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TRANSFORMATIONS ON TENSOR SPACES

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In this paper we consider those linear transformations from one tensor product of vector spaces to another which carry nonzero decomposable tensors into nonzero decomposable tensors. We obtain a general decomposition theorem for such transformations. If we suppose further that the transformation maps the space into itself then we have a complete structure theorem in the following two cases: (1) the transformation is onto, and (2) the field is algebraically closed and the tensor space is a product of finite dimensional vector spaces. The main results are contained in Theorems 3.5 and 3.8 which state that the transformation $T: U_1 \otimes \cdots \otimes U_n \rightarrow U_1 \otimes \cdots \otimes U_n$ has the form $T(x_1 \otimes \cdots \otimes x_n) = T_1(x_{\pi(1)}) \otimes \cdots \otimes T_n(x_{\pi(n)})$ where $T_i: U_{\pi(i)} \rightarrow U_i$ are nonsingular and π is a permutation. Case (2) generalizes a theorem of Marcus and Moyls.

Let F be a field and $\{U_a: a \in A\}$ be a finite set of vector spaces over F. Let $(U, t) = (\otimes(U_a: a \in A), t)$ be a tensor product. Then U is a vector space over F, $t: \prod (U_a: a \in A) \rightarrow U$ is multilinear and, for any vector space V over F and multilinear map $f: \prod (U_a: a \in A) \rightarrow V$, there is a unique linear transformation $g: U \rightarrow V$ such that $g \cdot t = f$. The decomposable tensors of U are defined to be the vectors $t(\prod (u_a: a \in A))$, denoted by $\otimes(u_a: a \in A)$, where $u_a \in U_a$ for $a \in A$.

The proofs of the main theorems are based on the purely combinatorial results of the following section.

2. Adjacency preserving functions.

In this section we define the adjacency preserving functions and find a decomposition theorem for them.

Let A be a nonempty finite set and for each $a \in A$ let S_a be a nonempty set. If J is a nonempty subset of A we let p_J denote the projection of $\prod (S_a: a \in A)$ onto $\prod (S_a: a \in J)$. If $J = \{a\}$ we write p_a for p_J.

For each $J \subseteqq A$ we define an equivalence relation, denoted by $\equiv (\text{mod } J)$, on $\prod (S_a: a \in A)$ by setting $x \equiv y \pmod J$ if and only if $p_a(x) = p_a(y)$ for all $a \in A \backslash J$. If $X \subseteqq \prod (S_a: a \in A)$ is a nonempty subset of equivalent elements relative to $\equiv (\text{mod } J)$ then we call X a J-subset. If $J = \{a\}$ is a singleton we use a-subset for J-subset. We note the following

2.1. LEMMA. *A subset $X \subseteqq \prod (S_a: a \in A)$ is an equivalence class relative to $\equiv (\text{mod } J)$ if and only if $p_J(X) = \prod (S_a: a \in J)$ and $p_a(X)$*

is a singleton for each $a \in A \backslash J$.

The equivalence classes will be called maximal J-subsets.

On the set of subsets of $\prod (S_a: a \in A)$ we define, for each $J \subsetneqq A$, the relation adj (mod J) by setting $X \operatorname{adj} Y (\operatorname{mod} J)$ if and only if X and Y are J-subsets and for precisely one $a \in A \backslash J$ we have $p_a(X) \neq p_a(Y)$. If $x, y \in \prod (S_a: a \in A)$ and $\{x\} \operatorname{adj} \{y\} (\operatorname{mod} \phi)$ then we say that x and y are adjacent. The relation adj (mod J) is symmetric but neither reflexive non transitive.

2.2. LEMMA. *Let* $J \subsetneqq A$ *and let* X *and* Y *be distinct maximal* J*-subsets of* $\prod (S_a: a \in A)$. *Then there is a finite sequence* $X_1, \cdots, X_n$ *of maximal* J*-subsets such that* $X = X_1$, $Y = X_n$, *and* $X_i \operatorname{adj} X_{i+1} (\operatorname{mod} J)$ *for* $i = 1, \cdots, n-1$.

