PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 3, 1967

MULTIPLIERS AND H^* ALGEBRAS

WAI-MEE CHING AND JAMES S. W. WONG

Let A be a normed algebra and $B(A)$ the algebra of all bounded linear operators from A into itself, with operator norm. An element $T \in B(A)$ is called a multiplier of A if $(Tx)y = x(Ty)$ for all $x, y \in A$. The set of all multipliers of A is denoted by $M(A)$. In the present paper, it is first shown that $M(A)$ is a maximal commutative subalgebra of $B(A)$ if and only if A is commutative. Next, $M(A)$ in case A is an H^*-algebra will be represented as the algebra of all complexvalued functions on certain discrete space. Finally, as an application of the representation theorem of $M(A)$, the set of all compact multipliers of compact H^*-algebras is characterized.

In case A is commutative, the general notion of multipliers was first studied by Helgason [7], followed by Wang [12] and Birtel [2], [3], [4]. In the special case when $A = L_1(G)$, the group algebra over an arbitrary locally compact abelian group, the problem of multipliers has also been studied by Helson [8] and Edwards [5]. (Cf. also Rudin [11].) Helgason [7] called a function g on the maximal ideal space $\mathscr{M}$ of A a multiplier if $g\hat{A} \subseteqq \hat{A}$ where $\hat{A}$ is the Gelfand transform of A. Later Wang [12] and Birtel [2] carried out more systematic studies on multipliers. In case A is semi-simple, Wang [12] proved that there exists a norm-decreasing isomorphism between $M(A)$ and $C^\infty(\mathscr{M})$, the algebra of bounded continuous functions of $\mathscr{M}$. In particular if $A = L_1(G)$, then $M(A) = M(G)$, the algebra of all bounded regular Borel measures on G. In the noncommutative case, Wendel [13] first studied multipliers[1] for noncommutative group algebras, followed by Kellogg [9] for H^*-algebras. However, since Kellogg's proofs rely heavily on the representation theorem of Wang [12] for multipliers on general commutative semi-simple Banach algebras, revelent results on multipliers of H^*-algebras were obtained only for the commutative case.

2. Multiplier algebras. Let A be a normed algebra. A is said to *without order* if either $xA = \{0\}$ or $Ax = \{0\}$ implies $x = 0$. Clearly, if A is semi-simple or A has a unit, then A is without order. In the sequal, we assume all normed algebras under consideration are without order. An element $T \in B(A)$ is called a *right* (*left*) multiplier

[1] Both Kellogg [**9**] and Wendel [**13**] used the terminology "centralizers" instead of "multipliers".

of A if $T(xy) = (Tx)y(T(xy) = x(Ty))$. We denote the set of all right (left) multipliers of A by $R(A)(L(A))$. We first observe the following:

PROPOSITION 1. $R(A) \cap L(A) = M(A)$.

Proof. Clearly, we have $R(A) \cap L(A) \subseteqq M(A)$. Let $T \in M(A)$. Note that $(T(xy))z = (xy)Tz = x(y(Tz)) = x((Ty)z)$ for all $x, y, z \in A$. Since A is without order, $T(xy) = x(Ty)$, i.e. $T \in R(A)$. Similarly, one easily shows that $T \in L(A)$, completing the proof.

A commutative subalgebra Y of an algebra X is called *maximal commutative subalgebra* of X if Y is not properly contained in any proper commutative subalgebra of X. If X has an identity element e, e belongs to any maximal commutative subalgebra of X. Using an argument based upon Zorn's lemma, one easily shows that $M(A)$ is contained in some maximal commutative subalgebra of $B(A)$, say $MC(A)$.

