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A CHARACTERIZATION OF RESTRICTIONS OF FOURIER-STIELTJES TRANSFORMS

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The main result that we prove here is as follows: Let E be a Lebesgue measurable subset of R, the real line, and let φ be a bounded measurable function defined on E. Then the first of the following conditions implies the second:

(1) **There exists a constant K, so that**

$$\left|\sum_{j=1}^{n} c_j \varphi(x_j)\right| \leqq K \| P \|_\infty$$

for all trigonometric polynomials of the form

$$P(y) = \sum_{j=1}^{n} c_j e^{ix_j y}, \quad \text{where } x_j \in E \text{ for all } 1 \leqq j \leqq n.$$

(2) **φ is E-almost everywhere a Stieltjes transform. Precisely, there exists a finite (complex Borel) measure μ, so that**

$$\varphi(x) = \hat{\mu}(x) = \int_{-\infty}^{\infty} e^{-ixy} d\mu(y)$$

for almost all $x \in E$. Moreover, μ may be chosen such that $\| \mu \| \leqq K$, where K is the constant in (1). ($\| \mu \|$ denotes the total variation of μ.)

In 1934 (c.f. [3]), Bochner established this result for the case when E is the entire real line. Our result also generalizes a theorem of Krein. Indeed Krein proved (c.f. [1] pp. 154–159) that (1) and (2) are equivalent for the case when E is an interval and φ is a continuous function defined on E. Now if we assume that E is closed and of uniformly positive measure, (meaning that if U is an open subset of $\boldsymbol{R}$ with $U \cap E$ nonempty, then the measure of $U \cap E$ is positive), and if $\varphi \in C(E)$ and satisfies (1), then our result implies that (2) holds for all $x \in E$. (i.e. $\varphi \equiv \hat{\mu} \mid E$ for some finite measure μ on $\boldsymbol{R}$). (It is trivial that (2) implies (1) under these hypotheses.)

Note finally that it E is a closed subset of $\boldsymbol{T}$, the circle group, of uniformly positive measure, and if $\varphi \in C(E)$ and satisfies (2), then $\varphi \in A(E)$. That is, φ can be extended to a function defined on all of $\boldsymbol{T}$, with absolutely convergent Fourier series. (We identify $\boldsymbol{T}$ with the real numbers modulo 1; in this case, the polynomials of condition (2) are almost-periodic functions defined on the integers.)

We obtain our main result by first proving the result mentioned in the above paragraph in Theorem 3; next by establishing the analogue of the main result for $\boldsymbol{T}$ in Theorem 4, and finally by passing from the circle to the real line in § 3.

The core of the proof of Theorem 3 is found in Lemma 2; the technique used there was suggested by a method due to C. S. Herz, as exposed in Théorème VII, pp. 124–126 of [4]. An essential step in the proof of Lemma 2 is Lemma 1, where we show that a measurable subset of $\boldsymbol{T}$ may be approximated in measure by nicely-placed closed subsets[1].

1. **Preliminaries.** The following two results are not essential for the main result, but they do provide some motivation for it. We let $\boldsymbol{Z}$ denote the integers; if μ is a finite measure on $\boldsymbol{R}$ (resp. $\boldsymbol{T}$), $\|\hat{\mu}\|_\infty = \sup_{x\in \boldsymbol{R}} |\hat{\mu}(x)|$ (resp. $\sup_{n\in \boldsymbol{R}} |\hat{\mu}(n)|$ where $\hat{\mu}(n) = \int_0^1 e^{-i2\pi nt} d\mu(t)$ for all $n \in \boldsymbol{Z}$).

PROPOSITION A. *Let E be an arbitrary subset of $\boldsymbol{R}$ (resp. $\boldsymbol{T}$), and let φ be a bounded function defined on E. Then the following two conditions are each equivalent to* (1).

(3) *There exists a constant K, so that*

$$\left| \int \varphi d\mu \right| \leqq K \|\hat{\mu}\|_\infty$$

for all discrete measures μ supported on E.

(4) *There exists a finite (complex regular Borel) measure ν defined on the Bohr compactification of $\boldsymbol{R}$ (resp. of $\boldsymbol{Z}$), so that $\varphi(x) = \hat{\nu}(x)$ for all $x \in E$.*

The fact that (1) and (3) are equivalent is a triviality. The equivalence of (1) and (4) is a consequence of the Riesz-representation theorem, together with the fact that the almost-periodic polynomials on $\boldsymbol{R}$ (resp. $\boldsymbol{Z}$) may be regarded as being dense in the space of continuous functions on the Bohr compactification of the respective groups. (See [5], pp. 30–32, for these and other properties of the Bohr compactification).

For the next proposition, we recall that for E a closed subset of $\boldsymbol{T}$, $A(E)$ is the set of all $\varphi \in C(E)$ for which there exists an $f \in C(\boldsymbol{T})$, such that $f(x) = \varphi(x)$ for all $x \in E$, and $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. $A(E)$ is a Banach algebra under the norm

$$\|\varphi\|_{A(E)} = \inf\left\{\sum_{-\infty}^{\infty} |\hat{f}(n)| : f \in A(\boldsymbol{T}) \text{ with } f \mid E = \varphi\right\}.$$

PROPOSITION B. *Let E be a closed subset of $\boldsymbol{T}$ such that if*

[1] Benjamin Halpern independently discovered a different proof of Lemma 1, and we are indebted to him for a stimulating discussion concerning this result.