Proof. Let $a_1, \cdots, a_n$ be the distinct elements of $A \backslash J$ for which $p_{a_i}(X) \neq p_{a_i}(Y)$. Then the maximal J-subsets X_i for which $P_{a_j}(X_i) = p_{a_j}(Y)$ for $j \leqq i$ and $p_{a_j}(X_i) = p_{a_j}(X)$ for $j > i$ will suffice.

2.3. DEFINITION. A function from one cartesian product of sets into another is an *adjacency preserving* function if and only if the images of adjacent elements are adjacent.

2.4. LEMMA. *Let* $f: \prod (S_a: a \in A) \rightarrow \prod (R_b: b \in B)$ *be an adjacency preserving function. For each* $a \in A$ *let* S_a *contain at least three elements. Then there is a function* $\sigma: A \rightarrow B$ *such that for any* $c \in A$ *and any maximal* c*-subset* X *of* $\prod (S_a: a \in A)$, $f(X)$ *is a* $\sigma(c)$*-subset of* $\prod (R_b: b \in B)$.

Proof. Let $c \in A$ and let X be a maximal c-subset of $\prod (S_a: a \in A)$. Then $f(X)$ is a d-subset of $\prod (R_b: b \in B)$ for some $d \in B$, where d depends on c and X. For, let x_1 and x_2 be distinct elements of X and let $d \in B$ be that element of B for which $p_d(f(x_1)) \neq p_d(f(x_2))$. Then, for any $x \in X$, $p_d(f(x))$ differs from at least one of the $p_d(f(x_i))$ and so $p_b(f(x))$ is independent of $x \in X$ for $b \neq d$. Therefore $f(X)$ is a d-subset. We let $\sigma(c, X) = d$. We show that $\sigma(c, X)$ is independent of the maximal c-subset X. Suppose the contrary. Then, from Lemma 2.2 it follows easily that there is a pair of maximal c-subsets X and Y for which $X \operatorname{adj} Y (\operatorname{mod} \{c\})$ and $\sigma(c, X) = d_1 \neq d_2 = \sigma(c, Y)$. Let c' be the unique element of A for which $p_{c'}(X) \neq p_{c'}(Y)$. Let $q: X \rightarrow Y$ be defined on each $x \in X$ by $p_a(q(x)) = p_a(x)$ if $a \neq c'$ and $p_{c'}(q(x)) = p_{c'}(Y)$. Then q is one-to-one, onto, and for each $x \in X$ the pair x and $q(x)$ are adjacent. Since S_c has at least three elements there are at least two elements $x \in X$ such that $p_{d_1}(f(x)) \neq p_{d_1}(f(Y))$, and of these, at least one for which $p_{d_2}(f(q(x))) \neq p_{d_2}(f(X))$. Now, for any $x \in X$ satisfying both of these inequalities, $f(x)$ and $f(q(x))$ are not adjacent, contrary to the hypothesis on f. Therefore $\sigma(c, X)$ is independent of the c-subset X and so $\sigma(c) = \sigma(c, X)$ is a well defined function satisfying the conclusion of the lemma.

2.5. THEOREM. *Let* $f: \prod (S_a: a \in A) \rightarrow \prod (R_b: b \in B)$ *be an adjacency preserving function and suppose that each* S_a *contains at least three elements. Then there is a partition of* A *into subsets* $A_1, \cdots, A_k$ *and distinct elements* $b_1, \cdots, b_k$ *of* B *such that for each* $i = 1, \cdots, k$ *there is a function* $f_i: \prod (S_a: a \in A_i) \rightarrow R_{b_i}$ *satisfying* $p_{b_i} \cdot f = f_i \cdot p_{A_i}$. *Furthermore, the image of* $\prod (S_a: a \in A)$ *under* f *is the set* $\prod (Q_b: b \in B)$ *where* Q_b *is the image of* $\prod (S_a: a \in A)$ *under* $(p_b \cdot f)$.

