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CHARACTERIZATION OF THE CONTINUOUS IMAGES OF ALL PSEUDO-CIRCLES

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The purpose of this paper is to establish a characterization of the continuous images of all pseudo-circles, and develop techniques which may be used in the further investigation of the mapping properties of pseudo-circles. The principal conclusions drawn in this paper from this characterization are the theorems that every planar circularly chainable continuum is the continuous image of a pseudo-circle and every snake-like continuum is the continuous image of a pseudo-circle.

The class of pseudo-circles may be defined to be the class of hereditarily indecomposable circularly chainable continua such that, if M is a pseudo-circle, then M is the intersection of the sets of points of a sequence $D_1, D_2, D_3, \cdots$ of circular chains having the properties that: (1) D_{i+1} is crooked in D_i, $i = 1, 2, 3, \cdots$, (2) D_{i+1} has unit winding number in D_i, $i = 1, 2, 3, \cdots$ and (3) the mesh of D_i approaches zero as i increases without bound. Thus, every nondegenerate proper subcontinuum of a pseudo-circle is a pseudo-arc and every pseudo-circle can be embedded in the plane. In view of the relationships between pseudo-arcs and pseudo-circles and the fact that pseudo-arcs are known to have important mapping properties, a number of questions have been raised in the literature regarding the mapping characteristics of pseudo-circles.

We now amplify the foregoing statements. In a recent paper [6] the author has established a global characterization of the continuous images of the pseudo-arc [1], [8], [11] which is similar in certain respects to the well known Hahn-Mazurkiewicz characterization of the continuous images of the arc. This result, which was also established independently by A. Lelek [9], constituted an answer to a question raised by R. H. Bing at the Summer Institute on Set Theoretic Topology, 1955 [3]. In addition, this characterization proved to be useful in [6] in showing that there does not exist any local topological property which characterizes the continuous images of the pseudo-arc. Furthermore, in a subsequent paper of this author [7] the characterization of the continuous images of the pseudo-arc was used to establish properties of topological operations on the class of continuous images of all snake-like continua.

The purpose of this present paper is to establish a corresponding characterization of the continuous images of all pseudo-circles [2, p. 48] This characterization will be expressed in a manner which is formally similar to that given for the continuous images of the pseudo-arc in [6]. However, the development of this result involves establishing preliminary theorems on cyclic orderings, winding numbers of circular p-chain refinements and the composition properties of crooked and noncrooked cyclic r-patterns. In addition a concept of linear representation of a cyclic r-pattern which is similar to the concept of universal covering space is introduced and used strongly in proving the principal preliminary theorem of this paper. Thus the development of this characterization of the continuous images of all pseudo-circles is substantially different from that of the characterization of the continuous images of the pseudo-arc. Among the further results presented in this paper is the theorem that every planar circularly chainable continuum is a continuous image of a pseudo-circle. This latter result is analogous to the corresponding theorem for the pseudo-arc obtained by J. Mioduszewski [10] and this author [6]. It is also proved that all snake-like continua are continuous images of pseudo-circles.

2. Preliminaries. The more standard terms used in this paper are defined in [12] or in the other appropriately indicated references. In addition, we shall define a number of special terms to be used throughout this paper. In general, these terms and notations were suggested by those used by Bing in [1] and [2] and those used by the author in [6].

It will be convenient in this development to use a modified form of the standard modular notation for cyclic systems. Specifically, the notation $k \bmod n$, where k is a nonnegative integer and n is a positive integer, will be used to denote the remainder obtained in dividing k by n. Thus, for example, the relationship $n = 0 \pmod n$ will be written $n \bmod n = 0$ in this paper to facilitate the presentation of results which involve more than one cyclic system.

DEFINITION 1. A *p-chain* will be defined to be a finite sequence of sets each of which, except the last, intersects its successor in the sequence. A *circular p-chain* will be defined to be a p-chain in which the first and last links intersect. The members of the p-chain or circular p-chain will be called *links* and the notations $P = P(0, n) = (p_0, p_1, \cdots, p_n)$ will be used to denote the p-chain or circular p-chain whose links are $p_0, p_1, \cdots, p_n$.

DEFINITION 2. A function f defined by a collection of ordered pairs of integers $((k, f(k)), (k+1, f(k+1)), \cdots, (k+n, f(k+n)))$ will be said to be an *r-pattern* if $|f(i) - f(j)| \leqq 1$ whenever $|i - j| \leqq 1$, $k \leqq i, j \leqq k+n$. If $g = ((0, g(0)), (1, g(1)), \cdots, (n, g(n)))$ has range $(0, 1, \cdots, m)$, for some positive integer m, and $|g(i) - g(j)| \bmod m \leqq 1$

whenever $|i - j| \bmod n \leqq 1$, $0 \leqq i, j \leqq n$, then g will be said to be a *cyclic r-pattern.*

DEFINITION 3. If $P = (p_0, p_1, \cdots, p_n)$ and $Q = (q_0, q_1, \cdots, q_m)$ are p-chains such that each link p_i of P is a subset of some link $q_{f(i)}$ of Q and the sequence f of ordered pairs $((0, f(0)), (1, f(1)), \cdots, (n, f(n)))$ is an r-pattern with range $(0, 1, \cdots, m)$ then f will be said to be an *r-pattern* of P in Q. If P and Q are circular p-chains and f is a cyclic r-pattern then f will be said to be a *cyclic r-pattern* of P in Q.

DEFINITION 4. Two cyclic r-patterns

$$f = ((0, f(0)), (1, f(1)), \cdots, (n, f(n)))$$

and

$$g = ((0, g(0)), (1, g(1)), \cdots, (n, g(n)))$$

will be defined to be *similar* if f and g have the same set $(0, 1, \cdots, m)$ as range and these is an integer h such that one of the following two conditions holds:

(a) $f(i) = (h + g(i)) \bmod (m + 1)$, $0 \leqq i \leqq n$,

(b) $f(i) = (h - g(i)) \bmod (m + 1)$, $0 \leqq i \leqq n$.

DEFINITION 5. Two cyclic r-patterns

$$f = ((0, f(0)), (1, f(1)), \cdots, (n, f(n)))$$

and

$$g = ((0, g(0)), (1, g(1)), \cdots, (n, g(n)))$$

will be defined to be *equivalent* if f and g have the same set $(0, 1, \cdots, m)$ as range and there is an integer h such that one of the following two conditions holds:

(a) $f(i) = g((h + i) \bmod (n + 1))$, $0 \leqq i \leqq n$, ·

(b) $f(i) = g((h - i) \bmod (n + 1))$, $0 \leqq i \leqq n$.

If f and g are cyclic r-patterns such that either g is equivalent to a cyclic r-pattern similar to f or g is similar to a cyclic r-pattern equivalent to f, then g will be referred to as an *adjustment* of f. We note that each of the relationships "similarity", "equivalence" and "adjustment" are equivalence relationships.