For an arbitrary normed algebra X, we denote its centre by $Z(X)$. One can easily verify the following inclusions:

$$Z(B(A)) \subseteqq Z(M(A)) \subseteqq M(A) \subseteqq MC(A) \subseteqq B(A) .$$

Kellogg [9] proved that $M(A)$ is a closed commutative subalgebra of $B(A)$, consequently we always have $M(A) = Z(M(A))$. More precisely, we can prove the following:

PROPOSITION 2. Let A be a normed algebra. Then the algebra $M(A)$ of all multipliers of A is a closed commutative sub-algebra of $B(A)$, the algebra of all bounded linear operators in A with operator norm.

Proof. Let $T_n \in M(A)$ and $\|T_n - T\| \to 0$, for $n = 1, 2, 3, \cdots$. We note that for any $x, y \in A$,

$$\begin{aligned} \|x(Ty) - (Tx)y\| &\leqq \|x(Ty) - x(T_n y)\| + \|(T_n x)y - (Tx)y\| \\ &\leqq 2\|x\|\,\|y\|\,\|T_n - T\| . \end{aligned}$$

Letting n tend to infinity, we have $x(Ty) = (Tx)y$. Thus $T \in M(A)$, and $M(A)$ is closed. These remarks together with the result of Kellogg complete the proof of the assertion.

From Proposition 2, we may easily deduce that all subalgebras of $B(A)$ occurring (*) are closed in $B(A)$.

PROPOSITION 3. Let $\mathscr{S}_{pA}(x)$ denote the spectrum of an element $x \in A$. Then $\mathscr{S}_{pB(A)}(T) = \mathscr{S}_{pM(A)}(T)$.

Proof. Since both $B(A)$ and $M(A)$ contain the identity, we need only to prove that for $T \in M(A)$ if T^{-1} exists and is in $B(A)$, then $T^{-1} \in M(A)$. For any $x, y \in A$, we observe that

$$(T^{-1}x)y = (T^{-1}x)(TT^{-1}y) = (TT^{-1}x)(T^{-1}y) = x(T^{-1}y) ,$$

THEOREM 1. *$M(A)$ is maximal commutative subalgebra of $B(A)$ if and only if A is commutative.*

Proof. Let A be commutative, and for each $x \in A$, we write T_x, ${}_xT$ the left and right regular representations of x in $B(A)$. Since A is commutative, $[A] = \{T_x: x \in A\} = \{{}_xT: x \in A\} \subseteqq M(A)$. Suppose A is not maximal, and let $MC(A)$ be some maximal commutative subalgebra containing A. Since A is not maximal, we may pick $T \in MC(A)\backslash M(A)$. On the other hand, $T \in MC(A)$ implies that T commutes with all elements of $[A]$, i.e., for all $x, y \in A$ $(Tx)y = (TT_x)y = (T_xT)y = x(Ty)$, proving that $T \in M(A)$. This contradiction establishes that A is maximal. Conversely let $M(A)$ be a maximal commutative algebra. Thus $T \in B(A)$, and $ST = TS$ for all $S \in M(A)$ imply $T \in M(A)$. In particular, $(T_xS)y = x(Sy) = (Sx)y = (ST_x)y$ and hence $T_x \in M(A)$ for all $x \in A$. Thus $(xy)z = T_x(yz) = y(T_xz) = (yx)z$ for all $x, y, z \in A$. Since A is without order, $xy = yx$ for all $x, y \in A$, i.e., A is commutative.

We will see from § 3 and § 4 that in case A is a simple H^*-algebra, then $M(A) = Z(B(A))$.

REMARK 1. If A is in addition complete, then $M(A)$ is also a Banach algebra. In this case, we may define $T \in M(A)$ as any mapping of A into itself satisfying the condition that $(Tx)y = x(Ty)$ for all $x, y \in A$. From the fact that A is without order, it is easily seen that T is linear. As a consequence of closed graph theorem, we may also show that T is bounded (see Wang [12]). The way we choose to define multipliers is just a matter of convenience. Note that throughout all of our discusson, we do not assume A to be complete.