$\varphi \in C(E)$ and φ satisfies (3), then $\varphi \in A(E)$. Then there exists a finite constant K, so that for all $f \in A(E)$,

$$\|f\|_{A(E)} \leqq K |||f|||, \quad \text{where}$$

$$|||f||| = \sup \left| \int f d\mu \right|, \quad \text{the supremum}$$

being taken over all discrete measures μ supported on E with $\|\hat{\mu}\|_\infty \leqq 1$.

Proof. $|||\cdot|||$ defines a new norm on $A(E)$, and we have that $|||f||| \leqq \|f\|_{A(E)}$, for all $f \in A(E)$. Now our hypotheses imply that $A(E)$ is complete under this norm also. Indeed, suppose $\{f_n\}$ is a Cauchy sequence in the norm $|||\cdot|||$. Fix $x \in E$, and let μ_x be the measure assigning a mass of one to x. Then $\|\hat{\mu}_x\|_\infty = 1$, so we have that

$$|||f_n - f_m||| \geqq \left| \int (f_n - f_m) d\mu_x \right| = |f_n(x) - f_m(x)|$$

for all integers n and m. Hence, $\{f_n\}$ is a Cauchy sequence in $C(E)$, so $\{f_n\}$ converges uniformly to a continuous function φ. Also, since $\{f_n\}$ is a Cauchy sequence in $|||\cdot|||$, there exists a constant K so that $|||f_n||| \leqq K$ for all n. This means that

$$\left| \int f_n d\mu \right| \leqq K \|\hat{\mu}\|_\infty$$

for all discrete measures μ. Now fixing μ a discrete measure, we have that

$$\lim_{n\to\infty} \left| \int f_n d\mu \right| = \left| \int \varphi d\mu \right| .$$

Hence φ satisfies (3), so $\varphi \in A(E)$ by hypothesis, whence

$$\lim_{n\to\infty} |||f_n - \varphi||| = 0 .$$

Thus, since $A(E)$ is a Banach space under the weaker norm $|||\cdot|||$, we have that $|||\cdot|||$ is equivalent to the norm $\|\cdot\|_{A(E)}$.

REMARK 1. Walter Rudin has constructed a closed independent set E which supports a measure whose Stieltjes transform vanishes at infinity (see [6]). Such a set does not satisfy the conclusion of Proposition B, since the independence of E implies that $|||f||| = \|f\|_\infty$ for all $f \in A(E)$, whereas the set cannot have its $C(E)$ and $A(E)$ norms equivalent (cf. [5], pp. 114–120).

REMARK 2. It follows from a theorem of Banach (Theorem 2, p. 213 of [2]), that the conclusion of Proposition B is equivalent to the following: if $F \in A(E)^*$, then there exists a sequence of discrete measures μ_n such that μ_n tends to F in the weak * topology of $A(E)$. ($A(E)^*$ denotes the conjugate space of A; the definition of $A(E)$ imples that if μ is a measure supported on E, then $\|\mu\|_{A(E)^*} = \|\hat{\mu}\|_\infty$, where $\|\mu\|_{A(E)^*} = \sup \left|\int f d\mu\right|$, the supremum being taken over all $f \in A(E)$ with $\|f\|_{A(E)} \leqq 1$.

In the terminology of [4] (cf. p. 115), our Theorem 3 thus implies that if E is of spectral synthesis and of uniformly positive measure, and if S is a pseudo-measure carried by E, there exists a sequence of linear combinations of point masses carried by E and tending weakly to S.

We note finally, that Proposition A holds for arbitrary locally compact abelian groups, and Proposition B holds for compact subsets of l.c.a. groups.

2. Throughout this section, E denotes a subset of $\boldsymbol{T}$ of positive Lebesgue measure; m denotes Lebesgue measure on $\boldsymbol{T}$ (with $m(\boldsymbol{T}) = 1$); if S and T are two subsets of $\boldsymbol{T}$,

$$S + T = \{s + t \colon s \in S \text{ and } t \in T\} .$$

If ψ is a Lebesgue-integrable function defined on a closed set E_1, and if φ is a bounded measurable function defined on a closed set E_2, we recall that the continuous function $\varphi * \psi$, defined by

$$(\varphi * \psi)(y) = \int_0^1 \varphi(y - x)\psi(x)dx \qquad \text{for all } y \in \boldsymbol{T} ,$$

is supported on the set $E_1 + E_2$.

Finally, if S is a subset of $\boldsymbol{T}$, χ_s denotes the characteristic function of S. i.e.

$$\chi_s(y) = 1 \quad \text{if} \quad y \in S \; ; \quad \chi_s(y) = 0 \quad \text{otherwise} .$$

LEMMA 1. *Given E and $\delta > 0$, for all sufficiently large integers N there exists a closed subset $F' \subset E$, depending on N, with $m(F') \geqq (1 - \delta)m(E)$, so that for some $0 \leqq \beta < 1/N$, each of the numbers $\beta + k/N, k = 0, 1, \cdots, N - 1$; either belongs to F', or is a distance at least $1/N$ away from F'.*

