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FIXED POINTS IN A CLASS OF SETS

D. R. SMART

THEOREM. A set of the form $X = A \cup \bigcup_{i \in J} B_i$ has the fixed point property if

(i) A is a closed simplex and each B_i is a closed simplex;

(ii) $A \cap B_i$ is a single point p_i for each i;

(iii) any arc in X joining a point in some B_i to a point in $X - B_i$ must pass through p_i.

(J can be any index set. The topology on X can be given by any metric satisfying (i) and (iii).)

The statement that X has the fixed point property means that each continuous mapping of X into X has a fixed point. The theorem applies to many sets which are not locally connected so that even Lefschetz's fixed point theorem is inapplicable. Instead of assuming that the subsets A and B_i are simplices we could merely assume that each of these subsets is locally arcwise connected and has the fixed point property. The result should still be true if each point p_i is replaced by a simplex P_i but this generalization would require altogether different methods.

Proof of the theorem. Let T be a continuous mapping of X into X. We distinguish three cases.

Case 1. Suppose $Tp_i \in B_i - \{p_i\}$ for some i. Then we will show that T has a fixed point in B_i.

Define
$$S : B_i \to B_i \text{ by}$$
$$Sx = Tx \text{ if } Tx \in B_i$$
$$Sx = p_i \text{ if } Tx \notin B_i .$$

Then S is continuous by Lemma 2 below.

Since B_i has the fixed point property, $Sx = x$ for some x in B_i. Now $x \neq p_i$ (for $x = p_i$ would give $Sp_i = p_i$, impossible since $Sp_i = Tp_i \in B_i - \{p_i\}$). Thus $Sx \neq p_i$ so that $Tx = Sx = x$.

Case 2. Suppose $Tp_i = p_i$ for some i. Then p_i is a fixed point.

Case 3. Suppose $Tp_i \in X - B_i$ for all i. Then we will show that T has a fixed point in A. Define $R : A \to A$ by

$$Rx = Tx \text{ if } Tx \in A .$$

$$Rx = p_i \text{ if } Tx \in B_i \, .$$

Then R is continuous by Lemma 2. Since A has the fixed point property, R has a fixed point in A. The fixed point ξ cannot be a point p_i since $Rx = p_i$ only if $Tx \in B_i$; and $Tp_i \notin B_i$. Since the fixed point is not p_i, $T\xi \notin B_i$. Thus $T\xi \in A$ so that $T\xi = R\xi = \xi$.

Thus in each case T has a fixed point, which proves the theorem. The above proof depends on two lemmas.

LEMMA 1. *If $z(t)$ is a continuous function on $[0, 1]$ to a metric space and either*

(i) *$w(t)$ is a constant, or*

(ii) *$w(t) \equiv z(t)$ except on a non-overlapping sequence of intervals $[t_{2n-1}, t_{2n}]$ $(n \geqq 1)$ such that*

$$t_1 = 0 \text{ and } w(t) \equiv z(t_2) \text{ on } [t_1, t_2]$$
$$t_4 = 1 \text{ and } w(t) \equiv z(t_3) \text{ on } [t_3, t_4]$$

and for $n > 2$, $z(t_{2n-1}) = z(t_{2n})$ and $w(t) \equiv z(t_{2n})$ on $[t_{2n-1}, t_{2n}]$. Then $w(t)$ is continuous on $[0, 1]$.

Proof. Obvious. (One proof is: if z_n is the function obtained from z by changing its value to that of w on the first n intervals, then z_n is continuous. Also $z_n \to w$ uniformly on $[0, 1]$ since the length of $[t_{2n-1}, t_{2n}]$ must tend to 0.

LEMMA 2. *Let Y be a closed simplex contained in a metric space X. Suppose that $X - Y$ is the union of disjoint sets Z_i, that $Z_i \cap Y$ is a one-point set $\{q_i\}$, and that any path from a point in a Z_i to a point in $X - Z_i$ must pass through q_i. Let U be continuous on Y X. Define T by*

$$Ty = Uy \text{ if } Uy \in Y$$
$$Ty = q_i \text{ if } Uy \in Z_i \, .$$

Then T is continuous.

Proof. If $y_n \to y$ in Y we must show that $Ty_n \to Ty$. Consider a path $g(t)$ in $Y (0 \leqq t \leqq 1)$ such that $g(0) = y$ and $g(1/n) = y_n$. Writing $Ug(t) = z(t)$ and $Tg(t) = w(t)$ the conditions of Lemma 1 are satisfied. For if $w(t)$ differs from $z(t)$ the possibilities are: $z(t)$ could be in some Z_i for all t, in which case $w(t)$ is a constant; otherwise, there is an initial interval $[0, t_2]$ where $z(t)$ is in some Z_i, and/or some intermediate intervals $[t_{2n-1}, t_{2n}]$ where $z(t)$ is in some $Z_{i(n)}$ and/or a final interval $[t_3, 1]$ where $z(t)$ is in some Z_j. By Lemma 1, $w(t)$ is continuous. Thus $Tg(1/n) \to Tg(0)$ as required.

The theorem can be used to establish some pathological examples. (It seems that all of these are already known.)

I. There exists a noncompact set having the fixed point property.

Take
$$A = \{(x, y) : 0 \leqq x \leqq 1, y = 0\}$$
$$B_n = \left\{(x, y) : x = \frac{1}{n}, 0 \leqq y \leqq 1\right\}.$$

(In this case $\overline{X}$ also has the fixed point property.)

II. There exists an unbounded set having the fixed point property. Take A as above.

$$B_n = \left\{(x, y) : x = \frac{1}{n}, 0 \leqq y \leqq n\right\}.$$

III. There exists a set with the fixed point property whose closure lacks this property. Take X as in II.

IV. There exists a precompact set with the fixed point property, whose closure lacks this property.

Take
$$A = \left\{e^{i\theta} : \frac{\pi}{2} \leqq \theta \leqq 2\pi\right\}$$
$$B_n = \left\{\left(1 + \frac{\theta}{n}\right)e^{i\theta} : 0 \leqq \theta \leqq \frac{\pi}{2}\right\}.$$

Several sets which have some interest in other contexts have the fixed point property in consequence of our theorem :-

V. The set

$$A \cup \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{n=1}^{\infty} C_n$$

where A is the unit interval, B_n is a unit line segment sloping up from (0,0) with slope $1/n$, and C_n is a unit line segment sloping up to (0,1) with slope $1/n$. (This is a non contractible set.)

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UNIVERSITY OF CAPE TOWN, RONDEBOSCH