3. Lemmata on matrix algebras. Let X_S be the algebra of all matrices $(x_{\alpha\beta})$, $\alpha, \beta \in S$, where S is a fixed set of indices and $x_{\alpha\beta}$'s are complex numbers satisfying the condition $\sum_{\alpha,\beta} |x_{\alpha\beta}|^2 < \infty$. The multiplication is defined by

$$z = (z_{\alpha\beta}) = x \cdot y = (x_{\alpha\beta})(y_{\gamma\delta}) ,$$

where

$$z_{\alpha\beta} = \sum_{\gamma \in S} x_{\alpha\gamma} y_{\gamma\beta} .$$

This multiplication is well defined since

$$\sum_{\alpha,\beta} |Z_{\alpha\beta}|^2 = \sum_{\alpha,\beta} \left|\sum_{\gamma} x_{\alpha\gamma} y_{\gamma\beta}\right|^2 \leqq \left(\sum_{\alpha,\beta} |x_{\alpha\beta}|^2\right)\left(\sum_{\gamma,\delta} |y_{\gamma\delta}|^2\right) < \infty .$$

We define an inner product on X_S by $(x, y) = w \sum_{\alpha,\beta} x_{\alpha\beta}\bar{y}_{\alpha\beta}$, where w is a fixed constant $\geqq 1$. X_S becomes a Banach algebra if the norm is induced by the inner product in the usual manner, i.e. $\|x\|^2 = (x, x)$. In this cases, $B(X_S)$ can be identified with a subalgebra of all matrices $T = (t_{\alpha\beta\gamma\delta})$ over $S \times S$ such that $Tx = y$ is defined by

$$y_{\alpha\beta} = \sum_{(\gamma,\delta)} t_{\alpha\beta\gamma\delta} x_{\gamma\delta}$$

with $\sum_{\alpha,\beta} |y_{\alpha\beta}|^2 < \infty$. (We refer to Naimark [10] for more detailed discussion of X_S.)

LEMMA 1. *$T \in M(X_S)$ if and only if T is a scalar multiple of the identity operator.*

Proof. Let $T = (t_{\alpha\beta\gamma\delta}) \in M(X_S)$, so $(Tx)y = T(xy)$ for all $x,y \in X_S$. For any fixed pair of indices $(\sigma, \tau) \in S \times S$, let $x_{\sigma\tau} = 1$, $x_{\alpha\beta} = 0$ if $(\alpha, \beta) \neq (\sigma, \tau)$ and $y_{\sigma\sigma} = 1$, $y_{\tau\tau} = -1$, $y_{\alpha\beta} = 0$ otherwise. Denote $z = (z_{\alpha\beta}) = (Tx)y = T(xy)$. Observe from $z = (Tx)y$ that

$$\sum_{\xi} \left(\sum_{(\gamma,\delta)} t_{\alpha\xi\gamma\delta} x_{\gamma\delta}\right)(y_{\xi\beta}) = \sum_{\xi} t_{\alpha\xi\sigma\tau} y_{\xi\beta} ,$$

and hence $z_{\alpha\sigma} = t_{\alpha\sigma\sigma\tau}$, $z_{\alpha\tau} = -t_{\alpha\tau\sigma\tau}$, $z_{\alpha\beta} = 0$ otherwise. On the other hand, from $z = T(xy)$ we have

$$\sum_{(\gamma,\delta)} t_{\alpha\beta\gamma\delta}\left(\sum_{\xi} x_{\gamma\xi} y_{\xi\delta}\right) = \sum_{\sigma} t_{\alpha\beta\sigma\delta} y_{\tau\delta} = -t_{\alpha\beta\sigma\tau} .$$