Proof. Let $\varepsilon > 0$ be given. Then we may choose a closed set $F \subset E$, so that $m(F) \geqq m(E)(1 - \varepsilon)$, and so that for all N sufficiently large,

$$m\left(F + \left[-\frac{1}{N}, \frac{1}{N}\right]\right) \leqq m(F)(1 + \varepsilon) .$$

(We may accomplish this by simply choosing a finite number of disjoint closed intervals which approximate E closely in measure. Precisely, if S and T are two subsets of $\boldsymbol{T}$, let

$$S \Delta T = (S \cap \mathscr{C} T) \cup (\mathscr{C} S \cap T) .$$

First, choose F_1 a closed subset of E, with $m(E \Delta F_1) < (\varepsilon/2)m(E)$. Next, choose $I_1, \cdots, I_p$ disjoint closed intervals with

$$m\left(F_1 \Delta \bigcup_{j=1}^{p} I_j\right) < \frac{\varepsilon'}{2} m(F_1) ,$$

where $\varepsilon' = \min\{\varepsilon, 2\varepsilon/(2 + \varepsilon)\}$. Finally, let

$$F = \bigcup_{j=1}^{p} I_j \cap F_1 ;$$

then the desired inequalities hold for all integers $N \geqq (4p/\varepsilon m(F))$.

Now fix such an N; then

$$m(F) = \sum_{k=1}^{N} m\left(F \cap \left[\frac{k-1}{N}, \frac{k}{N}\right]\right) .$$

Let g be defined on $[0, 1/N)$ by

$$g(x) = \frac{1}{N} \sum_{k=0}^{N-1} \chi_F\left(x + \frac{k}{N}\right) .$$

Then

$$N \int_0^{1/N} g(x) dm(x) = m(F) .$$

Since $g(x) \geqq 0$ for all x, we must have that $g \geqq (1 - \varepsilon)m(F)$ on a set of positive measure; thus, we may choose a $\beta, 0 \leqq \beta < (1/N)$, with

$$g(\beta) \geqq (1 - \varepsilon)m(F) .$$

Now consider the family of intervals,

$$I_k = \left[\beta + \frac{k}{N}, \beta + \frac{k+1}{N}\right], \qquad \text{for } k = 0, 1, \cdots, N-1 .$$

We remark that if $f \in F$ belongs to one of these intervals, then the entire interval is contained in the set $F + [-1/N, 1/N]$. (Of course,

$\boldsymbol{T}$ equals the union of these intervals).

Thus, let $\mathscr{K}$ be the subset of $\{0, 1, \cdots, N-1\}$ so that $k \in \mathscr{K}$ if and only if I_k contains a point of F. Then

$$F \subset \bigcup_{k \in \mathscr{K}} I_k \subset F + \left[-\frac{1}{N}, \frac{1}{N}\right].$$

Hence if r is the number of elements in $\mathscr{K}$, we have that

$$m(F) \leqq \frac{r}{N} \leqq m(F)(1+\varepsilon) .$$

Now, let

$$\mathscr{J} = \{I_k : k \in \mathscr{K} \text{ and both end points of } I_k \text{ belong to } F\} .$$

We shall show that $\mathscr{J}$ is nonempty; in fact, letting l be the cardinality of $\mathscr{J}$, we shall show that l is very close to r.

First, let

$\mathscr{K}' = \{k \in \mathscr{K} : \beta + (k/N) \in F\}$; let q be the cardinality of $\mathscr{K}'$: Then $(q/N) = g(\beta)$.

Now, let

$$\mathscr{K}'' = \left\{k \in \mathscr{K}' : \beta + \frac{k+1}{N} \notin F\right\},$$

and let s be the cardinality of $\mathscr{K}''$. Noticing that $k \in \mathscr{K}''$ if and only if $\beta + (k/N)$ is *not* a left-hand end point of an interval in $\mathscr{J}$, we thus have that $q - s = l$.

Now to each $k \in \mathscr{K}''$ corresponds a unique member of $\mathscr{K} \cap \mathscr{C}\mathscr{K}'$, namely the least of the numbers $q \in \mathscr{K}$ such that $q > k$ if there are such numbers; otherwise the least number in $\mathscr{K}$. (Recall that $\beta = \beta + 1$, as members of $\boldsymbol{T}$.) Thus

$$\operatorname{card} \mathscr{K}'' \leqq \operatorname{card} (\mathscr{K} \cap \mathscr{C}\mathscr{K}') .$$

But

$$\mathscr{K}'' \cup (\mathscr{K} \cap \mathscr{C}\mathscr{K}') \cup (\mathscr{K}' \cap \mathscr{C}\mathscr{K}'') \subset \mathscr{K} .$$

Hence $s + s + q - s \leqq r$. Thus, $q + s \leqq r$. Hence, since $s = q - l$, we obtain that $r - l \leqq 2(r - q)$. Now, let

$$F' = F \cap \bigcup_{J \in \mathscr{J}} J .$$

Then F' has the property that each number $\beta + (k/N)$ belongs to F', or is a distance at least $1/N$ away from F'. For if $\beta + (k/N)$ is not an endpoint of an interval $J \in \mathscr{J}$, then $\beta + (k/N)$ is at least distance $1/N$ away from the nearest point in $\mathscr{J}$. Moreover, F' was obtained by removing at most $r - l$ intervals from F', each of length $1/N$. Thus, recalling that

$$\frac{r}{N} \leqq m(F)(1 + \varepsilon) \quad \text{and} \quad \frac{q}{N} \geqq m(F)(1 - \varepsilon) ,$$

we have that

$$\begin{aligned} m(F') \geqq m(F) - \frac{r - l}{N} &\geqq m(F) - 2\left(\frac{r - q}{N}\right) \\ &\geqq m(F)[1 - 2[(1 + \varepsilon) - (1 - \varepsilon)]] \\ &= m(F)(1 - 4\varepsilon) \geqq m(E)(1 - 4\varepsilon)(1 - \varepsilon) . \end{aligned}$$