Proof. Let σ be given as in Lemma 2.4. Let $\{b_1, \cdots, b_k\} = \sigma(A)$ and let $A_i = \sigma^{-1}(b_i)$. Then $A_1, \cdots, A_k$ is a partition of A. Let J be one of the A_i and b the corresponding b_i. Let X be a maximal J-subset of $\prod (S_a: a \in A)$. We define $f_X: \prod (S_a: a \in J) \rightarrow R_b$ by

$$f_X = p_b \cdot f \cdot (p_J \mid X)^{-1} .$$

Then f_X is well defined since $(p_J \mid X)$ is a one-to-one function from X onto $\prod (S_a: a \in J)$. We prove that $f_X = f_Y$ for any two maximal J-subsets X and Y. Suppose the contrary. Then, from Lemma 2.2, it follows that we can choose maximal J-subsets X and Y such that $X \operatorname{adj} Y (\operatorname{mod} J)$ and $f_X \neq f_Y$. Let $a' \in A \backslash J$ be that element for which $p_{a'}(X) \neq p_{a'}(Y)$. Choose $s \in \prod (S_a: a \in J)$ such that $f_X(s) \neq f_Y(s)$. Let $x = (p_J \mid X)^{-1}(s)$ and $y = (p_J \mid Y)^{-1}(s)$. Then $x \in X$ and $y \in Y$ are a pair of adjacent elements of $\prod (S_a: a \in A)$. If we let $b' = \sigma(a')$ then $b' \neq b$ since $a' \notin J = \sigma^{-1}(b)$. Therefore, $f(x)$ and $f(y)$ are adjacent and b' is the only element of B for which $p_{b'}(f(x)) \neq p_{b'}(f(y))$. But $p_b(f(x)) = f_X(s) \neq f_Y(s) = p_b(f(y))$, a contradiction.

For each i we set $f_i = f_X$ where X is any maximal A_i-subset of $\prod (S_a: a \in A)$. Then, if $x \in \prod (S_a: a \in A)$, we choose a maximal A_i-subset X containing x and note that

$$(f_i \cdot p_{A_i})(x) = (p_{b_i} \cdot f \cdot (p_{A_i} \mid X)^{-1})(p_{A_i}(x)) = (p_{bi} \cdot f)(x) .$$

If $b \notin \sigma(A)$ then the image of $\prod (S_a: a \in A)$ under $(p_b \cdot f)$ consists of one element of R_b. In fact, suppose x and y are adjacent elements of $\prod (S_a: a \in A)$. Then $f(x)$ and $f(y)$ are adjacent and the element $b' \in B$ for which $p_{b'}(f(x)) \neq p_{b'}(f(y))$ is in $\sigma(A)$. Then $(p_b \cdot f)(x) = (p_b \cdot f)(y)$.

It is clear that $f(\prod (S_a: a \in A) \subseteqq \prod (Q_b: b \in B)$. To show that we have equality, suppose that $x \in \prod (Q_b: b \in B)$. Choose $y_b \in \prod (S_a: a \in A)$ such that $(p_b \cdot f)(y_b) = p_b(x)$. Choose $y \in \prod (S_a: a \in A)$ such that $p_{A_i}(y) = p_{A_i}(y_{b_i})$ for $i = 1, \cdots, k$. This can be done since the sets $A_1, \cdots, A_k$ are pairwise disjoint. If $b \notin \sigma(A)$ then $(p_b \cdot f)(y) = p_b(x)$ since the image is independent of y. For b_i we have $(p_{b_i} \cdot f)(y) = f_{A_i}(p_{A_i}(y)) = f_{A_i}(p_{A_i}(y_{b_i})) = p_{b_i}(f(y_{b_i})) = p_{b_i}(x)$. Therefore $f(y) = x$, and this completes the proof.