DEFINITION 6. If $f = ((0, f(0)), (1, f(1)), \cdots, (n, f(n)))$ is a cyclic r-pattern with range $(0, 1, \cdots, m)$, then the sequence of ordered pairs of integers $((f(0), f(1)), (f(1), f(2)), \cdots, (f(n - 1), f(n)), (f(n), f(0)))$ will be said to be the *characteristic sequence* of f and will be denoted by

$C(f)$. The occurences of the ordered pairs $(m, 0)$ and $(0, m)$ in $C(f)$ will be referred to as the *positive transitions* of f and the *negative transitions* of f, respectively. The number of positive transitions of f minus the number of negative transitions of f will be said to be the *winding number* of f. In this development the cyclic r-patterns whose winding numbers have unit absolute value will be of particular importance. A cyclic r-pattern whose winding number has absolute value equal to 1 will be said to be a *monocyclic r-pattern.*

DEFINITION 7. If f is a cyclic r-pattern such that $f(0) = 0$, then f will be said to have *canonical form*. Let f be a cyclic r-pattern having canonical form and let a and b be elements of the domain of f such that $0 < a < b$ and such that no integer of the sequence $f(a), f(a+1), \cdots, f(b)$ is zero. Then the sequence of ordered pairs of integers $(a, f(a)), (a+1, f(a+1)), \cdots, (b, f(b))$ will be defined to be a *primary r-pattern* of f.

DEFINITION 8. If $f = ((0, f(0)), (1, f(1)), \cdots, (n, f(n)))$ is a cyclic r-pattern with range $(0, 1, \cdots, m)$ and

$$g = ((k, g(k)), (k+1, g(k+1)), \cdots, (k+t, g(k+t)))$$

is an r-pattern such that (1) $n \leqq t$ and (2) $|f(i) - g(j)| \bmod (m+1) = 0$ whenever i is the least nonnegative integer such that

$$|i - j| \bmod (n+1) = 0,\ k \leqq j \leqq k + t\ ,$$

then g is defined to be a *linear representation* of f.

DEFINITION 9. A p-chain P will be said to be a *refinement* of a p-chain Q if there is an r-pattern of P in Q. If P is a circular p-chain which has a cyclic r-pattern in a circular p-chain Q then P will be said to be a *circular refinement* of Q.

The following types of refinements and circular refinements will be distinguished:

DEFINITION 10. *A* p-chain P will be said to be a *normal refinement* of a p-chain Q if there is an r-pattern

$$f = ((k, f(k), (k+1, f(k+1)), \cdots, (k+n, f(k+n)))$$

with range $(h, h+1, \cdots, h+m)$ of P in Q such that $f(k) = h$ and $f(k+n) = h+m$. If a p-chain P has an r-pattern f in a p-chain Q and each link p_i of P is the same set as the link $q_{f(i)}$ of Q, then P will be defined to be a *principal refinement* of Q. A p-chain P will be said to be a *crooked refinement* of a p-chain Q if there is an

r-pattern f of P in Q such that if i and j are integers of the domain of f, $i < j$ and $|f(i) - f(j)| > 2$, then there are integers u and v with the properties that $i < u < v < j$, $|f(u) - f(j)| \leqq 1$ and $|f(v) - f(i)| \leqq 1$. If f is an r-pattern having the properties described in the definition of "crooked refinement" we shall refer to f as a *crooked r-pattern.*

DEFINITION 11. If P is a circular p-chain having a cyclic r-pattern f in a circular p-chain. Q, and each link p_i of P is the same set as the link $q_{f(i)}$ of Q, then P will be said to be a *principal circular refinement* of Q. A circular p-chain P will be said to be *crooked* in a circular p-chain Q if there is a cyclic r-pattern f of P in Q such that whenever g is an adjustment of f having canonical form and r is a primary r-pattern of g, then r is a crooked r-pattern. We shall refer to a cyclic r-pattern f having the properties described in the definition of "crooked refinement" for circular p-chains as a *crooked cyclic r-pattern.*

In any type of refinement of a p-chain P in a p-chain Q, there may be several r-patterns of P in Q having the appropriate properties. However, in referring to a refinement whose existence has been hypothesized or otherwise established, we will assume that a particular r-pattern has been chosen and that this r-pattern will remain fixed throughout the given argument. Thus, in these circumstances, we will speak of "the" r-pattern of a p-chain P in a p-chain Q. A similar assumption will also be made with respect to circular refinements.

The concept of circular p-chainability of a continuum is now introduced.

DEFINITION 12. A continuum H will be said to be *circularly p-chainable* if there is a sequence $P_1, P_2, P_3, \cdots$ of circular p-chains such that for each positive integer i:

(a) The union of the elements of P_i is H.

(b) There is a monocyclic r-pattern P_{i+1} in P_i.

(c) The diameter of each link of P_i is less than $1/i$.

(d) The closure of each link of P_{i+1} is a subset of the link of P_i to which it corresponds under the monocyclic r-pattern of P_{i+1} in P_i.

A sequence of circular p-chains $P_1, P_2, P_3, \cdots$ having these properties with respect to the continuum H will be said to be *cyclically associated* with H.

It will be shown that the property of circular p-chainability is a characterizing property of the class of continuous images of all pseudo-circles. This property will also be of fundamental importance in the development of the related results mentioned in the introductory section of this paper.

3. Properties of cyclic r-patterns. In this section, we shall establish the principal combinatorial properties of cyclic r-patterns. These results will be used strongly in § 4 in establishing the major theorems of this paper.

THEOREM 3.1. *The absolute value of the winding number of a cyclic r-pattern is invariant under the operations of similarity and equivalence.*

Proof. Let $f = ((0, f(0)), (1, f(1)), \cdots, (n, f(n)))$ be a cyclic r-pattern with range $(0, 1, \cdots, m)$ and winding number w. We first prove that if g is a cyclic r-pattern similar to f, then g has winding number $\pm w$. Two cases will be considered.

Case 1. There is an integer k such that, for each integer i in the domain of f $f(i) = (k + g(i)) \bmod (m + 1)$. In this case we may assume that k is a positive integer less than $m + 1$. Then the positive transitions of g are in one-to-one correspondence with the occurences of the ordered pair $(k - 1, k)$ in the characteristic sequence $C(f)$ of f. The negative transitions of g are in one-to-one correspondence with the occurences of $(k, k - 1)$ in $C(f)$. We now proceed to obtain a relationship involving the number u of occurences of $(k - 1, k)$ in $C(f)$, the number v of occurence of $(k, k - 1)$ in $C(f)$ and the winding number w of f.