Thus, given $\delta > 0$, we simply choose ε so that

$$(1 - 4\varepsilon)(1 - \varepsilon) \geqq (1 - \delta) .$$

REMARKS. We note incidentally that l/N provides a good approximation to $m(E)$, since

$$m(E)(1 + \varepsilon) \geqq \frac{r}{N} \geqq \frac{l}{N} \geqq m(F') \geqq m(E)(1 - 4\varepsilon)(1 - \varepsilon) .$$

This shows that given $\varepsilon > 0$, we may, for all N sufficiently large, give an upper estimate to $m(E) - \varepsilon$ by considering some system of equally spaced intervals of length $1/N$, then adding up the lengths of all these intervals such that both their endpoints belong to E.

The next lemma is directed toward showing that if φ is a measurable function satisfying (3), then φ also satisfies (3) for a larger class of measures supported on E. (See the first line of the proof of Theorem 3.)

LEMMA 2. *Let φ be a bounded measurable function defined on E. Then there exists a sequence of discrete measures $\{\nu_M\}$ supported on E, so that*

$$\begin{aligned} &\|\nu_M\| \leqq \|\varphi\|_\infty && \textit{for all } M, \textit{ with} \\ &\|\hat{\nu}_M\|_\infty \leqq \left(1 + \frac{1}{M}\right)\|\hat{\varphi}\|_\infty && \textit{and} \\ &\lim_{M\to\infty} \hat{\nu}_M(l) = \hat{\varphi}(l) && \textit{for all integers } l. \end{aligned}$$

Proof. Fix M an integer. Since φdm is absolutely continuous with respect to m, we may choose a $\delta > 0$ so that if K is a Lebesgue measurable set with $m(K) \leqq \delta$, then

$$\int_K |\varphi| \, dm < \frac{1}{M} \| \hat{\varphi} \|_\infty .$$

(Of course we assume that $\|\varphi\|_1 > 0$.) Now by Lemma 1, we may choose a closed set $F \subset E$, an integer $N \geqq M$, and a number $0 \leqq \beta < (1/N)$, so that $m(E \cap \mathscr{C}F) \leqq \delta$, and so that each of the numbers $\beta + (k/N)$, for $k = 0, 1, \cdots, N-1$, either belongs to F, or is a distance at least $1/N$ from F. Let φ' be the restriction of φ to F, i.e. $\varphi' = \varphi\chi_F$.

Let $m_{N\beta}$ be the discrete measure supported on $\{\beta + (k/N)\}_{k=0}^{N-1}$, and which assigns mass $1/N$ to each of the points $\beta + k/N$.

Now let Δ_N be the function whose graph is an isosceles triangle of height N and base $[-1/N, 1/N]$. Finally, let

$$\nu_M = (\Delta_N * \varphi') m_{N\beta} .$$

Now, since $\Delta_N^* \varphi'$ is supported on $F + [-1/N, 1/N]$, it follows that ν_M is supported on F. Moreover,

$$\| \Delta_N \|_1 = 1, \ \| \varphi' \|_\infty \leqq \| \varphi \|_\infty, \quad \text{and} \quad \| m_{N\beta} \| = 1 ;$$

hence

$$\| \nu_M \| \leqq \| \Delta_N * \varphi' \|_\infty \| m_{N\beta} \| \leqq \| \varphi \|_\infty .$$

For the next two assertions of the Lemma, we need the following easily established properties of $\hat{\Delta}_N$ and $\hat{m}_{N\beta}$:

(a) $\hat{\Delta_N}(j) \geqq 0$ for all j.

(b) $\sum_{l=-\infty}^{\infty} \hat{\Delta_N}(l) = N$.

(c) $\sum_{j=-\infty}^{\infty} \hat{\Delta}_N(l + jN) = 1$ for all integers l.

(d) $\lim_{j \to \infty} \hat{\Delta}_j(l) = 1$ for all l.

(e) $\hat{m}_{N\beta}(j) = 0$ if j is not a multiple of N; otherwise, $\hat{m}_{N\beta}(j) = e^{-i2\pi\beta j}$.