From these computation, we obtain that $t_{\alpha\beta\sigma\tau} = 0$ if $\beta \neq \sigma$ and $\beta \neq \tau$. In case $\beta = \sigma$, we have $t_{\alpha\sigma\sigma\tau} = -t_{\alpha\sigma\sigma\tau}$ and so again $z_{\alpha\beta} = 0$. Hence we conclude that $t_{\alpha\beta\sigma\tau} = 0$ unless $\beta = \tau$. Similarly, from $x(Ty) = T(xy)$ we obtain $t_{\alpha\beta\sigma\tau} = 0$ unless $\alpha = \sigma$. Since σ, τ are arbitrary, we have $t_{\alpha\beta\sigma\tau} \neq 0$ only if $(\alpha, \beta) = (\sigma, \tau)$. Next we choose $x_{\sigma\tau} = 1$, $x_{\alpha\beta} = 0$ if $(\alpha, \beta) \neq (\sigma, \tau)$ and $y_{\mu\nu} = 1$, $y_{\alpha\beta} = 0$ if $(\alpha, \beta) \neq (\mu, \nu)$ in the equation $(Tx)y = x(Ty)$. It is readily seen from a similar computation that $t_{\alpha\beta\alpha\beta} = t_{\gamma\delta\gamma\delta}$ for all $\alpha, \beta, \gamma, \delta \in S$. Thus if $T \in M(X_S)$, then T must be a scalar multiple of the identity operator.

LEMMA 2. $M(X_S) = Z(B(X_S))$.

Proof. In view of the inclusion relation (*), we need only to show that if $T \in Z(B(X_S))$, then $T \in M(X_S)$. Let $T = (t_{ij})$, $i, j \in S \times S$, such that for two fixed distinct indices $k, h \in S \times S$, $t_{kk} = a \neq t_{hh} = b$ and $t_{ij} = 0$ otherwise. From Lemma 1, we clearly have $T \notin M(A)$. Define $T_1 \in B(A)$, $T_1 = (t'_{ij})$, by $t'_{kk} = 1$, and $t'_{ij} = 0$ otherwise. It is readily seen by a direct computation that $TT_1 \neq T_1T$, hence $T \notin Z(B(X_S))$, proving the assertion.

4. **H^*-algebras.** An H^*-algebra A is a Banach $*$-algebra (a Banach algebra with involution) and a Hilbert space, where the Banach algebra norm coincides with the Hilbert space norm, with the the crucial connecting property $(xy, z) = (y, x^*y)$. It is assumed that for each $x \in A$, $\|x^*\| = \|x\|$ and $x^*x \neq 0$ if $x \neq 0$. A simple example of an H^*-algebra is the matrix algebra X_S introduced in § 3. In fact, X_S is a simple H^*-algebra, and indeed every simple H^*-algebra is isometric and $*$-isomorphic to some matrix algebra X_S. In general, Ambrose [1] proved that every H^*-algebra is the direct, and at the same time orthogonal, sum of its closed minimal two-sided ideals which are simple H^*-algebras. (Naimark [10], p. 331).

LEMMA 3. *Let A be a normed algebra which is the direct sum of closed two-sided ideals $\{I_\alpha : \alpha \in \mathscr{E}\}$ in A. If $T \in M(A)$, then T maps each I_α into itself.*

Proof. Let $x \in I_\alpha$ for some fixed $\alpha \in \mathscr{E}$. Suppose that $(Tx)_\beta \neq 0$, i.e. The projection of Tx into I_β, for some $\beta \neq \alpha, \beta \in \mathscr{E}$. We may choose $y \in I_\beta$, $y \neq 0$, such that $(Tx)y = (Tx)_\beta y = 0$. (For otherwise, if $(Tx)_\beta I_\beta = 0$, then

$$(Tx)_\beta A = (Tx)_\beta \Big(\oplus \sum_{\alpha \in \mathscr{E}} I_\alpha \Big) = (Tx)_\beta I_\beta = 0 \ ,$$

contradicting the fact that A is without order.) But on the other hand, $T(xy) = T \cdot 0 = 0$, violating the multiplier condition. Thus, $(Tx)_\beta = 0$, i.e. T maps each I_α into itself.

