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CRITERIA FOR ZERO CAPACITY OF IDEAL BOUNDARY COMPONENTS OF RIEMANNIAN SPACES

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Capacities of ideal boundary components of Riemannian spaces are introduced to measure their magnitude with respect to harmonic functions on the spaces. The main purpose of this paper is to find zero capacity criteria.

The modular criterion, well-known for Riemann surfaces, i.e. for 2-dimensional Riemannian spaces, is shown to be valid for general Riemannian spaces. The so-called metric criterion, however, brings forth entirely new aspects for higher dimensions.

CAPACITY OF A SUBBOUNDARY

1. **Subboundaries.** Throughout this paper we denote by R a noncompact orientable connected C^∞ Riemannian space. A relatively compact region whose relative boundary is smooth will be called a regular region. A sequence $\{R_n\}_1^\infty$ of regular regions $R_n \subset R$ such that $\bar{R}_n \subset R_{n+1}$ and $R = \bigcup_1^\infty R_n$ is called an exhaustion of R.

The ideal boundary component of Kerékjártó and Stoïlow may be related to $\{R_n\}$; here $R - \bar{R}_n$ can be assumed to consist of a finite number of relatively noncompact regions F_{ni} with corders β_{ni}. Choose a sequence $F_1 = F_{1i_1}$, $F_2 = F_{2i_2}$, $\cdots$ such that $\bar{F}_{n+1} \subset F_n$. Then $\{F_n\}_1^\infty$ defines a boundary component γ. We denote by γ_n the relative boundary ∂F_n of F_n.

A *subboundary*, also to be denoted by γ, is a union of boundary components.

2. **Capacity function.** Let B be a parametric ball about $a \in R$ with compact $\bar{B}$. Suppose γ is a subboundary of R, and γ_n the union of all ∂F_n such that $\{F_n\}_1^\infty$ defines a boundary component belonging to γ.

Consider the family $P = \{p\}$ of harmonic functions p on $R - a$ such that (a) $p = -g_a + h$ in B where g_a is the Green's function of B with pole at a, and h a harmonic function on B with $h(a) = 0$, (b) $\int_{\gamma_n} *dp = 1$ and $\int_{\beta_{ni} \not\subset \gamma_n} *dp = 0$ for large n, where the β_{ni} are components of ∂R_n.

We use the conventional notations

$$\int_\gamma *dp = \lim_{n\to\infty} \int_{\gamma_n} *dp \ .$$

$$\int_{\beta} p * dp = \lim_{n\to\infty} \int_{\beta_n} p * dp ,$$

β being the entire boundary of R.

Amalgamating the method of Sario [1] with the existence theorem of principal functions in Sario-Schiffer-Glasner [2], we can easily see that P is not empty and that there exists a function $p_\gamma \in P$ such that

$$k_\gamma = \min_P \int_\beta p * pd = \int_\beta p_\gamma * dp_\gamma . \tag{1}$$

Here $-\infty < k_\gamma \leqq \infty$, and if $k_\gamma < \infty$, then p_γ is unique. This follows from the identity

$$\int_\beta p * dp = D(p - p_\gamma) + \int_\beta p_\gamma * dp_\gamma , \tag{2}$$

where D indicates the Dirichlet integral.

The function p_γ shall be referred to as a *capacity function* for γ. The quantity $c_\gamma = e^{-k_\lambda}$ for $\dim R = 2$, and $k_\gamma^{-(m-2)}$ for $\dim R = m \geqq 3$ will be called the *capacity* of γ.

Modular Criterion

3. Moduli. Let Ω be a union of disjoint regular regions Ω_j, $j = 1, \cdots, k$. Suppose that $\partial\Omega_j$ consists of two nonempty disjoint sets β'_j and β''_j which are unions of components of $\partial\Omega_j$. Set $\beta' = \bigcup_1^k \beta'_j$ and $\beta'' = \bigcup_1^k \beta''_j$. Let u_0 be the continuous function on $\bar{\Omega}$ which is harmonic on Ω with $u_0 \mid \beta' = 0$, $u_0 \mid \beta'' = \log \mu$, and $\int_{\beta'} * du_0 = 1$. The constant $\mu > 1$ is called the *modulus* of the configuration $(\Omega, \beta', \beta'')$,

$$\mu = \operatorname{mod}(\Omega, \beta', \beta'') .$$

The function u_0 is referred to as the *modulus function.*

Consider the family $U = U(\Omega, \beta', \beta'')$ of C^1-functions u on $\bar{\Omega}$ which are harmonic on Ω with $\int_{\beta'} * du = 1$. Then we have

$$\min_U D_\Omega(u) = D_\Omega(u_0) . \tag{3}$$

This follows from the identity

$$D_\Omega(u) = D_\Omega(u_0 - u) + D_\Omega(u_0) \tag{4}$$

for every $u \in U$.

