weierstrass points we must have $n \geqq p+1$, but at each Weierstrass point we may choose an $n \leqq p$. We shall return to this interpretation later. The connection between the two definitions is easily established, e.g. see [2].

There is still a third fundamental definition of Weierstrass points which we shall use. This definition employs the "Noether mapping" υ of a nonhyperelliptic Riemann surface $\mathscr{R}$ of genus p into $\mathbf{P}_{p-1}(\mathrm{C})$, the complex projective space of dimension $p-1$ (see [1]), and can only be employed if $\mathscr{R}$ is nonhyperelliptic. The mapping υ is accomplished by selecting a basis of Abelian differentials of the first kind on $\mathscr{R}$ and considering them as homogeneous coordinates in $\mathbf{P}_{p-1}(\mathrm{C})$. Then $\upsilon(\mathscr{R})$, the image of $\mathscr{R}$ in $\mathbf{P}_{p-1}(\mathrm{C})$, is a nonsingular curve of degree $2p-2$. The Weierstrass points of $\mathscr{R}$ are those point of $\upsilon(\mathscr{R})$ at which the osculating hyperplane is hyperosculating, and the degree of the hyper-osculation is the order of the Weierstrass point (see [2]). So, for example, if $\mathscr{R}$ is of genus 3 and nonhyperelliptic then $p-1=2$ and $\upsilon(\mathscr{R})$ is a plane curve of degree 4 and the Weierstrass points are the inflection points of $\upsilon(\mathscr{R})$.

Actually, what we are going to consider are Weierstrass points on plane domains, and we have only defined them so far on compact Riemann surfaces. But these definitions can be extended to plane domains by employing a technique due to Schottky, viz. the technique of doubling a plane domain (see [5] and [4]). We consider here only domains of finite connectivity which are bounded by analytic Jordan curves. Thus, if $\mathscr{D}$ is a plane domain with $p+1$ boundary curves, its double is a compact Riemann surface $\mathscr{R}$ of genus p. $\mathscr{D}$ and its boundary curves are contained in $\mathscr{R}$ as "half" of the Riemann surface; the other "half" $\mathscr{D}^*$ has the same boundary curves. There is an anti-analytic involution of $\mathscr{R}$ onto itself which maps $\mathscr{D}$ anti-conformally onto $\mathscr{D}^*$ and leaves the boundary curves pointwise fixed. All of those Riemann surfaces of genus p which are the doubles of plane domains can be characterized in a way which will prove very useful in this study (see [1]).

Let us denote the involution on $\mathscr{R}$, referred to above, by * and

$z \in \mathscr{D}$, we shall denote its image in $\mathscr{D}^*$ under $*$ by z^*. Thus, if B denotes the set of points of the boundary curves of $\mathscr{D}$, then $z = z^*$ is equivalent to saying that $z \in B$. We can now utilize this fact to show that if $f(z)$ is any function on $\mathscr{R}$ we can construct another function $g(z)$ on $\mathscr{R}$ which is real on B. We do this as follows: define

$$g(z) = f(z) + \overline{f(z^*)} .$$

Then

$$g(z^*) = \overline{g(z)} ,$$

therefore, if $z_1 \in B$, $g(z_1)$ must be real.

If $v_1, \cdots, v_p$ is a basis of Abelian integrals of the first kind on $\mathscr{R}$, then so is $u_1, \cdots, u_p$, where

$$u_j(z) = v_j(z) + \overline{v_j(z^*)}, j = 1, \cdots, p .$$

But then each $u_j(z)$ is real on B, up to an additive constant; therefore, $(du_j/dz)\cdot(dz/ds)$ is real on B, where $dz/ds = \dot{z}$ is the derivative of z with respect to arc length, i.e. the unit tangent to B at the point in question (each curve of the boundary is oriented naturally by the orientation of $\mathscr{D}$).

