ON THE RELATIONSHIP BETWEEN HAUSDORFF DIMENSION AND METRIC DIMENSION

A. C. VOSBURG

The definitions of the Hausdorff dimension $\dim_h X$, upper metric dimension $\overline{\dim} X$ and lower metric dimension $\underline{\dim} X$ of a metric space X all depend upon asymptotic characteristics of diameters of sets in covers of X. We relate these notions. First we note that $\dim_h X \leq \underline{\dim} X$ holds for all totally bounded metric spaces X, while on the other hand there exist perfect subsets A of $[0, 1]$ satisfying $\dim_h A = 0$ and $\underline{\dim} A = 1 = \overline{\dim}$ $[0,1]$. Finally we show that there exist perfect subsets S of $[0,1]$ which satisfy $\dim_h S = 0$ and $\overline{\dim} S = 1$ even when strong local conditions are imposed.

The notions of Hausdorff dimension (see 1, 2) and metric dimension (see 5 p. 296, 8) are closely related; in fact most compact metric spaces encountered in analysis have the same Hausdorff and metric dimensions. In this paper we investigate some aspects of the relationship between these two concepts.

By the Hausdorff dimension of a subset E of a metric space is meant the number $\dim_h E = \sup \{p: \mu_p^*(E) = +\infty\}$, where $\mu_p^*(E)$ is defined to be $+\infty$ if $p = 0$ and $\mu_p^*(E) = \sup_{\varepsilon > 0} l(E, p; \varepsilon)$ if $p > 0$,

$$(1) \qquad l(E, p; \varepsilon) = \inf \left\{ \sum_{i=1}^{+\infty} (\operatorname{diam} E_i)^p : E \subset \bigcup_{i=1}^{+\infty} E_i, \operatorname{diam} E_i \leq \varepsilon \text{ for each}\right.$$
$$\left. i = 1, 2, \cdots \right\}.$$

For each totally bounded subset A of a metric space (i.e. each subset which for each $\varepsilon > 0$ can be covered by a finite number of sets of diameter not exceeding ε) the upper metric dimension $\overline{\dim} A$ and lower metric dimension $\underline{\dim} A$ of A are defined as follows (all logarithms have base 2):

$$(2) \qquad \overline{\dim} A = \varlimsup_{\varepsilon \to 0+} (\log N_\varepsilon(A))/\log(\varepsilon^{-1})$$

and

$$(3) \qquad \underline{\dim} A = \varliminf_{\varepsilon \to 0+} (\log N_\varepsilon(A))/\log(\varepsilon^{-1}) ,$$

where, for each $\varepsilon > 0$, $N_\varepsilon(A)$ denotes the smallest number of sets in any cover of A by sets of diameter not exceeding 2ε. It is customary (see 5, p. 280) to abbreviate $\log N_\varepsilon(A)$ by $H_\varepsilon(A)$; this function has

been called (see 5, p. 280) the ε-entropy (or metric entropy) function of A. When $\underline{\dim} A = \overline{\dim} A$ this common value is called the metric dimension of $\overline{A}$ and denoted dim A. The asymptotic behavior of $H_\varepsilon(A)$ as $\varepsilon \to 0 +$ in some sense reflects the massiveness of A, and various applications of this notion have appeared in recent years (see 3, 5, 7, 10). For a tour of the subject of metric entropy see Lorentz (6, 7).

We will use the elementary inequality $N_\varepsilon(A) \geqq M_{2\varepsilon}(A)$ which is valid (see 6, p. 151) for all totally bounded sets A, where $M_\varepsilon(A)$ denotes, for each $\varepsilon > 0$, the maximal number of points in any subset S of A having the property that each pair of points of S are separated by distance greater than ε.

$\underline{\dim} A \leqq \overline{\dim} A \leqq \dim A$ being obvious whenever meaningful, we wish to further compare $\dim_h A$, $\dim A$, $\overline{\dim} A$, $\underline{\dim} A$, and investigate some related questions. First (Theorem 1) we will note from elementary considerations that the inequality

$$(4) \qquad\qquad \dim_h A \leqq \underline{\dim} A$$

holds for each totally bounded subset A of a metric space. On the other hand we will note that there exist (even perfect) subsets A of $[0, 1]$ for which

$$(5) \qquad \dim_h A = 0 \text{ and } \dim A = \dim [0, 1] = 1 .$$

Our main result (Theorem 2) involves looking more deeply into the relationship between (4) and (5) by imposing certain additional local conditions.