3. The preservers of decomposable tensors. In this section we require the

3.1. LEMMA. *Let $U = \otimes (U_a: a \in A)$ be a tensor product where the U_a are vector spaces over a field F. Let $x_a, x'_a \in U_a$ for $a \in A$. Then*

(1) *$\otimes (x_a: a \in A) = 0$ if and only if $x_a = 0$ for some $a \in A$.*

(2) *If $x = \otimes (x_a: a \in A)$ and $x' = \otimes (x'_a: a \in A)$ are nonzero decomposable tensors then,*

(a) *$\langle x \rangle = \langle x' \rangle$ if and only if $\langle x_a \rangle = \langle x'_a \rangle$ for $a \in A$.*

(b) *$x + x'$ is a decomposable tensor if and only if $\langle x_a \rangle = \langle x'_a \rangle$ for all except possibly one $a \in A$.*

Proof. The statements (1) and (2)(a) are elementary properties of U. The sufficiency of the condition in (2)(b) is clear. We prove the necessity of this condition by the following indirect argument. Suppose $x + x' = \otimes (y_a: a \in A)$ where $\langle x_b \rangle \neq \langle x'_b \rangle$ and $\langle x_c \rangle \neq \langle x'_c \rangle$ for a pair of indicies b and c. We may suppose that $\langle y_b \rangle \neq \langle x_b \rangle$. We define a function $f: \prod (U_a: a \in A) \rightarrow F$ as follows. For each $a \in A$ we choose a linear functional $f_a \in \mathscr{L}(U_a, F)$ such that

(1) $f_a(x_a) \neq 0$ for all $a \in A$,

(2) $f_b(y_b) = f_c(x'_c) = 0$,

and set $f(u) = \prod (f_a(p_a(u)): a \in A)$. Then f is multilinear and it induces a linear transformation $f': \otimes (U_a: a \in A) \rightarrow F$. But $0 = f'(y) = f'(x + x') = f'(x) + f'(x') = f'(x) \neq 0$, which is impossible.

Throughout the rest of this section we let $U = \otimes (U_a: a \in A)$ and $W = \otimes (W_b: b \in B)$ where the U_a and W_b are vector spaces over a field F. We also assume that $\dim (U_a) \geqq 2$ and that A, B are finite sets. We let $T: U \rightarrow W$ be a linear transformation mapping nonzero decomposable tensors into nonzero decomposable tensors.

Let S_a and R_b be the sets of one dimensional subspaces of U_a and W_b respectively. We define a function $f: \prod (S_a: a \in A) \rightarrow \prod (R_b: b \in B)$ as follows. Let $x_a \in U_a$ be nonzero and let $T(\otimes (x_a: a \in A)) = \otimes(y_b: b \in B)$. Let $x \in \prod (S_a: a \in A)$ and $y \in \prod (R_b: b \in B)$ such that $p_a(x) = \langle x_a \rangle$ and $p_b(y) = \langle y_b \rangle$. We set $f(x) = y$ and note that by the above lemma, f is well defined. We prove next the

3.2. LEMMA. *The function f above is an adjacency preserving function.*

Proof. Let $u, u' \in \prod (S_a: a \in A)$ be adjacent and choose $x_a, x'_a \in u_a$ such that $p_a(u) = \langle x_a \rangle$, $p_a(u') = \langle x'_a \rangle$. Let $x = \bigotimes (x_a: a \in A)$ and $x' = \bigotimes (x'_a: a \in A)$. Then, by (2)(b) of Lemma 3.1, $x + x'$ is a decomposable tensor and by (2)(a) we have that $\langle x \rangle \neq \langle x' \rangle$. Let $y = T(x) = \bigotimes (y_b: b \in B)$ and $y' = T(x') = \bigotimes (y'_b: b \in B)$. Then $y + y'$ is a decomposable tensor. Let $w, w' \in \prod (R_b: b \in B)$ be such that $p_b(w) = \langle y_b \rangle$ and $p_b(w') = \langle y'_b \rangle$. Then w and w' are either adjacent or equal. If they were equal then $y = ey'$ for some $e \in F$, from which we get $T(x - ex') = 0$, a contradiction since $x - ex'$ is a nonzero decomposable tensor for all $e \in F$.