To do this, let $i_1, i_2, \cdots, i_t$ be the maximal increasing sequence of integers of the domain of f such that, for each number i_j of this sequence, $(f(i_j), f((i_j + 1) \bmod (n + 1)))$ is either a positive transition of f or a negative transition of f. Next observe that u is equal to the number of occurences of $(k - 1, k)$ in the collection

$$((f(i_1), f(i_1 + 1), (f(i_1 + 1), f(i_1 + 2)), \cdots, (f(n - 1), f(n)), \\ (f(n), f(0)), (f(0), f(1)), \cdots, (f(i_1 - 1), f(i_1))) .$$

Thus u is equal to the sum of the integers $u_1, u_2, \cdots, u_t$ which are, respectively, the number of occurences of $(k - 1, k)$ in the collections:

$$C_1 = ((f(i_1 + 1), f(i_1 + 2)), (f(i_1 + 2), f(i_1 + 3)), \cdots, (f(i_2 - 1), f(i_2))) ,$$
$$C_2 = ((f(i_2 + 1), f(i_2 + 2)), (f(i_2 + 2), f(i_2 + 3)), \cdots, (f(i_3 - 1), f(i_3))) ,$$
$$\vdots$$
$$C_t = ((f(i_t + 1), f(i_t + 2)), (f(i_t + 2), f(i_t + 3)), \cdots, (f(n - 1), f(n)), \\ (f(n), f(0)), (f(0), f(1)), \cdots, (f(i_1 - 1), f(i_1))) .$$

In a similar manner, v can be expressed as the sum of the numbers $v_1, v_2, \cdots, v_t$ of occurences of $(k, k - 1)$ in the collections

$C_1, C_2, \cdots, C_t$, respectively. Now, for each integer j, $1 \leqq j \leqq t$, the two elements of a given ordered pair of the collection C_j differ by at most 1, the second member of each ordered pair of C_j is equal to the first member of the succeeding ordered pair of C_j, if any. Furthermore, if $(f(i_j), f(i_j + 1))$ and $(f(i_{j+1}), f(i_{j+1} + 1))$ are both positive transitions of f, then $f(i_j + 1) = 0$ and $f(i_{j+1}) = m$ so that $u_j - v_j = 1$. Similarly, if $((f(i_j), f(i_j + 1))$ and $(f(i_{j+1}), f(i_{j+1} + 1))$ are both negative transitions of f, then $u_j - v_j = -1$. In both of the remaining situations we obtain the result $u_j - v_j = 0$. Thus, we conclude that the winding number w of f is equal to the winding number $u - v$ of g and the proof for this case is complete.

Case 2. There is an integer k such that, for each integer i in the domain of f, $f(i) = (k - g(i)) \bmod (m + 1)$. In this case we may assume that $m \leqq k \leqq 2m + 1$. Furthermore, if $k = m$ or $k = 2m + 1$, then the occurences of the positive transitions and negative transitions of f are in a one-to-one correspondence with the negative transitions and positive transitions, respectively, of g. In this situation, the winding number of g is opposite in sign but equal in absolute value to the winding number of f. If $m < k < 2m + 1$, the positive transitions of g are in one-to-one correspondence with the occurences of $(k + 1, k)$ in $C(f)$, and the negative transitions of g are in one-to-one correspondence with the occurences of $(k, k + 1)$ in $C(f)$. These situations, in reverse order, were considered in Case 1 and we conclude, in this case, that the winding number of g is equal in absolute value, although opposite in sign, to the winding number of f.

Finally, we consider the relationships of the winding numbers of equivalent cyclic r-patterns f and g. In the case that f and g satisfy condition (a) of Definition 5, the sequence of integers $g(0), g(1), \cdots, g(n)$ is a cyclic permutation of the sequence of integers $f(0), f(1), \cdots, f(n)$. If f and g satisfy condition (b) of Definition 5, the sequence $g(0), g(1), \cdots, g(n)$ is a cyclic permutation of the sequence $f(n), f(n - 1), \cdots, f(0)$. If follows, in both cases, that the absolute value of the winding number of f is equal to the absolute value of the winding number of g.

Therefore, the absolute value of the winding number of a cyclic r-pattern is invariant under the operations of similarity and equivalence.

COROLLARY. *If f is a cyclic r-pattern with range $(0, 1, \cdots, m)$ and k is an integer, $0 \leqq k \leqq m$, then the integer obtained by subtracting the number of occurences of $((k + 1) \bmod (m + 1), k)$ in $C(f)$ from the number of occurences of $(k, (k + 1) \bmod (m + 1))$ in $C(f)$ is equal to the winding number of f.*

THEOREM 3.2. *If f and g are cyclic r-patterns such that fg is a defined composite function, then fg is a cyclic r-pattern. Furthermore,* *the winding number of fg is equal to the product of the winding numbers of f and g.*

Proof. Let $f = ((0, f(0)), (1, f(1)), \cdots, (n, f(n)))$ be a cyclic r-pattern with range $(0, 1, \cdots, m)$ and let

$$g = ((0, g(0)), (1, g(1)), \cdots, (t, g(t)))$$

be a cyclic r-pattern with range $(0, 1, \cdots, n)$. Then, since f is a cyclic r-pattern, $|fg(i) - fg(j)| \bmod m \leqq 1$ whenever

$$|g(i) - g(j)| \bmod n \leqq 1,\ 0 \leqq g(i), g(j) \leqq n\ .$$

Furthermore, since g is a cyclic r-pattern, $|i - j| \bmod t \leqq 1$ implies that

$$|g(i) - g(j)| \bmod n \leqq 1,\ 0 \leqq i, j \leqq t\ .$$

Thus $|fg(i) - fg(j)| \bmod m \leqq 1$ whenever

$$|i - j| \bmod t \leqq 1,\ 0 \leqq i, j \leqq t\ ,$$

so that fg is a cyclic r-pattern.

Now, let h and k be the winding numbers of f and g, respectively, and let i be an integer such that an occurence of

$$(fg(i), fg((i + 1) \bmod (t + 1)))$$

is either a positive or negative transition of fg. Then $g(i)$ and $g((i + 1) \bmod (t + 1))$ are distinct integers such that

$$|g(i) - g((i + 1) \bmod (t + 1))| \bmod n \leqq 1\ .$$

In general, if an occurence of an ordered pair of integers (a, b) in $C(fg)$ is a transition of fg, there is an integer u in the domain of f, and hence in the range of g, such that either $f(u) = a$ and $f((u + 1) \bmod (n + 1)) = b$ or $f(u) = b$ and $f((u + 1) \bmod (n + 1)) = a$. From the Corollary to Theorem 3.1, the number of occurences of $(u, (u + 1) \bmod (n + 1))$ in $C(g)$ minus the number of occurences of $((u + 1) \bmod (n + 1), u)$ in $C(g)$ is equal to the winding number k of q. In addition, the number of occurences of (a, b) in $C(f)$ minus the number of occurences of (b, a) in $C(f)$ is equal to either h or $-h$ according as an occurence of (a, b) in $C(f)$ is a positive or negative transition of f. Therefore, the winding number of fg is equal to the product hk.