2. THEOREM 1. *Let A be a totally bounded subset of a metric space X. Then A satisfies (4).*

Proof. Let $\dim A = s \geqq 0$, and assume $\delta > 0$. By the definition (3), there exists a sequence $\varepsilon_1, \varepsilon_2, \cdots$ which decreases to zero such that $\log N_{\varepsilon_n}(A) \leqq (s + \delta) \log (\varepsilon_n^{-1})$, and hence $N_{\varepsilon_n}(A) \leqq (1/\varepsilon_n)^{s+\delta}$ for each $n = 1, 2, \cdots$. This implies the existence of a family of no more than $(1/\varepsilon_n)^{s+\delta}$ sets of diameter not exceeding $2\varepsilon_n$ which covers A. Taking $\varepsilon = 2\varepsilon_n$ in the infimum l in (1), this yields

$$l(A, p; 2\varepsilon_n) \leqq (1/\varepsilon_n)^{s+\delta}(2\varepsilon_n)^p \leqq 2^p \text{ if } p \geqq s + \delta .$$

Hence $\mu_p^*(A) \leqq 2^p < +\infty$ if $p \geqq s + \delta$, which, as desired, implies $\dim_h A \leqq s$.

While Theorem 1 states that the Hausdorff dimension of a compact set A must be small if the ε-entropy of A approaches $+\infty$ sufficiently slowly, i.e., if $\underline{\dim} A$ is small, it is natural to ask what can be said

in the converse direction.   Quite elementary considerations show that the reverse inequality need not hold.   In fact, it is easy to construct (even perfect) sets A of real numbers for which (5) holds.   For example the range R of the sequence $\{a_n\}$ defined by

$$a_1 = 10,\ a_k = a_{k-1} - 1/(k \log^2 k) \text{ if } k > 1$$

satisfies (5), and R can be used to build a perfect set satisfying (5). An example is any perfect set A defined by $A = \bigcup_{k=1}^{+\infty} A_k \cup \{\lim_{k \to +\infty} a_k\}$, where each set A_k is taken to be a perfect set satisfying

$$\dim_h A_k = 0,\ \inf A_k > a_{k+1} + (1/2)(a_k - a_{k+1}) \text{ for } k = 1, 2, \cdots$$

and

$$\sup A_k < a_{k-1} - (1/2)(a_{k-1} - a_k) \text{ for } k = 2, 3, \cdots.$$

While in a sense this answers the converse question to Theorem 1, much is left to be desired, for the largeness of dim A is really a consequence of that of dim R, and R is only a countable set.   We show that the differences between the notions of Hausdorff and metric dimensions run deeper than might be suggested by the above.   To do this we introduce the notion of the metric dispersion of a totally bounded subset of a metric space.

DEFINITION 1.   Let A be a totally bounded subset of a metric space and let $x_0 \in A$.   Then by the upper metric dimension of A at x_0, denoted $\overline{\dim}\,(A, x_0)$, is meant the number

$$\overline{\dim}\,(A, x_0) = \inf \left\{ \overline{\lim_{\varepsilon \to 0+}} H_\varepsilon(A \cap U)/\log (\varepsilon^{-1})\colon U \varepsilon U(x_0) \right\}, \text{ where } U(x_0)$$

denotes the class of open neighborhoods of x_0.

DEFINITION 2.   By the metric dispersion of A, denoted disp A, is meant disp $A = \inf_{x_0 \varepsilon A} \overline{(\dim}\,(A, x_0))$.

THEOREM 2.   *There exist perfect sets S of real numbers satisfying*

$$(6) \qquad\qquad \dim_h S = 0 \text{ and } \operatorname{disp} S = 1.$$

Proof.   Abbreviating 2^a by exp a, we let

$$(7) \qquad \delta_1 = 1,\ \delta_n = \exp \{- (n!)^2\}$$
$$\varepsilon_n = \exp \{-n((n - 1)!)^2\}$$
$$m_n = \exp \{(n - 2)((n - 1)!)^2\},\ n = 2, 3, \cdots.$$

For each $n = 3, 4, \cdots$ and each closed interval $I = [a, b]$ of length at

least δ_{n-1}, we denote by $F(I; n)$ the subset of I which is the union of all intervals of the family

$$\mathfrak{J}(I; n) = \{[a + i(\delta_n + \varepsilon_n), a + i(\delta_n + \varepsilon_n) + \delta_n]: i = 0, 1, \cdots, m_n - 1\}.$$

From (7) it follows that

$$\delta_n \leqq \varepsilon_n,$$

and

$$m_n(\delta_n + \varepsilon_n) \leqq 2m_n\varepsilon_n \leqq \delta_{n-1},$$

so indeed $F(I; n)$ is contained in I. Further, for each I and n, the distance between any two intervals of $\mathfrak{J}(I; n)$ is no smaller than ε_n, and the distance between any pair of left end points of intervals of $\mathfrak{J}(I; n)$ is no smaller than $\delta_n + \varepsilon_n > \varepsilon_n$.