We thus have, for all integers l, that

$$\begin{aligned} \hat{\nu}_M(l) &= [(\Delta_N * \varphi') m_{N\beta}]\hat{\ }(l) \\ &= \sum_{j=-\infty}^{\infty} \hat{\Delta}_N(l - jN) \hat{\varphi}'(l - jN) e^{-2\pi i \beta j N} . \end{aligned}$$

Hence,

$$| \hat{\nu}_M(l) | \leqq \sup_j | \hat{\varphi}'(l - jN) | \sum_{j=-\infty}^{\infty} | \hat{\Delta}_N(l - jN) | \leqq \| \hat{\varphi}' \|_\infty .$$

By the first two statements of this proof, we have that

$$\| \varphi - \varphi' \|_1 < \frac{1}{M} \| \hat{\varphi} \|_\infty ,$$

from which it follows that

$$\|\hat{\varphi}'\|_\infty \leqq \left(1 + \frac{1}{M}\right) \|\hat{\varphi}\|_\infty ;$$

hence the second assertion follows. Finally, we fix l an integer; then

$$\begin{aligned}
&|\hat{\nu}_M(l) - \hat{\varphi}(l)| \\
&\quad = |\hat{\Delta}_N(l)\hat{\varphi}'(l) - \hat{\varphi}(l) + \sum_{j\neq 0} \hat{\Delta}_N(l - jN)\hat{\varphi}'(l - jN)e^{-2\pi i\beta jN}| \\
&\quad \leqq \hat{\Delta}_N(l)\,|\hat{\varphi}'(l) - \hat{\varphi}(l)| + (1 - \hat{\Delta}_N(l))\,|\hat{\varphi}(l)| \\
&\qquad\qquad + \sup_{j\neq 0} |\hat{\varphi}'(l - jN)| \sum_{j\neq 0} \hat{\Delta}_N(l - jN) \\
&\quad < \frac{1}{M} \|\hat{\varphi}\|_\infty + 3\|\hat{\varphi}\|_\infty(1 - \hat{\Delta}_N(l)) .
\end{aligned}$$

(The last inequality follows from (c) and the fact that $\|\hat{\varphi}'\|_\infty \leqq 2\|\hat{\varphi}\|_\infty$.) Hence by (d), we have that $\lim_{M\to\infty} \hat{\nu}_M(l) = \hat{\varphi}(l)$ for all integers l.

THEOREM 3. *Let E be a closed subset to $\boldsymbol{T}$ of uniformly positive measure. Then if $\psi \in C(E)$ and if ψ satisfies condition* (3) *with the constant K, there exists an $f \in A$ with $\|f\|_A \leqq K$, and with $f|_E = \psi$.*

Proof. First, the hypotheses together with Lemma 2 show that

$$\left|\int \psi\varphi dm\right| \leqq K \|\hat{\varphi}\|_\infty$$

for all bounded measurable functions φ supported on E.

Indeed, fix such a φ, and choose $\{\nu_M\}$ a sequence of discrete measures supported on E and satisfying the conclusion of Lemma 2. Since the total variations of the sequence are uniformly bounded, it follows that ν_M tends to φ in the weak* topology of $C(E)^*$. (Some subsequence converges by Alaoglu's theorem, but any convergent subsequence must converge to φ by the uniqueness of Fourier-Stieltjes transforms.) Hence,

$$\lim_{M\to\infty} \int \psi d\nu_M = \int \varphi\psi dm .$$

Thus,

$$\left|\int \varphi\psi dm = \lim_{M\to\infty} |\psi d\nu_M| \leqq \overline{\lim_{M\to\infty}} K \|\hat{\nu}_M\|_\infty \leqq K \|\hat{\varphi}\|_\infty .\right.$$

Now, let X be the subspace of $c_0(\boldsymbol{Z})$, the sequences on the integers vanishing at infinity, defined as

$$X = \{\hat{\varphi} : \varphi \text{ is a bounded measurable function, defined on } E\} .$$

Now define F a linear functional on X by

$$F(\hat{\varphi}) = \int \psi \varphi dm .$$

(Since $\hat{\varphi}_1 = \hat{\varphi}_2$ if and only if $\varphi_1 = \varphi_2$ a.e., F is well defined.) Thus F is a bounded linear functional on X; so by the Hahn-Banach theorem and the fact that $c_0(\boldsymbol{Z})^*$ may be identified with $L^1(\boldsymbol{Z})$ (the space of all absolutely convergent sequences), there exists an $f \in A$, with $\|f\|_A \leqq K$, so that

$$F(\hat{\varphi}) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n)\hat{f}(-n) = \int f \varphi dm$$

for all bounded measurable φ supported on E. The last equality shows that $f = \psi$ a.e.; since ψ is continuous and E is of uniformly positive measure, this implies that $f|_E = \psi$.

We are finally prepared to establish the analogue of our main result for the circle group $\boldsymbol{T}$.

THEOREM 4. *Let ψ be a bounded measurable function defined on E, and satisfying* (3) *with constant K. Then there exists an $f \in A$ with $\|f\|_A \leqq K$, and such that*

$$f(e) = \psi(e) \quad \textit{for almost all} \quad e \in E .$$

Proof. By Lusin's theorem, given an integer N, we may choose F a closed subset of E, with $m(E \cap \mathscr{C}F) < (1/N)$, so that $\psi \mid F$ is continuous; let ψ_N denote $\psi|_F$. We may also assume that F is of uniformly positive measure, by simply taking N large enough and replacing F by the support of the measure $\chi_F dm$, if necessary.