We note that the class of monocyclic r-patterns is closed under the operation of function composition. The theorem which follows is the principal theorem involving the combinatorial and refinemental relationships among planar circular p-chains. In particular, it is an important preliminary theorem to the major theorems of this paper.

THEOREM 3.3. *If f and g are monocyclic r-patterns having identical ranges, then there exist monocyclic r-patterns r and s such that fr and gs are defined composite functions and $fr = gs$.*

The proof of this theorem is somewhat involved and it will be convenient to first establish the following two lemmas.

LEMMA 3.3.1. *Let*

$$f = ((0, f(0)), (1, f(1)), \cdots, (h, f(h)))$$

and

$$g = ((0, g(0)), (1, g(1)), \cdots, (k, g(k)))$$

be monocyclic r-patterns having identical ranges, let f_1 and g_1 be linear representations of f and g, respectively, and let r_1 and s_1 be r-patterns having the same set $(0, 1, \cdots, n)$ as domain and satisfying the following conditions.

(1) *$f_1 r_1$ and $g_1 s_1$ are defined composite functions and $f_1 r_1 = g_1 s_1$*

(2) $|r_1(0) - r_1(n)| = h + 1$.

(3) $|s_1(0) - s_1(n)| = k + 1$.

Then there are monocyclic r-patterns r and s such that fr and gs are defined and $fr = gs$.

Proof. We shall define r and s in a parallel manner. First, r is defined to have the same domain $(0, 1, \cdots, n)$ as r_1 and, for each integer i in the domain of r, $r(i)$ is defined to be the least nonnegative integer such that $|r(i) - r_1(i)| \bmod (h + 1) = 0$. The function s is then defined by replacing the symbols r, r_1 and h in the preceding definition by s, s_1 and k, respectively.

Now, the range of r_1 is identical with the domain of f_1 and f_1 is a linear representation of f, so that the range of r_1 includes integers of the form $a, a + 1, \cdots, a + h$. Thus, it follows that the range of r is identical with the domain of f. Similarly, the domain of g is identical with the range of s. Hence fr and gs are defined composite functions.

To see that r is a monocyclic r-pattern, we note first, since r_1 is an r-pattern, that if i and j are integers such that $0 \leqq i, j \leqq n$, and $|i - j| \leqq 1$, then $|r_1(i) - r_1(j)| \leqq 1$. Hence $|r(i) - r(j)| \bmod h \leqq 1$. If $|i - j| = n$ it follows from condition (2) of the hypothesis that $|r_1(i) - r_1(j)| \bmod (h + 1) = 0$ so that in this case also $|r(i) - r(j)| \bmod h \leqq 1$. To complete the proof that r is a monocyclic r-pattern we need to show that the absolute value of the winding number of r is equal to 1. We do this by observing that r_1 is a linear representation of r and that $|r_1(0) - r_1(n)| = h + 1$. Thus the absolute value of the winding number of r is equal to 1. Therefore r is a monocyclic r-pattern. A similar argument shows that s is a monocyclic r-pattern.

Finally, let i be an integer in the common domain of r and s. We shall show that $fr(i) = gs(i)$. From the definition of r, $r(i)$ is the least nonnegative integer such that $|r(i) - r_1(i)| \bmod (h+1) = 0$. Thus, if $(0, 1, \cdots, m)$ is the common range of f and g, it follows from the definition of "linear representation" that

$$|fr(i) - f_1r_1(i)| \bmod (m+1) = 0 .$$

Furthermore, in a similar manner $|gs(i) - g_1s_1(i)| \bmod (m+1) = 0$. In addition, from condition (1) of the hypothesis of the lemma, $f_1r_1(i) = g_1s_1(i)$ and $0 \leqq fr(i), gs(i) \leqq m$. Therefore we conclude that $fr(i) = gs(i)$ and the proof is complete.

LEMMA 3.3.2. *Let f and g be monocyclic r-patterns having identical ranges, let f_1 and g_1 be monocyclic r-patterns equivalent to f and g, respectively, and suppose r_1 and s_1 are monocyclic r-patterns such that f_1r_1 and g_1s_1 are defined composite functions and $f_1r_1 = g_1s_1$. Then, there exist monocyclic r-patterns r and s such that fr and gs are defined composite functions and $fr = gs$.*

Proof. Let $f = ((0, f(0)), (1, f(1)), \cdots, (h, f(h)))$, let $g = ((0, g(0)), (1, g(1)), \cdots, (k, g(k)))$ and let $(0, 1, \cdots, m)$ be the common range of f and g. Then, since f_1 is equivalent to f, there exists an integer u such that one of the following two conditions is satisfied:

(1) $f_1(i) = f((u+i) \bmod (h+1))$, $0 \leqq i \leqq h$,

(2) $f_1(i) = f((u-i) \bmod (h+1))$, $0 \leqq i \leqq h$.

Similarly, since g_1 is equivalent to g, there exists an integer v sucn that one of the following two conditions is satisfied.

(3) $g_1(i) = g((v+i) \bmod (h+1))$, $0 \leqq i \leqq k$,

(4) $g_1(i) = g((v-i) \bmod (h+1))$, $0 \leqq i \leqq k$.

We shall assume that f_1 satisfies condition (1) and that g_1 satisfies condition (3) since it will be seen that a similar proof may be followed in the remaining three cases. In this case we may also assume that u is an integer such that $0 \leqq u \leqq h$ and v is an integer such that $0 \leqq v \leqq k$.

Now, let $r_1 = ((0, r_1(0)), (1, r_1(1)), \cdots, (n, r_1(n)))$ and let

$$s_1 = ((0, s_1(0)), (1, s_1(1)), \cdots, (n, s_1(n))) .$$

We define r and s by the following two relationships:

(5) $r(i) = (u + r_1(i)) \bmod (h+1)$, $0 \leqq i \leqq n$, and

(6) $s(i) = (v + s_1(i)) \bmod (k+1)$, $0 \leqq i \leqq n$.

Then the domain of f and the range of r are identical, and the domain of g and the range of s are identical. Thus fr and gs are defined composite functions. Furthermore, by Theorem 3.1, since r and r_1 are similar and s and s_1 are similar, it follows that r and s are monocyclic r-patterns. Finally, if i is an integer of the set $(0, 1, \cdots, n)$, then from (1) and (5),

$$fr(i) = f((u + r_1(i)) \bmod (h + 1)) = f_1 r_1(i) .$$

Similarly, from (3) and (6),

$$gs(i) = f((v + s_1(i)) \bmod (k + 1)) = g_1 s_1(i) .$$

Therefore, since $f_1 r_1 = g_1 s_1$, we conclude that r and s are monocyclic r-patterns such that $fr = gs$.