3.3. DEFINITION. For each $A' \subsetneqq A$ and $x \in \prod (U_a: a \notin A')$ we define a multilinear function $N_x: \prod (U_a: a \in A') \rightarrow U$ by setting $N_x(u) = \bigotimes (v_a: a \in A)$ where $v_a = p_a(x)$ for $a \notin A'$ and $v_a = p_a(u)$ for $a \in A'$. We let $M_x: \bigotimes (U_a: a \in A') \rightarrow U$ be the linear transformation induced by N_x.

Since $\dim (U_a) \geqq 2$, each S_a contains at least three elements, and therefore we can apply Lemma 2.4 to obtain the function $\sigma: A \rightarrow B$ satisfying the conclusions of that lemma. Let $\sigma(A) = \{b_1, \cdots, b_k\}$ and $A_i = \sigma^{-1}(b_i)$. Let $V_i = \bigotimes (U_a: a \in A_i)$, $V = \bigotimes (V_i: i = 1, \cdots, k)$ and let $\varphi: U \rightarrow V$ be the canonical isomorphism.

3.4. THEOREM. *The decomposable tensor preserver T has the form $M_y \cdot (T_1 \otimes \cdots \otimes T_k) \cdot \varphi$ where*

(1) *$y \in \prod (W_b: b \notin \sigma(A))$, or M_y is deleted if $\sigma(A) = B$,*

(2) *$T_i: V_i \rightarrow W_{b_i}$ is a linear transformation mapping nonzero decomposable tensors of V_i into nonzero vectors of W_{b_i}.*

Proof. Let $x_i \in \prod (U_a: a \notin A_i)$ be chosen for $i = 1, \cdots, k$. Consider $T \cdot M_{x_i}: V_i \rightarrow W$. For each $v \in V_i$, $M_{x_i}(v) = \bigotimes (v_a: a \in A)$ where $\langle v_a \rangle$ does not depend on v whenever $a \notin A_i$. Therefore, since $(p_b \cdot f)(s)$, for $s \in \prod (S_a: a \in A)$, does not depend on the coordinates of s in $A \backslash \sigma^{-1}(b)$, there are fixed $w_b \in W_b$ for each $b \in B$, $b \neq b_i$, such that the image of V_i under $T \cdot M_{x_i}$ has the form $\{\bigotimes (w'_b: b \in B) : w'_b = w_b$ for $b \neq b_i$ and $w'_{b_i} \in W'_{b_i}\}$, where W'_{b_i} is a subspace of W_{b_i}. Then $T \cdot M_{x_i}$ induces a linear transformation $T_i: V_i \rightarrow W_{b_i}$ defined by $T_i(v) = w'_{b_i}$ where $T \cdot M_{x_i}(v) = \bigotimes (w'_b: b \in B)$, $w'_b = w_b$ for $b \neq b_i$. If $x'_i \in \prod (U_a : a \notin A_i)$ is another element and $T'_i: V_i \rightarrow W_{b_i}$ is induced as above by $T \cdot M_{x'_i}$, then for each decomposable tensor $x \in V_i$ there is a $c_x \in F$ such that $T_i(x) = c_x T'_i(x)$. It then follows easily that there is a $c \in F$ such that $T_i = cT'_i$.

By Theorem 2.5 the image of $\prod (S_a: a \in A)$ under f is $\prod (Q_b: b \in B)$ where $Q_b \subseteqq R_b$ can be given explicitly. For each $b \notin \sigma(A)$, Q_b consists of one element and we let $Q_b = \langle y_b \rangle$. Let $y \in \prod (W_b : b \notin \sigma(A))$ where $p_b(y) = y_b$, and let $T' = M_y \cdot (T_1 \otimes \cdots \otimes T_k) \cdot \varphi$. If $x \in U$ is a decomposable tensor then $T(x) = e_x T'(x)$ for some $e_x \in F$, and therefore it follows that $T = eT'$ on all of U for some $e \in F$.