For each N, ψ_N satisfies the hypotheses of Theorem 3, with constant K. Hence we may choose an $f_N \in A$, with $\|f_N\|_A \leqq K$ and $f_N|_F = \psi_N$. Again by Alaoglu's theorem, since the $\hat{f}_N$'s are uniformly bounded in $c_0(\boldsymbol{Z})^*$, there exists a function τ defined on $\boldsymbol{Z}$ and a subsequence $\hat{f}_{N_j}$ of the $\hat{f}_N$'s, so that

$$\|\tau\|_{L^1(\mathbf{Z})} = \sum_{n=-\infty}^{\infty} |\tau(n)| \leqq K ,$$

and so that

$$\lim_{j \leftarrow \infty} \sum_{n=-\infty}^{\infty} \hat{f}_{N_j}(n)\beta(-n) = \sum_{n=-\infty}^{\infty} \tau(n)\beta(-n)$$

for all $\beta \in c_0(\boldsymbol{Z})$. Thus, let

$$f(x) = \sum_{-\infty}^{\infty} \tau(n) e^{2\pi i n x}$$

for all $x \in \boldsymbol{T}$; then $\|f\|_A \leqq K$, and

$$\lim_{j\to\infty} \int f_{N_j} \varphi dm = \int f \varphi dm$$

for all bounded measurable functions φ defined on E. But fix such a φ; then

$$\lim_{N\to\infty} \int f_N \varphi dm = \int \psi \varphi dm \;;$$

indeed, for fixed N, taking the corresponding F as in the first statement of this proof, we have that

$$\int |f_N - \psi| \varphi dm = \int_{E \cap \mathscr{C} F} |f_N - \psi| \varphi dm \leqq \frac{1}{N}(K + \|\psi\|_\infty) \|\varphi\|_\infty \,.$$

Hence, $\psi = f$ a.e. an E.

3. Proof of the main result. We first have need of the following lemma, showing that the Stieltjes transform of a finite compactly supported measure on the real line may be nicely approximated by its values on a discrete subset.

LEMMA 5. *Given ε and $N > 0$, there exists an $M > 0$, so that if $L \geqq M$ and if ν is a finite measure supported on $[-N, N]$,*

$$\sup_{x\in \boldsymbol{R}} |\hat{\nu}(x)| \leqq (1 + \varepsilon) \sup_{j \in \boldsymbol{Z}} \left| \hat{\nu}\left(\frac{\pi j}{L}\right)\right| \,.$$

Proof. We first note that given λ real number, there exists $f \in L^1(\boldsymbol{R})$ with $\hat{f}(x) = e^{i\lambda x} - 1$ for all $|x| \leqq N$, and such that $\|f\|_1 \leqq 6|\lambda| N$. For example, let

$$k(x) = \frac{1}{2N} (\chi_{[-N,N]})\hat{}\,(x) (\chi_{[-2N,2N]})\hat{}\,(x)$$

for all real x, and set

$$f(x) = \frac{1}{2\pi} (k(x + \lambda) - k(x))$$

for all real x.

(To see that f has the desired properties, one may use an argument analogous to that given in the proof of 2.6.3, page 49 of [5]. Briefly, for $|y| \leqq N$, we have that

$$\frac{1}{2N}\chi_{[-N,N]}*\chi_{[-2N,2N]}(y) = 1; \quad \text{hence}$$

$$\hat{f}(y) = (e^{i\lambda y} - 1)\frac{\hat{k}(y)}{2\pi} = e^{i\lambda y} - 1$$

by the inversion theorem. Now

$$f(x) = \frac{1}{2\pi}\frac{1}{2N}(e^{i\lambda\cdot}\chi_{[-N,N]})\hat{\ }(x)((e^{i\lambda\cdot} - 1)\chi_{[-2N,2N]})\hat{\ }(x)$$

$$+ \frac{1}{2\pi}\frac{1}{2N}((e^{i\lambda\cdot} - 1)\chi_{[-N,N]})\hat{\ }(x)(\chi_{[-2N,2N]})\hat{\ }(x) .$$

Hence by the Plancherel theorem and the Schwartz inequality,

$$\|f\|_1 \leqq \frac{1}{2N}\|\chi_{[-N,N]}\|_2 \sup_{|y|\leqq 2N}|e^{i\lambda y} - 1|\,\|\chi_{[-2N,2N]}\|_2$$

$$+ \frac{1}{2N}\sup_{|y|\leqq N}|e^{i\lambda y} - 1|\,\|\chi_{[-N,N]}\|_2\,\|\chi_{[-2N,2N]}\|_2$$

$$\leqq 3\sqrt{2}\,|\lambda|\,N ;$$

thus the constant "6" could be replaced by the constant "$3\sqrt{2}$".)

Now, suppose $L > 6\pi N$, ν is supported on $[-N, N]$, and fix x a real number. Let j be the integer such that

$$\frac{\pi j}{L} \leqq x < \frac{\pi(j+1)}{L} .$$

Next, choose f as in the first statement of the proof, with $\lambda = (\pi j/L) - x$, and let $f_1(y) = f(y - (\pi j/L))$ for all real y. Then

$$\begin{aligned}
&\left|\hat{\nu}(x) - \hat{\nu}\left(\frac{\pi j}{L}\right)\right| \\
&= \left|\int_{-N}^{N}(e^{-ixt} - e^{-i(\pi j/L)t})d\nu(t)\right| \\
&= \left|\int_{-\infty}^{\infty}\hat{f}_1(t)d\nu(t)\right| \\
&= \left|\int_{-\infty}^{\infty}\hat{\nu}(t)f_1(t)dt\right| \\
&\leqq \|\hat{\nu}\|_\infty\|f_1\|_1 \leqq 6\,|\lambda|\,N\|\hat{\nu}\|_\infty \leqq 6N\frac{\pi}{L}\|\hat{\nu}\|_\infty .
\end{aligned}$$