Denote by T_α the restriction of T to I_α. It is clear that if $T \in M(A)$, then $T_\alpha \in M(I_\alpha)$ for each $\alpha \in \mathscr{E}$. Hence we may write

$$TA = T\Big(\oplus \sum_{\alpha \in \mathscr{E}} I_\alpha \Big) = \oplus \sum_{\alpha \in \mathscr{E}} TI_\alpha = \oplus \sum_{\alpha \in \mathscr{E}} T_\alpha I_\alpha \ .$$

We note that for each $T \in M(A)$, there corresponds a unique set $\{T_\alpha\}$ where $T_\alpha \in M(I_\alpha)$.

THEOREM 2. *Let A be an H^*-algebra, and $\{I_\alpha : \alpha \in \mathscr{E}\}$ the set of all minimal closed two-sided ideals in A. Denote by E the topological space of the set of all minimal closed two-sided ideals in A with the* *discrete topology. Then there exists a $*$-isomorphism which is at the same time an isometry of $M(A)$ onto $C^\infty(E)$, the space of all bounded continuous complex functions on E.*

Proof. From the structure theorem of H^*-algebras, we know that $A = \bigoplus \sum_\alpha I_\alpha$ of all its closed minimal ideals which are simple H^*-algebras, $*$-isomorphic and isometric to some matrix algebras X_{s_α}. For each $T \in M(A)$, let $\{T_\alpha: \alpha \in \mathscr{E}\}$ be the corresponding set of multipliers of I_α. By Lemma 1, T_α must be a scalar multiple of the identity operator P_α, say $T_\alpha = t(\alpha)P_\alpha$, for some complex number $t(\alpha)$ depending on T. Define $\Phi: M(A) \to C(E)$, the space of all complex-valued functions on E by $\Phi(T)(\alpha) = t(\alpha)$ for each $\alpha \in E$. Clearly Φ is linear, multiplicative and preserves involution. (i.e., $*$ operations for elements in A, complex conjugation for elements in $C^\infty(E)$ and operator adjoint for elements in $M(A)$.) To show that Φ is isometric, we observe

$$\begin{aligned} \| Tx \|^2 &= \left\| T\Big(\bigoplus \sum_\alpha x_\alpha\Big)\right\|^2 = \left\| \bigoplus \sum_\alpha T_\alpha x_\alpha \right\|^2 \\ &= \sum_\alpha \| T_\alpha x_\alpha \|^2 = \sum_\alpha \| t(\alpha) x_\alpha \|^2 \leqq \| \Phi(T) \|^2 \| x \|^2 \end{aligned}$$

and hence $\| T \| \leqq \| \Phi(T) \|$. Conversely, we have for some $x_\alpha \neq 0$,

$$| \Phi(T)(\alpha) | = | t(\alpha) | = \frac{\| T_\alpha x_\alpha \|}{\| x_\alpha \|} \leqq \| T_\alpha \| \leqq \| T \| ,$$

proving $\| \Phi(T) \| \leqq \| T \|$. Thus, Φ is indeed an isometry, and being linear, it is one-to-one. On the other hand for each $f \in C^\infty(E) \subseteqq C(E)$, let $T_\alpha = f(\alpha)P_\alpha$. It is readily seen that the mapping T determined by $\{T_\alpha\}$ belongs to $M(A)$ and satisfies $\Phi(T) = f$. Thus, we conclude that Φ is an isometric $*$-isomorphism from $M(A)$ onto $C^\infty(E)$.

We note that the present proof differs from its commutative counterpart [9] in the use of Ambrose's structure theorem [1] for H^*-algebras instead of Gelfand's representation for general commutative Banach Algebras.