3.5. THEOREM. *A decomposable tensor preserve of U onto itself has the form $\otimes (T_a : a \in A)$ where $T_a : U_a \rightarrow U_{\sigma(a)}$ is an onto nonsingular linear transformation and $\sigma : A \rightarrow A$ is a permutation.*

Proof. Let $f : \prod (S_a : a \in A) \rightarrow \prod (S_a : a \in A)$ be the adjacency preserving function induced by the decomposable tensor preserver and let $\sigma : A \rightarrow A$ be the function induced by f. Then the image of $\prod (S_a : a \in A)$ under f has the form $\prod (Q_a : a \in A)$ and therefore the image of U is spanned by elements $\otimes (u_a : a \in A)$ where u_a belongs to the smallest subspace of U_a which contains all the subspaces making up Q_a. Since $\dim (U_a) \geqq 2$ and the tensor preserver is assumed to be onto, no Q_a can consist of only one element. Therefore σ is a permutation. The theorem now follows from Theorem 3.4. That T_a is onto and nonsingular is clear.

3.6. DEFINITION. If V is a vector space over a field F and if $\mathscr{L}(V)$ is the vector space of linear transformations of V into itself, then a subspace of $\mathscr{L}(V)$ is called a nonsingular subspace if each of its nonzero elements is a nonsingular linear transformation.

3.7. THEOREM. *Let $k \geqq 2$ be an integer and let $W_1, \cdots, W_k$ be vector spaces over a field F where $2 \leqq \dim (W_1) \leqq \cdots \leqq \dim (W_k) < \infty$. Then there is a linear transformation $L : \otimes (W_i : i = 1, \cdots, k) \rightarrow W_k$ mapping nonzero decomposable tensors into nonzero vectors if and only if $\mathscr{L}(W_k)$ contains a nonsingular subspace with dimension equal to $\dim (W_{k-1})$.*

Proof. Suppose that L exists. Let $w_i \in W_i$ for $i = 1, \cdots, k-2$ be nonzero vectors. For each $x \in W_{k-1}$ we let

$$\mathscr{W}_x = \{w_1 \otimes \cdots \otimes w_{k-2} \otimes x \otimes y : y \in W_k\} ,$$

and note that $\mathscr{W}_x$ is a subspace consisting of decomposable tensors. Let $L_x = (L \mid \mathscr{W}_x)$ and let $I_x : W_k \rightarrow \mathscr{W}_x$ be defined by

$$I_x(y) = w_1 \otimes \cdots \otimes w_{k-2} \otimes x \otimes y$$

for $y \in W_k$. Then $L_x I_x \in \mathscr{L}(W_k)$ is nonsingular for $x \neq 0$. Let $\dim (W_{k-1}) = t$ and let $\{x_1, \cdots, x_t\}$ be a basis of W_{k-1}. Then $\{L_{x_1} I_{x_1}, \cdots, L_{x_t} I_{x_t}\}$ spans a nonsingular subspace of $\mathscr{L}(W_k)$. For, suppose $\sum_{i=1}^{t} a_i L_{x_i} I_{x_i}$ is singular. Choose $y \neq 0$ such that

$$\sum_{i=1}^{t} a_i L_{x_i} I_{x_i}(y) = 0 .$$

Then, since $\sum_{i=1}^{t} a_i L_{x_i} I_{x_i}(y) = L(w_1 \otimes \cdots \otimes w_{k-2} \otimes \sum_{i=1}^{t} a_i x_i \otimes y)$, we must have that $\sum_{i=1}^{t} a_i x_i = 0$. Therefore $a_1 = \cdots = a_t = 0$.