We can now consider

$$W(z) = \begin{vmatrix} \dfrac{du_1}{dz}, \cdots, \dfrac{du_p}{dz} \\ \vdots \qquad \vdots \\ \dfrac{d^pu_1}{dz^p}, \cdots, \dfrac{d^pu_p}{dz^p} \end{vmatrix} ,$$

and we see that $W(z)\ (dz/ds)^{p(p+1)/2}$ is real on B. The Weierstrass points of $\mathscr{D}$ are by definition those points of $\mathscr{D}$ for which $W = 0$. But it is clear that if $z \in \mathscr{D}$ is a Weierstrass point of $\mathscr{R}$, so is $z^* \in \mathscr{D}^*$. So at most, half of the Weierstrass points of $\mathscr{R}$ can lie in $\mathscr{D}$. We say "at most" half instead of "exactly" half, because some of the Weierstrass points may lie on B and so do not lie either in $\mathscr{D}$ or $\mathscr{D}^*$. It is just this point which we wish to discuss.

2. A question. The question has been asked whether or not all Weierstrass points of plane domains lie on the boundary. Indeed, if the plane domain is hyperelliptic (i.e. if its double is a hyperelliptic Riemann surface), this is actually the case. Since all previous examples (of which the author knows) of plane domains, in which Weierstrass points have been located, are hyperelliptic, the question might be expected to have an affirmative answer on the basis of experience.

But let us phrase the question more explicitly in another way.

QUESTION. *Do plane domains exist which have interior points as Weierstrass points? What connectivity can they have?*

3. An answer. In this section we shall construct an example of a plane domain which answers the question in § 2 above. In fact, we shall show that there exist plane domains of every finite connectivity greater than three which have interior Weierstrass points.

We construct the first example, a domain of connectivity four, from an algebraic plane curve—the domain is to be conformally equivalent to "half" of the Riemann surface of this curve. The construction proceeds as follows. Let us define

$$\Phi_1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 ,$$

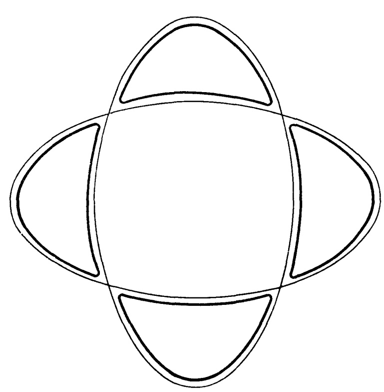
$$\Phi_2 = \frac{x^2}{b^2} + \frac{y^2}{a^2} - 1 ,$$

$$\Psi_\varepsilon = \Phi_1 \Phi_2 + \varepsilon ,$$

where a, b, ε are real, $\varepsilon > 0$ and $a \neq b$ (for example let $a = 2,\ b = 1$). Then the curve

$$\Psi_\varepsilon = 0 ,$$

which is pictured below (with the curve $\Phi_1\Phi_2 = 0$ drawn in lightly to show how $\Psi_\varepsilon = 0$) arises is a nonsingular plane curve of degree 4. Of course $\Psi_\varepsilon = 0$ gives rise to a nonsingular algebraic curve of degree 4 in $\boldsymbol{P}_2(C)$ obtained by making the polynomial Ψ_ε homogeneous and using complex homogeneous coordinates. Let us denote this curve by $\mathscr{C}$ and the curve in $\boldsymbol{P}_2(\boldsymbol{R})$, the real projective plane, by $\mathscr{C}_{\boldsymbol{R}}$. Thus $\mathscr{C}_{\boldsymbol{R}}$ arises from



$\Psi_\varepsilon = 0$ upon introducing homogeneous coordinates. Also

$$\mathscr{C}_{\boldsymbol{R}} = \mathscr{C} \cap \boldsymbol{P}_2(\boldsymbol{R}) ,$$

i.e. it is that part of $\mathscr{C}$ which lies in $\boldsymbol{P}_2(\boldsymbol{R})$.

According to a theorem of Harnack, the curve $\mathscr{C}_{\boldsymbol{R}}$ (and hence $\mathscr{C}$) must be of genus at least 3 (because $\mathscr{C}_{\boldsymbol{R}}$ has four components). On the other hand $\mathscr{C}_{\boldsymbol{R}}$, or $\mathscr{C}$, is of degree 4 so can have genus at most equal to 3. Since the genus must then be exactly 3 and the degree is 4, it is clear that $\mathscr{C}$ is nonsingular. For the theorem of Harnack, see [3, p. 257].