REMARK 2. We note that the orthogonal complement of each minimal closed two-sided ideal is a maximal closed two-sided ideal, and vice versa. Hence the space of all minimal closed two-sided ideals is homeomorphic to the space of all maximal closed two-sided ideals. Thus, in case A is commutative, the above representation theorem reduces to that of Kellogg's (Theorem (4.1), [9]).

REMARK 3. From Lemma 2 and the above theorem, it is easily seen that if A is a H^*-algebra then $M(A) = Z(B(A))$ if and only if A is simple.

REMARK 4. The result of Theorem 2 remains valid for any algebra which is the direct sum of ideals $\{I_\alpha\}$ such that each ideal is isomorphic and isometric to some matrix algebra. The isometry of $M(A)$ and $C^\infty(E)$ can be proved without using the orthogonality of the direct sum in an H^*-algebra.

REMARK 5. Since $M(A)$ is a commutative involutory algebra, it is also contained in the set of all normal operators on A.

REMARK 6. Since $M(A)$ is $*$-isomorphic and isometric to $C^\infty(E)$, its maximal ideal space is homeomorphic to the Stone-Čech compactification of the discrete space E. (See [6], Chapter 6).

REMARK 7. A Banach $*$-algebra A with identity e is called *completely symmetric* if for each $x \in A, (e + x^*x)^{-1} \in A$. (See Naimark [10], p. 299.) It is clear that $C^\infty(E)$ and hence $M(A)$ is completely symmetric. In particular, the Shilov boundary of $M(A)$ coincides with its maximal ideal space. (Cf. Naimark [10], p. 218.)

Another interesting example of H^*-algebras is the group algebra $L_2(G)$, where G is an arbitrary compact group. In this case, all the minimal closed two-sided ideals of $L_2(G)$ are isomorphic and isometric to finite dimensional simple H^*-algebras, or equivalently X_{S_α}, with S_α finite for each $\alpha \in \mathscr{E}$ (see [1].). In the following, we will prove a result for the set of all multipliers which are at the same time compact operators in case A is a H^*-algebra whose minimal closed two-sided ideals are finite-dimensional. (Such an algebra will be called *compact H^*-algebra*. Clearly, every commutative H^* algebra is a compact H^*-algebra.)

THEOREM 3. *Let A be a H^*-algebra whose minimal closed two-sided ideals are finite dimensional, and $M_C(A)$ the set of all compact operators in $M(A)$. Then $\Phi(M_C(A)) = C_0(E)$, the algebra of all continuous functions on E which vanish at infinity.*

Proof. Since every I_α is finite dimensional, each $T_\alpha \in M(I_\alpha)$ is a scalar multiple of the identity operator P_α, and hence compact. For any finite set $F \subseteq E$, if we define

$$T = \sum_{\alpha \in F} T_\alpha = \sum_{\alpha \in F} c_\alpha P_\alpha ,$$

where c_α are complex constants, T is the finite sum of compact operators and thus again compact. Let $C_K(E)$ be the algebra of all continuous functions on E with compact support. We have just seen that $\Phi^{-1}(C_K(E)) \subset M_C(A)$. Since $\overline{C_K(E)} = C_0(E)$, thus $\overline{\Phi^{-1}(C_K(E))} = \overline{\Phi^{-1}(C_K(E))}$. However, $M_C(A)$ is the intersection of the closed subalgebra $M(A)$ and the closed ideal of all compact operators in $B(A)$, and is thus closed. As a consequence, we have $\overline{\Phi^{-1}(C_K(E))} \subseteqq M_C(A)$. On the other hand, suppose that there exists a $T \in M_C(A)$ such that $\Phi(T) = f \notin C_0(E)$, i.e., there exists $\varepsilon > 0$ such that the set $G = \{\alpha \in E: |f(\alpha)| \geqq \varepsilon\}$ is infinite. For each $\alpha \in \mathscr{E}$, choose $x_\alpha \in I_\alpha$ with $\|x_\alpha\| = 1$. Note that $\{x_\alpha\}$ is a bounded sequence, but $\{Tx_\alpha\} = \{f(\alpha)x_\alpha\}$ is an orthogonal sequence with $\|Tx_\alpha\| \geqq \varepsilon$ which cannot have any convergent subsequence. This contradicts the fact that T is compact. Thus, $M_C(A) \subseteqq \Phi^{-1}(C_0(E))$, completing the proof.