Suppose that $\mathscr{L}(W_k)$ has a nonsingular subspace with dimension $t = \dim(W_{k-1})$. We construct L inductively. Suppose that

$$L_0 \colon W_2 \otimes \cdots \otimes W_k \to W_k$$

has been defined such that nonzero decomposable tensors of $W_2 \otimes \cdots \otimes W_k$ are mapped into nonzero vectors of W_k. Let $s = \dim(W_1)$ and let $\{L_1, \cdots, L_s\}$ be a basis of an s-dimensional nonsingular subspace of $\mathscr{L}(W_k)$. Such a basis exists since $s = \dim(W_1) \leqq \dim(W_{k-1})$. Let $\{x_1, \cdots, x_s\}$ be a basis of W_1. Let $N \colon W_1 \otimes W_k \to W_k$ be the linear transformation induced by the multilinear function $\bar{N} \colon W_1 \times W_k \to W_k$ where $\bar{N}(\sum_{i=1}^{s} a_i x_i, y) = \sum_{i=1}^{s} a_i L_i(y)$. Then $\bar{N}(x, y) = 0$ implies that $x = 0$ or $y = 0$ and therefore $N(x \otimes y) = 0$ if and only if $x \otimes y = 0$. Let $I \colon W_1 \to W_1$ be the identity and let $L = N \cdot (I \otimes L_0)$. Then $L(w_1 \otimes \cdots \otimes w_k) = N(w_1 \otimes L_0(w_2 \otimes \cdots \otimes w_k)) = 0$ implies that $w_1 = 0$ or $L_0(w_2 \otimes \cdots \otimes w_k) = 0$. Therefore, either $w_1 = 0$ or $w_2 \otimes \cdots \otimes w_k = 0$, and in both cases $w_1 \otimes \cdots \otimes w_k = 0$. This completes the proof.

3.8. THEOREM. *Let F be algebraically closed and let $T \colon U \to U$ be a decomposable tensor preserver where* $\dim(U_a)$ *is finite for all* $a \in A$. *Then* $T = \otimes(T_a \colon a \in A)$ *where* $T_a \colon U_a \to U_{\sigma(a)}$ *is a nonsingular linear transformation and* $\sigma \colon A \to A$ *is a permutation satisfying* $\dim(U_a) = \dim(U_{\sigma(a)})$ *for* $a \in A$.

Proof. We prove that σ is a permutation. By Theorem 3.4, $T = M_y \cdot (T_1 \otimes \cdots \otimes T_k) \cdot \varphi$ where the domain of T_i is $V_i = \otimes(U_a \colon a \in A_i)$ and $A_i = \sigma^{-1}(a_i)$ for some $a_i \in A$. For each $a \in A_i$, V_i contains a subspace with dimension equal to $\dim(U_a)$ which consists of decomposable tensors only. It follows that U_{a_i}, which is the range space of V_i under T_i, has dimension at least as large as the maximum of the $\dim(U_a)$ for $a \in A_i$. Therefore, for each $a \in A$, $\dim(U_a) \leqq \dim(U_{\sigma(a)})$. Suppose that σ is not one-to-one. Of those $a \in A$ for which $\sigma^{-1}(\sigma(a))$ consists of at least two elements, choose one, say b, for which $\dim(U_b)$ is maximal. Then $\dim(U_b) = \dim(U_{\sigma(b)})$. For, suppose that $\dim(U_b) < \dim(U_{\sigma(b)})$. Then σ maps the set $\{a \mid \dim(U_a) > \dim(U_b)\} \cup \{b\}$ into the set $\{a \mid \dim(U_a) > \dim(U_b)\}$, and consequently, there is a $b' \in A$ for which $\sigma^{-1}(\sigma(b'))$ has at least two elements and $\dim(U_{b'}) > \dim(U_b)$.

This contradicts the choice of b. Now, $\sigma(b) \in \sigma(A)$ so that $\sigma(b) = a_i$ for some i and $T_i: V_i \to U_{a_i}$. By Theorem 3.7, $\mathscr{L}(U_{a_i})$ contains a nonsingular subspace with dimension at least two. This is impossible since F is assumed to be algebraically closed (for nonsingular C and D, $C - eD$ is singular for any eigenvalue e of $D^{-1}C$). Therefore σ is a permutation and the theorem follows from Theorem 3.4.

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