Proof of Theorem 3.3. First note that for each monocyclic r-pattern there is an equivalent monocyclic r-pattern having positive winding number. Thus, by Lemma 3.3.2, we may assume without loss in generality that each of the cyclic r-patterns f and g have winding number equal to 1. Now, let $f = ((0, f(0)), (1, f(1)), \cdots, (h, f(h)))$, let $g = ((0, g(0)), (1, g(1)), \cdots, (k, g(k)))$ and let $(0, 1, \cdots, m)$ be the common range of f and g. We shall establish the existence of the monocyclic r-patterns r and s by constructing two finite sequences of r-patterns related to f and g, respectively, and showing that the last members of these sequences determine monocyclic r-patterns of the desired type.

We define r-patterns f_1 and g_1 in the following manner. Let f_1 be the r-pattern having domain $(0, 1, \cdots, h + 1)$ such that $f_1(0) = f(0)$ and, if $f_1(i)$ has been chosen, $0 \leqq i \leqq h$, then $f_1(i + 1)$ is defined to be:

$$f_1(i) + 1 \quad \text{if } f((i + 1) \bmod (h + 1)) = (f(i) + 1) \bmod (m + 1) ,$$
$$f_1(i) - 1 \quad \text{if } f((i + 1) \bmod (h + 1)) = (f(i) + 1) \bmod (m + 1) ,$$

or

$$f_1(i) \qquad \text{if } f((i + 1) \bmod (h + 1)) = f(i) .$$

The pattern g_1 is defined by replacing the symbols f, f_1 and h, in the preceding definition by g, g_1 and k, respectively. Then f_1 and g_1 are r-patterns which are linear representations of f and g, respectively, and $f_1(h + 1) - f_1(0) = g_1(k + 1) - g_1(0) = m + 1$.

Next we define r-patterns f_2 and g_2. Since we shall require, again, that the definitions have parallel form, it will be sufficient to define f_2. To do this, let u be an integer such that $f_1(u)$ is a minimal element of the range of f_1. We define f_2 to be the r-pattern having the same domain as f_1 such that $f_2(0) = f_1(u)$ and, if $f_2(i)$ has been chosen, $0 \leqq i \leqq h$, then we define $f_2(i + 1)$ to be equal to $f_1(i + u + 1)$ for $0 \leqq i \leqq h - u$ and to be equal to $f_1(i + u - h) + (m + 1)$ for

$h - u < i \leqq h$. It follows that f_2 and g_2 are r-patterns which are linear representations of monocyclic r-patterns equivalent to f_1 and g_1, respectively.

We now define r-patterns f_3 and g_3 which are modifications of f_2 and g_2, respectively, such that $f_3(0) = g_3(0)$. Specifically, observe that one of the r-patterns f_2 and g_2 has a member of its range which is equal to the least member of the range of the other r-pattern. Without loss in generality we may assume that $f_2(0)$ is equal to some element of the range of g_2. Let c be the greatest integer such that $f_2(0) = g_2(c)$, and let g_3 be defined by the relationships

$$g_3(i) = g_2(c + i) \qquad \text{for} \quad 0 \leqq i \leqq k - c \ ,$$

and

$$g_3(i) = g_2(c + i - k - 1) + (m + 1) \qquad \text{for} \quad k - c < i \leqq k + 1 \ .$$

We then define f_3 to be identical with f_2. Now, since g_2 is an r-pattern such that $g_2(k + 1) - g_2(0) = m + 1$, $g_2(0)$ is a minimal element of the range of g_2 and c is the greatest integer such that $f_2(0) = g_2(c)$, it follows that $g_3(0)$ is a minimal element of the range of g_3. Furthermore, since the definition of g_3 in terms of g_2 is analogous to the definition of g_2 in terms of g_1, g_3 is a linear representation of some monocyclic r-pattern equivalent to g. Thus, the r-patterns f_3 and g_3 have the following properties: (1) $f_3(0) = g_3(0)$, (2) $f_3(0)$ and $g_3(0)$ are minimal elements of the ranges of f_3 and g_3, respectively, (3) f_3 is a linear representation of a monocyclic r-pattern equivalent to f, (4) g_3 is a linear representation of a monocyclic r-pattern equivalent to g and (5) $f_3(h + 1) = g_3(k + 1)$.

Finally, we define r-patterns f_4 and g_4. To do this, note from properties (1), (2), (3) and (4) that either the range of f_3 is a subset of the range of g_3, or the range of g_3 is a subset of the range of f_3. Without loss in generality we may suppose that the range of f_3 is a subset of the range of g_3. Now, let v be an integer such that $g_3(v)$ is a maximal element of the range of g_3. Then, from property (3) and the fact that $f_3(h + 1) - f_3(0) = m + 1$, the following expression defines a linear representation of a monocyclic r-pattern equivalent to f:

$$\begin{aligned}&((0, f_3(0)), (1, f_3(1)), \cdots, (h, f_3(h)), (h + 1, f_3(0) + (m + 1)),\\ &(h + 2, f_3(1) + (m + 1)), \cdots, (2h + 1, f_3(h) + (m + 1)),\\ &(2h + 2, f_3(0) + 2(m + 1), (2h + 3, f_3(1) + 2(m + 1)), \cdots,\\ &(w_1, f_3(b_{w_1}) + d_{w_1}(m + 1))) \ ,\end{aligned}$$

where $0 \leqq b_{w_1} \leqq h$, $0 \leqq d_{w_1}$ and w_1 is the least integer such that $f_3(b_{w_1}) + d_{w_1}(m + 1) = g_3(v)$. We define f_4 to be the r-pattern determined by the foregoing conditions. The r-pattern g_4 is defined in a similar although not precisely parallel manner. Specifically, we define g_4 to be the linear representation of a monocyclic r-pattern equivalent to g determined by the following expression:

$$((0, g_3(0)), (1, g_3(1)), \cdots, (k, g_3(k)), (k+1, g_3(0) + (m+1)), \\ (k+2, g_3(1) + (m+1)), \cdots, (w_2, g_3(b_{w_2}) + (m+1))) ,$$

where $0 \leqq b_{w_2} \leqq k$, and w_2 is the least integer greater than or equal to $k+1$ such that $g_3(b_{w_2}) + (m+1) = g_3(v)$.