Let $\mathscr{R}$ denote the Riemann surface of $\mathscr{C}$; then $\mathscr{C}$ is the image of $\mathscr{R}$ under the Noether imbedding υ referred to earlier. This is easily seen as follows: let x_0, x_1, x_2 be homogeneous coordinates in $\boldsymbol{P}_2(\boldsymbol{C})$ such that

$$x = \frac{x_1}{x_0} , y = \frac{x_2}{x_0} ,$$

then x_0, x_1, x_2 are linearly independent homogeneous polynomials of degree 1; and since the degree n of the nonsingular curve $\mathscr{C}$ is 4, $4 - 3 = 1$, i.e. the adjoint curves of degree $n - 3$ are lines, we see that x_0, x_1, x_2 restricted to $\mathscr{C}$ form a basis of differentials of the first kind on $\mathscr{C}$, hence $\mathscr{C}$ is then the Noether imbedding, with respect to this basis, of its Riemann surface.

There is a characterization of those Riemann surfaces which are the doubles of plane domains [1]. This characterization says that a Riemann surface $\mathscr{R}$ of genus p is nonhyperelliptic and the double of a plane domain if and only if the image of $\mathscr{R}$ in $\boldsymbol{P}_{p-1}(\boldsymbol{C})$, under the Noether imbedding, intersects $\boldsymbol{P}_{p-1}(\boldsymbol{R}) \subset \boldsymbol{P}_{p-1}(\boldsymbol{C})$ in $p + 1$ mutually disjoint Jordan curves. This characterization shows us immediately that the Riemann surface $\mathscr{R}$ of the curve $\mathscr{C}$ is the double of a plane domain, and is nonhyperelliptic.

The points of $\mathscr{C}$ and the points of $\mathscr{R}$ are in a natural one to one correspondence and may therefore be identified. The Weierstrass points of $\mathscr{R}$ then coincide with the inflection points of $\mathscr{C}$. These number 24 when counted with proper multiplicity. The order of a Weierstrass point and the order of inflection of an inflection point are the same.

From the illustration of $\mathscr{C}_{\boldsymbol{R}}$ it is clear that it has exactly eight inflection points; and the curvature changes sign at each of these points. Since the curvature does change sign, the order of each inflection point must be odd. By the symmetries of $\mathscr{C}_{\boldsymbol{R}}$ all the inflection points must have the same order; therefore, the order must be either 1 or 3. If the order were 3, however, each Weierstrass point of $\mathscr{R}$ would be of maximum order, which would mean that $\mathscr{R}$ would be hyperelliptic. Since $\mathscr{R}$ is not hyperelliptic (as we have seen above), we know that the order of each inflection point is 1. This means that $\mathscr{C}$ has 16 inflection points (counted with proper multiplicity) which do not lie on $\mathscr{C}_R$.

Now let $\mathscr{D}$ be a plane domain whose double is $\mathscr{R}$. (How to make an actual determination of $\mathscr{D}$ from a knowledge of $\mathscr{R}$ or $\mathscr{C}$ is discussed in [1]). Then we have shown that $\mathscr{D}$ has 8 Weierstrass points (counted with multiplicity) in its interior. This answers the question of existence in the case of quadruply connected domains. But since a Weierstrass point will move an arbitrarily small distance under a sufficiently small deformation of the conformal structure of $\mathscr{D}$ (see [4]), we see that the existence question is answered for domains of connectivity n for every $n \geqq 4$. For we can punch as many holes as we please in $\mathscr{D}$ and if we make them all small enough, we can move our original 8 Weierstrass points as little as we please. (Of course, in the process of hole punching we acquire many more Weierstrass points.)

We have now answered the question asked in § 2. But the discussion there prompts us to ask a further question: If a plane domain has all of its Weierstrass points on the boundary, is it necessarily hyperelliptic? We do not as yet have an anwer to this question.

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Received July 8, 1966. This work was supported in part by a National Science Foundation grant GP 4069.

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