Now we can define the desired set S as a generalized Cantor set. Let the sequence $\{A_n\}_{n=1}^{+\infty}$ of sets be defined by

$$A_1 = [0, 1], \quad A_n = \cup \{F(C; n): C \in \underset{\sim}{A}_{n-1}\} \text{ if } n \geqq 2,$$

where, for each $n = 1, 2, \cdots, \underset{\sim}{A}_n$ denotes the family of all component intervals of A_n (the components of each A_n are intervals of length δ_n). For S we take $S = \bigcap_{n=1}^{+\infty} A_n$.

Since each A_n has $\prod_{k=1}^{n} m_k$ components, each of length δ_n, to prove that $\dim_h S = 0$ it suffices to observe that from (7) it follows that

$$\lim_{n \to +\infty} \{(\prod_{k=1}^{n} m_k)\delta_n^p\} = 0 \text{ for each } p > 0.$$

To establish disp $S = 1$, we consider any open interval I such that $S \cap I$ is nonempty, and estimate $N_\varepsilon(S \cap I)$ from below. If n is sufficiently large, I contains a component interval $[c, d]$ of A_{n-1}, and hence also contains all intervals of $\mathfrak{J}([c, d]; n)$. If we let $\underset{\sim}{C}$ denote a cover of $F([c, d]; n)$ by sets of diameter not exceeding 2ε, $\varepsilon = (1/2)\varepsilon_n$, then $\underset{\sim}{C}$ covers also the set of left end points of the intervals of $\mathfrak{J}([c, \tilde{d}]; n)$, and no covering set among the sets of $\underset{\sim}{C}$ covers more than one such endpoint. Hence the number of covering sets of the family $\underset{\sim}{C}$ cannot be less than m_n. This implies that

$$(8) \qquad\qquad N_{(\varepsilon_n/2)}(S \cap I) \geqq m_n$$

holds for all sufficiently large n. Since (7) implies that

$$(\log m_n)/\log (2/\varepsilon_n) = (n - 2)((n - 1)!)^2/(n((n - 1)!)^2) \to 1 \text{ as } n \to +\infty,$$

we have, from (8),

$$\varlimsup_{\varepsilon \to 0+} N_\varepsilon(S \cap I)/\log (\varepsilon^{-1}) \geqq 1.$$

Since this limit superior cannot be greater than 1, we have disp $S = 1$, which completes the proof of the theorem.

It remains an open question whether there exist perfect sets S satisfying $\dim_h S = 0$, $\underline{\dim}(S, x) > 0$ for all (or at least most, in some sense) of the points x of S, where $\underline{\dim}(S, x)$ is defined analogously to $\overline{\dim}(S, x)$ in Definition 1.

In conclusion we mention that Theorem 2 leaves open the question as to whether for two sets A, B with $\dim_h A \leqq \dim_h B$ any inequality necessarily follows for disp A, disp B. The answer is in the negative; that is, for each ω, $0 < \omega < 1$, there are perfect sets A, B of real numbers such that $\dim_h A < \dim_h B = \omega$ and $\omega = $ disp $B <$ disp $A = 1$. This follows from Theorem 2 together with the fact that for each ω, $0 < \omega < 1$, there exist compact perfect sets B such that $\dim_h B = \omega$ and $H_\varepsilon(B) \sim \omega \log 1/\varepsilon$. Such sets B are constructed in Hausdorff's paper (see [1], § 10).

REFERENCES

1. F. Hausdorff, *Dimension and outer Measure*, Math. Ann. **79** (1919), 157-119.
2. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, 1941.
3. A. N. Kolmogorov, *Asymptotic characteristics of some completely bounded metric spaces*, Dokl. Akad, Nauk SSSR **108** (1956) 585-589.
4. A. N. Kolmogorov and V.M. Tihomirov, *ε-entropy and ε-capacity of sets in function spaces*, Uspehi Mat. Nauk (86), **14** (1959), 3-86.
5. ―――, *ε-entropy and ε-capacity of sets in functional spaces*, Amer. Math. Soc. Translations (2) **17**, 277-364 (English translation of **4**).
6. G. G. Lorentz, *Approximation of functions*, Holt, Rinehart and Winston, New York, 1966.
7. ―――, *Metric entropy, widths, and superpositions of functions*, Amer. Math. Monthly **69** (1962), 469-485.
8. L. Pontrjagin and L. Schnirelmann, *Sur une propriété métrique de la dimension*, Annals of Math. (2) **33** (1932), 156-162.
9. A. G. Vitushkin, *Estimation of the complexity of tabulation problems*, Moscow 1959.
10. ―――, *Theory of the transmission and processing of information*, Pergamon Press, New York, 1961 (English translation of **9**).

SYRACUSE UNIVERSITY AND
THE GENERAL ELECTRIC COMPANY