It will now be shown that the r-patterns f_4 and g_4 determine monocyclic r-patterns r and s of the desired type. We shall use Lemmas 3.3.1 and 3.3.2 and certain properties of p-chains. Let $Q = Q(0, f_4(w_1))$ be a p-chain, let $P = P(0, w_1)$ be a p-chain which is a normal refinement of Q having r-pattern f_4 in Q and let $T = T(0, w_2)$ be a p-chain which is a principal normal refinement of Q having r-pattern g_4 in Q. Then, by Theorem 3.2 of [6], there is a p-chain $S_1 = S_1(0, z_1)$ such that S_1 is a normal refinement of P and a principal normal refinement of T. By property (2) of f_3 and the definition of f_4, the sub-p-chain $P(w_1, h+1)$ of $P(w_1, 0)$ is a normal refinement of $Q(f_4(w_1), f_4(0) + (m+1))$. Similarly, the sub-$p$-chain $T(w_2, k+1)$ of $T(w_2, 0)$ is a principal normal refinement of $Q(f_4(w_1), f_4(0) + (m+1))$. Hence there is a sub-$p$-chain $S_2 = S_1(z_1, z_2)$ of $S_1(z_1, 0)$ such that S_2 is a normal refinement of $P(w_1, h+1)$ and a principal normal refinement of $T(w_2, k+1)$. It follows that the p-chain sum $S = S(0, z) = S_1 + S_2$ is a refinement of P and a principal normal refinement of T such that the last link of S corresponds to the $(h+1)$-st link of P under the r-pattern of S in P, and the last like of S corresponds to the $(k+1)$-st link of T under the r-pattern of S in T.

Now, by Lemma 3.3.2, we may assume without loss in generality that f_4 and g_4 are linear representations of f and g, respectively. Next, we note that the condition that a refinement be a principal refinement in the proof of Theorem 3.2 of [6] and in the preceding paragraph is not essential and that equivalent results may be obtained by considering the corresponding r-patterns. In particular it may be concluded that if P and T are normal refinements of Q such that each link of a refining p-chain is contained in exactly one link of the refined p-chain, then there is a p-chain S which is a normal refinement of both P and T and is such that each link of the refining p-chain is contained in exactly one link of a refined p-chain. Thus, in this theorem, if r_1 is the r-pattern of S in P and s_1 is the r-pattern of S in T, it may be assumed that f_4, g_4, r_1 and s_1 have been chosen so that $f_4 r_1 = g_4 s_1$. Finally, $r_1(z) - r_1(0) = m+1$ and $s_1(z) - s_1(0) = m+1$. Therefore, by Lemma 3.3.1, there exist monocyclic r-patterns r and s such that $fr = gs$.

REMARK. The foregoing theorem cannot be modified to state that if f and g are arbitrary cyclic r-patterns having identical ranges then there are cyclic r-patterns r and s such that $fr = gs$. The following example shows that in the case of cyclic r-patterns f and g having winding numbers each equal to zero, such a theorem would be false.

Let

$$f = ((0, 0), (1, 1), (2, 2), (3, 1), (4, 0)) ,$$

and

$$g = ((0, 1), (1, 2), (2, 0), (3, 2), (4, 1)) .$$

If s is a cyclic r-pattern $((0, s(0)), (1, s(1)), \cdots, (n, s(n)))$ whose range is identical with the domain of g, there are integers h and k in the domain of s such that $|h - k| \bmod (n-1) = 1$, $s(h) = 2$ and $s(k) = 1$ or 3. Thus $gs(h) = 0$ and $gs(k) = 2$. Now, suppose that there exists a cyclic r-pattern $r = ((0, r(0)), (1, r(1)), \cdots, (n, r(n)))$ such that $fr = gs$. Then $|r(h) - r(k)| \bmod 4 \leqq 1$, $fr(h) = 0$ and $fr(k) = 2$, which is contrary to the definition of f.

We now consider the final theorem of this section.

THEOREM 3.4. *Let f be a monocyclic r-pattern and let g be a crooked monocyclic r-pattern such that fg is a defined composite function. Then fg is a crooked monocyclic r-pattern.*

Proof. First we show that if f_1 is an r-pattern and g_1 is a crooked r-pattern such that f_1g_1 is a defined composite function then f_1g_1 is a crooked r-pattern. Since g_1 is an r-pattern $|g_1(i) - g_1(j)| \leqq 1$ whenever i and j are integers of the domain of g_1 such that $|i - j| \leqq 1$. Furthermore, $|g_1(i) - g_1(j)| \leqq 1$ implies that $|f_1g_1(i) - f_1g_1(j)| \leqq 1$, since f_1 is an r-pattern. Thus $|f_1g_1(i) - f_1g_1(j)| \leqq 1$ whenever i and j are integers of the domain of f_1g_1 such that $|i - j| \leqq 1$. Hence f_1g_1 is an r-pattern. To show that f_1g_1 is a crooked r-pattern, let i and j be integers of the domain of f_1g_1 such that $i < j$ and $|f_1g_1(i) - f_1g_1(j)| > 2$. Then $|g_1(i) - g_1(j)| > 2$. Therefore, since g_1 is a crooked r-pattern, there are integers u and v such that $i < u < v < j$, $|g_1(u) - g_1(j)| \leqq 1$ and $|g_1(v) - g_1(i)| \leqq 1$. It follows that $|f_1g_1(u) - f_1g_1(j)| \leqq 1$ and $|f_1g_1(v) - f_1g_1(i)| \leqq 1$, $i < u < v < j$, so that f_1g_1 is a crooked r-pattern.

Now, let $f = ((0, f(0)), (1, f(1)), \cdots, (n, f(n)))$ be a monocyclic r-pattern with range $(0, 1, \cdots, m)$, let $g = ((0, g(0)), (1, g(1)), \cdots, (t, g(t)))$ be a crooked monocyclic r-pattern with range $(0, 1, \cdots, n)$ and note, by Theorem 3.2, that fg is a monocyclic r-pattern. We choose p to be a monocyclic r-pattern such that p is an adjustment of fg and p has canonical form. Then, since an adjustment is the composition of a similarity operation and an equivalence operation, there exist integers h and k such that

(1) $p(i) = (h + fg((k + i) \bmod (t + 1))) \bmod (m + 1), \ 0 \leqq i \leqq t$

or $p(i)$ is defined by an equation similar to (1) in which one or both of the first two plus signs are replaced by minus signs. Since a similar argument may be followed in each of these four cases, we shall assume that $0 \leqq h \leqq m$, $0 \leqq k \leqq t$ and that $p(i)$ is determined by equation (1).