4. Modular criterion. Let $\tilde{F}_n$ be the sum of those F_n for which $\{F_n\}_1^\infty$ defines a boundary component in the subboundary γ. Consider

$E_n = (R_{n+1} - \bar{R}_n) \cap \tilde{F}_n$ and set $\gamma_n = \partial F_n$, $\gamma'_n = \partial E_n - \gamma_n$. In terms of

$$\mu_{n\gamma} = \text{mod}\,(E_n, \gamma_n, \gamma'_n) \tag{5}$$

we state:

THEOREM 1. *If there exists an exhaustion of R with*

$$\prod_{n=1}^{\infty} \mu_{n\gamma} = \infty \ , \tag{6}$$

then the capacity of γ vanishes.

In fact, let p_n and k_n stand for p_γ and k_γ with respect to γ_n and R_n. By (1) we infer that

$$D_{R_{n+1}-\bar{R}_n}(p_{n+1}) \leqq k_{n+1} - k_n \ ,$$

and by (3) that

$$\log \mu_{n\gamma} \leqq D_{E_n}(p_{n+1}) \ .$$

Therefore $\log \mu_{n\gamma} \leqq k_{n+1} - k_n$, and we conclude that (6) implies

$$\lim_{n\to\infty} k_n = \infty \ .$$

On the other hand it is not difficult to see that $k_\gamma = \lim_n k_n$, whence $c_\gamma = 0$.

METRIC CRITERION

5. Conformally equivalent metric. Let λ be a positive C^∞-function on R. The new metric

$$d\sigma = \lambda ds \tag{7}$$

is conformally equivalent to the original metric ds on R. We fix a point $a \in R$ and assume that

$$R(r) = \{x \in R \mid \sigma(x, a) < r\} \tag{8}$$

is relatively compact in R for $0 < r < \infty$, with $R = \bigcup_{0<r<\infty} R(r)$. Consider the minimal union $\gamma(r)$ of components of $\beta(r) = \partial R(r)$ which separates γ from a. Let

$$S_\gamma(r) = \int_{\gamma(r)} dS_\sigma \ , \tag{9}$$

where dS_σ is the surface element induced by $d\sigma$.

THEOREM 2. *If there exists an admissible λ such that*

$$\int_{\varepsilon}^{\infty} \frac{dr}{S_\gamma(r)} = \infty \qquad (\varepsilon > 0) , \tag{10}$$

then γ *has vanishing capacity for* R *with* $\dim R = 2$. *If, moreover, there exists a constant* M *such that*

$$0 < \frac{1}{M} \leqq \lambda \leqq M , \tag{11}$$

then the same conclusion holds regardless of the dimension of R.

For the proof we choose a sequence $\{r_n\}_1^\infty$ such that $\varepsilon < r_n < r_{n+1} < \infty$ and $\lim_n r_n = \infty$, with $R_n = R(r_n)$ regular. As in § 4 we define $E_n = (R_{n+1} - \bar{R}_n) \cap \tilde{F}_n$ and $\mu_{n\gamma}$. We also denote by u_n the corresponding modulus function.

The proof in the case $\dim R = 2$ will be given in § 6 and that in the general case under the assumption (11), in § 7.

6. The case $\dim R = 2$. Observe that

$$\int_{E_n} |\nabla_\sigma u_n|^2 \, dV_\sigma = \int_{r_n}^{r_{n+1}} \left[\int_{\gamma(r)} |\nabla_\sigma u_n|^2 \, dS_\sigma \int_{\gamma(r)} dS_\sigma \right] \frac{dr}{S_\gamma(r)} . \tag{12}$$

By the Schwarz inequality we have

$$\int_{\gamma(r)} |\nabla_\sigma u_n|^2 \, dS_\sigma \int_{\gamma(r)} dS_\sigma \geqq \left(\int_{\gamma(r)} *_\sigma du_n \right)^2 . \tag{13}$$

Since $*_\sigma = *$ and $|\nabla_\sigma u_n|^2 \, dV_\sigma = |\nabla u_n|^2 \, dV$, it is seen that (12), (13), and $D_{E_n}(u_n) = \log \mu_{n\gamma}$ imply

$$\log \prod_1^n \mu_{k\gamma} \geqq \int_{r_1}^{r_{n+1}} \frac{dr}{S_\gamma(r)} . \tag{14}$$

We conclude that (10) implies (6), and consequently $c_\gamma = 0$.

7. The case $\dim R = m > 2$. By (11) we see that

$$\int_{E_n} |\nabla_\sigma u_n|^2 \, dV_\sigma \leqq M^{m-2} D_{E_n}(u_n) , \tag{15}$$

$$\int_{\gamma(r)} *_\sigma du_n \geqq M^{-(m-2)} \int_{\gamma(r)} * du_n . \tag{16}$$

Therefore (14) must be modified to give

$$\log \prod_1^n \mu_{k\gamma} \geqq M^{-3(m-2)} \int_{r_1}^{r_{n+1}} \frac{dr}{S_\gamma(r)} .$$

But this sufficient to conclude that $c_\gamma = 0$.

REMARK. Condition (11) cannot be suppressed in the case of higher dimensions.

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