Hence,

$$|\hat{\nu}(x)| - 6N\frac{\pi}{L}\|\hat{\nu}\|_\infty \leqq \left|\hat{\nu}\left(\frac{\pi j}{L}\right)\right| \leqq \sup_{k\in Z}\left|\hat{\nu}\left(\frac{\pi k}{L}\right)\right| .$$

Thus, since x was arbitrary,

$$\|\hat{\nu}\|_\infty \leqq \frac{1}{1-\dfrac{6N\pi}{L}} \sup_{k\in Z}\left|\hat{\nu}\left(\frac{\pi k}{L}\right)\right| .$$

So, given $\varepsilon > 0$, simply choose M so that $L \geqq M$ implies that

$$\frac{1}{1-\dfrac{6N\pi}{L}} \leqq 1+\varepsilon .$$

REMARK. Our proof shows that the conclusion of Lemma 5 holds not only for Stieltjes transforms, but for any bounded continuous function φ whose spectrum is supported on the interval $[-N, N]$, i.e. we obtain that

$$\sup|\varphi(x)| \leqq (1+\varepsilon)\sup_{j\in Z}\left|\varphi\left(\frac{\pi j}{L}\right)\right|$$

for all $L \geqq M$.

Proof of the main result. (All terms are as defined on the first page of this paper.)

Fix N an integer; by Lemma 5, we may choose $L > N$ so that if ν is a finite measure supported on $[-N, N]$, then

$$\sup_{x\in R}|\hat{\nu}(x)| \leqq \left(1+\frac{1}{N}\right)\sup_{j\in Z}\left|\hat{\nu}\left(\frac{\pi j}{L}\right)\right| .$$

We assume that φ satisfies condition (1), or equivalently, condition (3); let $\varphi_N = \varphi|_{E\cap[-N,N]}$. φ_N may be considered as being defined on a closed subset of the reals modulo $2L$; we then have that if ν is a discrete measure supported on $E\cap[-N, N]$

$$\left|\int\varphi d\nu\right| \leqq K\sup_{x\in R}|\hat{\nu}(x)| \leqq K\left(1+\frac{1}{N}\right)\sup_{j\in Z}\left|\hat{\nu}\left(\frac{\pi}{L}j\right)\right| .$$

Applying the obvious version of Theorem 4 for the reals modulo $2L$ instead of the reals modulo 1, we obtain that there exists a sequence $\{a_j\}$ with

$$\sum_{j=-\infty}^{\infty}|a_j| < \left(1+\frac{1}{N}\right)K ,$$

such that

$$\varphi_N(x) = \sum a_j e^{i(\pi/L)jx}$$

for almost all $x \in E \cap [-N, N]$.

Now let μ_N be the discrete measure which, for each integer j, assigns mass a_{-j} to the point $(\pi/L)j$; then $\varphi_N = \hat{\mu}_N$ a.e. on $E \cap [-N, N]$, and $\|\mu_N\| \leqq (1 + (1/N))K$.

Finally, by Alaoglu's theorem, since the finite measures on $\boldsymbol{R}$ may be identified with the adjoint of $C_0(\boldsymbol{R})$, the Banach space of continuous functions vanishing at infinity, we may choose a finite measure μ, with $\|\mu\| \leqq K$, and a subsequence $\{\mu_{N_j}\}$ so that

$$\int f d\mu = \lim_{j\to\infty} \int f d\mu_{N_j}$$

for all $f \in C_0(\boldsymbol{R})$. Now if g is a continuous function with compact support, then

$$\begin{aligned}
&\int_{-\infty}^{\infty} g(x)\varphi(x)dx \\
&\quad = \lim_{j\to\infty} \int_{-\infty}^{\infty} g(x)\varphi_{N_j}(x)dx \\
&\quad = \lim_{j\to\infty} \int_{-\infty}^{\infty} g(x)\hat{\mu}_{N_j}(x)dx \\
&\quad = \lim_{j\to\infty} \int_{-\infty}^{\infty} \hat{g}(x)d\mu_{N_j}(x) = \int_{-\infty}^{\infty} \hat{g}(x)d\mu(x) = \int_{-\infty}^{\infty} g(x)\hat{\mu}(x)dx \,.
\end{aligned}$$

Hence $\hat{\mu} = \varphi$ a.e.

REMARK. For the sake of simplicity in notation, we have only considered the one-dimensional case. However, all our results also hold in the context of $\boldsymbol{R}^p$ and $\boldsymbol{T}^p$ for all $p > 1$. We indicate briefly the necessary changes in the notation and arguments.