We now show that the cyclic r-pattern p can be expressed as a composite function $f'g'$, where f' is an adjustment of f having canonical form and g' is an adjustment of g having canonical form. To do this, let w be the integer such that

$$0 \leqq w \leqq n \quad \text{and} \quad (w + g(k)) \bmod (n + 1) = 0 \ .$$

Then, the cyclic r-patterns f' and g' are defined in the following manner:

(2) $f'(i) = (h + f((n + 1 - w + i) \bmod (n + 1))) \bmod (m + 1),$
$0 \leqq i \leqq n \ ,$

(3) $g'(i) = (w + g((k + i) \bmod (t + 1))) \bmod (n + 1), \ 0 \leqq i \leqq t \ .$

Thus from (1), (2) and (3) it follows that

(4) $p(i) = f'(g'(i)), \ 0 \leqq i \leqq t.$

Furthermore, by the choice of w, $g'(0) = 0$. In addition, from (4) and the fact that p and g' are cyclic r-patterns having canonical form, it follows that $f'(0) = 0$ so that f' also has canonical form.

To complete the proof of this theorem, noting the result obtained in the first paragraph of the proof, it is sufficient to show that each primary r-pattern of p is equivalent to the composite function of an arbitrary r-pattern and a crooked r-pattern. Let

$$r = ((u, p(u)), \ (u + 1, \ p(u + 1), \cdots, (v, p(v)))$$

be a primary r-pattern of p. Then, neither the domain nor the range of r contains the element zero. Hence, from (4) and the fact that f' and g' are cyclic r-patterns having canonical form, the sequence $g'(u), g'(u + 1), \cdots, g'(v)$ consists of integers each of which is nonzero. Thus, noting Definition 2, it follows that the sequence of ordered pairs of integers

(5) $r_2 = ((u, g'(u)), (u + 1, g'(u + 1)), \cdots, (v, g'(v)))$

is a primary r-pattern of g'. In a similar manner, if a and b are the minimum and maximum integers of the range of r_2, then

(6) $r_1 = ((a, f'(a)), (a + 1, f'(a + 1), \cdots, (b, f'(b)))$

is a primary r-pattern of f'. Now, g' is a crooked cyclic r-pattern, so that r_2 is a crooked r-pattern. Furthermore, $r = r_1r_2$. Therefore,

r is a crokeed r-pattern and we conclude that fg is a crooked monocyclic r-pattern.

4. Characterization of the continuous images of all pseudo-circles. The purpose of this section is to establish the two principal theorems of this paper, that the class of continuous images of all pseudo-circles consists of exactly those continua which are circularly p-chainable, and that every planar circularly chainable continuum is a continuous image of a pseudo-circle.

In the presentation of the theorems of this section it will be convenient to omit the qualifying term "circular" in the expression "circular refinement" where no confusion is likely to result.

THEOREM 4.1. *If a continuum C is a continuous image of a pseudo-circle then there exists a sequence of circular p-chains cyclically associated with C such that each refinement is determined by a monocyclic r-pattern having canonical form.*

Proof. Let M be a pseudo-circle and let f be a continuous transformation with domain M so that the continuous image of M under f is the continuum C. Then, from the definition of "pseudo-circle" given in [2, p. 48], there is a sequence of planar circular chains $D_1, D_2, D_3, \cdots$ such that for each positive integer i: (1) each link of D_i is an open circular disk having diameter less than $1/i$, (2) the closure of each link of D_{i+1} is a subset of some link of D_i, (3) each complementary domain of the union of the links D_{i+1} contains a complementary domain of the union of the links of D_i, (4) if E_i is a proper sub-chain of D_i and E_{i+1} is a sub-chain of D_{i+1} contained in E_i, then E_{i+1} is crooked in E_i, and (5) the intersection of the sets of points of the circular chains $D_1, D_2, D_3, \cdots$ is M.

From condition (1) the union of the links of each circular chain D_i is topologically equivalent to the interior of an annular ring. Hence, from conditions (3) and (5), M is a continuum which separates the plane into exactly two complementary domains having M as their common boundary. It follows that the circularly chainable continuum M is not a snake-like continuum. Now, noting condition (2), if i is a positive integer, there is a cyclic r-pattern g_i of D_{i+1} in D_i such that the closure of each link of D_{i+1} is a subset of the link of D_i to which it corresponds under g_i. Furthermore, it is easily seen that g_i can be chosen so that $g_i(0) = 0$. Therefore, since D_i and D_{i+1} are planar circular chains whose links are open disks and each complementary domain of D_i is a subset of a complementary domain of D_{i+1}, it follows from the proof of Theorem 3 of [4] that the winding number of g_i has unit absolute value.

We denote the sequence of links of D_i by $(d_{i0}, d_{i1}, \cdots, d_{in_i})$ and indicate the circular p-chain $(f(d_{i0} \cap M), f(d_{i1} \cap M), \cdots, f(d_{in_i} \cap M))$ by P_i, for $i = 1, 2, 3, \cdots$. Now, M is a compact continuum, so that the function f is uniformly continuous. Thus, a subsequence $P_{k_1}, P_{k_2}, P_{k_3}, \cdots$ of the sequence $P_1, P_2, P_3, \cdots$ may be chosen so that for each positive integer i, the diamenter of each link of P_{k_i} is less than $1/i$. In addition, noting Theorem 3.2, the composite function $g_{k_i} g_{k_i+1} \cdots g_{k_{i+1}-1}$ is a monocyclic r-pattern of $P_{k_{i+1}}$ in P_{k_i} having canonical form, $i = 1, 2, 3, \cdots$. In addition, the closure of each link of $P_{k_{i+1}}$ is a subset of the link of P_{k_i} to which it corresponds under the cyclic r-pattern of $P_{k_{i+1}}$ in P_{k_i}, $i = 1, 2, 3, \cdots$. Finally, from condition (5), above, if i is a positive integer then the union of the links of P_{k_i} is equal to the continuum C. Therefore, the sequence of circular p-chains satisfies each of the requirements of Definition 12 with respect to C and each refinement is determined by a cyclic r-pattern having canonical form.

The theorem which follows constitutes the characterization of the continuous images of all pseudo-circles mentioned in the first paragraph of this section and discussed in the Introduction of this paper. In the proof of this theorem, strong use will be made of the combinatorial properties of cyclic r-patterns established in the preceding section.

THEOREM 4.2. *In order that a continuum C be a continuous image of a pseudo-circle it is necessary and sufficient that C be circularly p-chainable.*

Proof of necessity. This follows form Theorem 4.1.

Proof of sufficiency. First we show that if $P_1, P_2, P_3, \cdots$ is a sequence of circular p-chains which is cyclically associated with the continuum C, then there is a corresponding sequence of circular p-chains $T_1, T_2, T_3, \cdots$ which satisfies the requirements of Definition 12 with respect to C and is also such that for each positive integer i:

(1) T_i is a principal refinement of P_i having a monocyclic r-pattern in P_i.

(2) If $i > 1$, then T_i is a crooked refinement of T_{i-1} having a crooked monocyclic r-pattern in T_{i-1}.