REMARK 8. We note that for every compact multiplier T of a compact H^*-algebra, there exists a net $T_\alpha \in B(A)$ with finite ranks, such that T_α converges to T in operator norm.

REMARK 9. For each $T \in M(A)$, let $\{T_\alpha\}$ be the collection of all restrictions of T to I_α. Clearly $\{T_\alpha\}$ is a family of mutually orthogonal projections, since $\{I_\alpha\}$ is an orthogonal family of subspaces. For each $T \in M_C(A)$, we observe that there are only countably many T_α different from zero. (Observe that the set $\{\alpha: f(\alpha) \neq 0, f = \Phi(T)\} = \bigcup_{n=1}^\infty S_n$, where $S_n = \{\alpha: |f(\alpha)| \geqq 1/n\}$, is countable since for each n, S_n is finite.) Hence, we may write

$$T = \sum_{i=1}^\infty f(\alpha_i)P_{\alpha_i}\,, \qquad \text{with} \qquad \lim_{i\to\infty} |f(\alpha_i)| = 0\,.$$

This decomposition of T into a sequence of orthogonal projections can be considered as an extension of the well-known spectral decomposition of a self-adjoint compact operators of H^*-algebras. In this case, T is not assumed to be self-adjoint.

REMARK 10. By a similar consideration as given in Remark 2, Theorem 3 may be considered as a generalization of Theorem (4.3) of [9]. Furthermore, the maximal ideal space of the algebra $M_C(A)$ of all compact multipliers of a compact H^*-algebra A is homeomorphic to E, the set of all minimal two-sided ideals in A with discrete topology.

REMARK 11. We remark that the specialization of general H^*-algebras to compact H^*-algebras is necessary since in case of X_S, the identity operator in $B(X_S)$ is compact if and only if X_S is finite-dimensional.

REFERENCES

1. W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. **57** (1945), 364-386.

2. F. T. Birtel, *Banach algebras of multipliers*, Duke Math. J. **28** (1961), 203-212.

3. ———, *Isomorphisms and isometric multipliers*, Proc. Amer. Math. Soc. **13** (1962), 204-210.

4. ———, *On a commutative extension of a Banach algebra*, Proc. Amer. Math. Soc. **13** (1962), 815-822.

5. R. E. Edwards, *On factor functions*, Pacific J. Math. **5** (1955), 376-378.

6. L. Gillman, and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

7. S. Helgason, *Multipliers of Banach algebras*, Ann. of Math. **64** (1956), 240-254.

8. E. Helson, *Isomorphisms of abelian group algebras*, Ark. Math. **2** (1953), 475-487.

9. C. N. Kellogg, *Centralizers and H^* algebras*, Pacific J. Math. **17** (1966), 121-129.

10. M. Naimark, *Normed Rings*, P. Noordhoff N. V., The Netherlands, 1964.

11. W. Rudin, *Fourier Analysis on Groups*, Interscience, New York, 1962.

12. J. K. Wang, *Multipliers of commutative Banach algebras*, Pacific J. Math. **11** (1961), 1131-1149.

13. J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. **2** (1952), 251-261.

Received June 7, 1966. The research of the first-named author was supported by the Summer Supplement of a National Research Council Studentship held at the University of Toronto. The research of the second-named author was supported by a Summer Research Fellowship from the Canadian Mathematical Congress, at Edmonton 1966, and also by NRC Grant A-3125.

UNIVERSITY OF TORONTO
TORONTO, CANADA
UNIVERSITY OF ALBERTA
EDMONTON, CANADA