We identify $\boldsymbol{T}^p$ with $\boldsymbol{R}^p/\boldsymbol{Z}^p$, and endow both $\boldsymbol{T}^p$ and $\boldsymbol{R}^p$ with the sup of coordinates metric. If a and b are real numbers, we define the half-open p-dimensional interval

$$[a, b)_p = \{x \in \boldsymbol{R}^p : x = (x_1, \cdots, x_p) \text{ and } a \leqq x_j < b \text{ for all } 1 \leqq j \leqq p\} \,.$$

Similarly, we define closed and open intervals. If $x \in \boldsymbol{R}^p$ and $n \in \boldsymbol{Z}^p$, we define

$$xn = nx = n_1x_1 + \cdots + n_px_p \,.$$

We then replace "$\boldsymbol{Z}$", "$\boldsymbol{R}$" and "$\boldsymbol{T}$" by "$\boldsymbol{Z}^p$", "$\boldsymbol{R}^p$", and "$\boldsymbol{T}^p$" respectively, throughout the paper. Where summation indices run over $\boldsymbol{Z}$, we thus allow them to run over $\boldsymbol{Z}^p$, and where integrals are taken over intervals, we take them over p-dimensional intervals. With these changes, the statements and proofs of Theorems 3, 4, and the main result are exactly the same; a few more modifications are required for the proofs of the three lemmas, as follows:

In Lemmas 1 and 2, we take β to be a point in $[0, 1/N)_p$. In the proof of Lemma 1, we allow the indices "k" to range over all $k \in \boldsymbol{Z}^p$ such that $k = (k_1, \cdots, k_p)$ and $0 \leqq k_j \leqq N-1$ for all $1 \leqq j \leqq p$. For each such k, we define

$$I_k = \beta + \frac{k}{N} + \left[0, \frac{1}{N}\right]_p .$$

$\mathscr{J}$ is then defined to be all intervals such that all of their endpoints belong to F; i.e.

$$\mathscr{J} = \Big\{ I_k \colon \text{for all } x \in \boldsymbol{Z}^p \text{ such that } x_j = 0 \text{ or } 1 \text{ for all } j, \text{ we have that } \beta + \frac{k+x}{N} \in F \Big\} .$$

Exactly the same definitions are given for $\mathscr{K}$ and $\mathscr{K}'$, then $\mathscr{K}''$ is defined as

$$\Big\{ k \in \mathscr{K}' \colon \text{there exists an } x \in \boldsymbol{Z}^p \text{ with } x_j = 0 \text{ or } 1 \text{ all } j, \text{ so that } \beta + \frac{k+x}{N} \notin F \Big\} .$$

We may then correspond to each member of $\mathscr{K}''$ a member of $\mathscr{K} \cap \mathscr{C}\mathscr{K}'$ as follows:

Given $k \in \mathscr{K}''$, choose $x \in \boldsymbol{Z}^p$ with $x_j = 0$ or 1 for all j, such that $\beta + ((k+x)/N) \notin F$. Now let l be the least integer with $1 \leqq l \leqq N-1$ so that there exists a $q \in \mathscr{K}$ and an $m \in \boldsymbol{Z}^p$ with $k + lx - q = Nm$ (i. e. such that $k + lx \equiv q \bmod N\boldsymbol{Z}^p$); then $q \in \mathscr{K} \cap \mathscr{C}\mathscr{K}'$, so we correspond q to k.

Given such a q and such an x, k is uniquely determined by the relation $k \equiv q - lx \bmod N\boldsymbol{Z}^p$, where l is chosen to be the least integer with $1 \leqq l \leqq N-1$, so that $\beta + ((q - lx)/N) \in F$.

However, for different x's, we may have different k's in $\mathscr{K}''$ corresponded to the same q in $\mathscr{K} \cap \mathscr{C}\mathscr{K}'$. Since there are at most $2^p - 1$ such x's (x_j must equal 1 for some j), it follows that

$$\frac{1}{2^p - 1} \operatorname{card} \mathscr{K}'' \leqq \operatorname{card} (\mathscr{K} \cap \mathscr{C}\mathscr{K}') .$$

We thus obtain that $r - l \leqq 2^p(r - q)$, where r, l, and q are as defined in Lemma 1; the term "4ε" must then be replaced by the term "$2^{p+1}\varepsilon$".

One other modification is required: in all rational numbers having N as denominator (and not having a "k" as a numerator!), we replace "N" by "N^p". Thus the function $g(x)$ is defined on $[0, 1/N))$, by

$$g(x) = \frac{1}{N^p} \sum_{\substack{k_j=0 \\ 1 \leqq j \leqq p}}^{N-1} \chi_F\left(x + \frac{k}{N}\right);$$

we then have that

$$N^p \int_{[0, 1/N)_p} g dm = m(F) .$$

For the proof of Lemma 2, we replace the function Δ_N by the function

$$\Delta_{N,p} = N^{2p} \chi_{[0,1/N]_p} * \chi_{[-1/N, 0]_p} .$$

$m_{N\beta}$ is then defined as the discrete measure which assigns mass $1/N^p$ to each of the points $\beta + (k/N)$, where $k = (k_1, \cdots, k_p)$ and $0 \leqq k_q \leqq N - 1$ for all $1 \leqq q \leqq p$. Exactly the same proof then holds.

Finally, in the proof of Lemma 5, the number "6" should be replaced by a constant K that depends only on p. (Of course, λ is taken as a point in $\boldsymbol{R}^p$, with $|\lambda| = \sup_{1 \leqq j \leqq p} |\lambda_j|$.) An example of a function with the property given in the first line of the proof of Lemma 5, may then be obtained by setting

$$k(x) = \frac{1}{2^p N^p} (\chi_{[-N,N]_p})^\wedge(x) (\chi_{[-2N,2N]})^\wedge(x)$$

for all $x \in \boldsymbol{R}^p$, and then putting

$$f(x) = \frac{1}{2^p \pi^p} (k(x + \lambda) - k(x)) \quad \text{for all} \quad x \in \boldsymbol{R}^p .$$

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