To do this, let T_1 be an arbitrary principal refinement of P_1 having a monocyclic r-pattern in P_1. Next, assume that for some positive integer n, T_i has been defined for each positive integer i less than or equal to n, and that T_i satisfies the requirements of Definition 12 and conditions (1) and (2), above. Now consider the case that $i = n + 1$.

Since the sequence $P_1, P_2, P_3, \cdots$ is cyclically associated with C there is a monocyclic r-pattern f_n of P_{n+1} in P_n. In addition, from condition (1), above, for $i = n$, there is a monocyclic r-pattern g_n of

T_n in P_n. Thus, f_n and g_n are monocyclic r-patterns whose ranges are identical. It follows, by Theorem 3.3, that there exist monocyclic r-patterns r_{n+1} and s_n such that the composite functions $f_n r_{n+1}$ and $g_n s_n$ are defined and $f_n r_{n+1} = g_n s_n$. We define a circular p-chain Q_{n+1} to be the principal refinement of P_{n+1} which has the monocyclic r-pattern r_{n+1} in P_{n+1} and show that then Q_{n+1} is a refinement of T_n having the monocyclic r-pattern s_n in T_n. It is also shown that the closure of each link of Q_{n+1} is a subset of the link of T_n to which it corresponds under s_n. In order to establish these two assertions it is noted that s_n is a cyclic r-pattern, the domain of s_n is identical with the set of subscripts of the links of Q_{n+1} and the range of s_n is identical with the set of subscripts of the links of T_n. Hence, both of these assertions will be established if it is shown that the closure of each link of Q_{n+1} is a subset of the link of T_n to which it corresponds under s_n. Now, Q_{n+1} is a principal refinement of P_{n+1} having the monocyclic r-pattern r_{n+1} in P_{n+1}, so that a link of Q_{n+1} with subscript i is the same set as the link of P_{n+1} with subscript $r_{n+1}(i)$. In addition, from condition (d) of Definition 12 for the sequence $P_1, P_2, P_3, \cdots$, the closure of the link of P_{n+1} with subscript $r_{n+1}(i)$ is a subset of the link of P_n with subscript $f_n r_{n+1}(i)$. Thus, since T_n is a principal refinement of P_n having the monocyclic r-pattern g_n in P_n, the closure of the link of Q_{n+1} with subscript i is a subset of any link of T_n with subscript in the set of integers $g_n^{-1} f_n r_{n+1}(i)$. Therefore, since $f_n r_{n+1} = g_n s_n$, the closure of the link of Q_{n+1} with subscript i is a subset of the link of T_n with subscript $s_n(i)$, as was to be shown.

We now define T_{n+1} to be a principal refinement of Q_{n+1} having a crooked monocyclic r-pattern in Q_{n+1}. It is easily seen that such a circular p-chain exists. In particular we can choose T_{n+1} to be the principal refinement of Q_{n+1} having the crooked monocyclic r-pattern in Q_{n+1} described in the example of a pseudo-circle in [2, p. 48]. To see that T_{n+1} satisfies conditions (1) and (2), above, let c_{n+1} denote the crooked monocyclic r-pattern of T_{n+1} in Q_{n+1}. Then, by Theorem 3.2, $r_{n+1}c_{n+1}$ is a monocyclic r-pattern. Furthermore, T_{n+1} is a principal refinement of P_{n+1} under the monocyclic r-pattern $r_{n+1}c_{n+1}$ of T_{n+1} in P_{n+1}. In addition, from Theorem 3.2 and Theorem 3.4, $s_n c_{n+1}$ is a crooked monocyclic r-pattern of T_{n+1} in T_n. Thus, the sequence of circular p-chains $T_1, T_2, T_3, \cdots$ satisfies the required conditions (1) and (2). It remains to show that the sequence $T_1, T_2, T_3, \cdots$ also satis fies the requirements of Definition 12 with respect to the conitnuum C. The requirements (a) and (c) of Definition 12 follow from property (1), above, and requirement (b) of Definition 12 follows from property (2), above. The final requirement (d) of Definition 12 is a consequence of the fact that, for each positive integer i, T_{i+1} is a principal refinement of

Q_{i+1}, together with the result established in the preceding paragraph that the closure of each link of Q_{i+1} is a subset of the link of T_i to which it corresponds under the cyclic r-pattern of Q_{i+1} in T_i.

In this last section of the proof we construct a pseudo-circle M and define a continuous transformation f of M onto C. To do this, let $T_1, T_2, T_3, \cdots$ be a sequence of circular p-chains cyclically associated with C such that for each positive integer i, T_{i+1} has a crooked monocyclic r-pattern in T_i. We construct a corresponding sequence of circular chains $D_1, D_2, D_3, \cdots$ in the plane in the following manner. Let D_1 be a circular chain having the same number of links as T_1 and such that the links of D_1 are open circular disks of diameters less than 1 whose union is contained in the plane. Next, suppose for each positive integer i less than or equal to some positive integer n that D_i has been chosen and consider the case that $i = n + 1$. Now, from the definition of T_{n+1} as a principal refinement crooked in a circular p-chain Q_{n+1}, it may be seen that T_{n+1} can be chosen to have any sufficiently large number of links. Thus, we may assume that T_{n+1} has a sufficient number of links that there is a circular chain D_{n+1} having the same cyclic r-pattern in D_n as the cyclic r-pattern of T_{n+1} in T_n. Since the winding number of D_{n+1} in D_n has unit absolute value, it follows from the proof of Theorem 4 of [4] that such a circular chain D_{n+1} can be constructed in the plane. In addition we choose D_{n+1} to be such that the links of D_{n+1} are open circular disks of diameter less than $1/n + 1$ and the closure of each link of D_{n+1} is a subset of the link of D_n to which it corresponds under the cyclic r-pattern of D_{n+1} in D_n. Then, the circular chains $D_1, D_2, D_3, \cdots$ have the properties required in the definition of "pseudo-circle" [2, p. 48] so that the intersection M of the sets of points of $D_1, D_2, D_3, \cdots$ is a pseudo-circle.

We now define the continuous transformation f of M onto C. To facilitate the description of f let D_i be represented by the sequence of links $(d_{i0}, d_{i1}, \cdots, d_{ik_i})$, $i = 1, 2, 3, \cdots$. Let x be a point of M and let the sequence of links d_{1u_1} of D_1, d_{2u_2} of D_2, d_{3u_3} of $D_3, \cdots$ be a sequence of open sets closing down on x such that, for each positive integer i, $d_{i+1u_{i+1}}$ corresponds to d_{iu_i} under the cyclic r-pattern of D_{i+1} in D_i. We define

$$f(x) = \bigcap_{i=1}^{\infty} t_{iu_i}$$

and note by conditions (c) and (d) of Definition 12 that $f(x)$ exists and is a single point. If a second sequence of links d_{1v_1} of D_1, d_{2v_2} of D_2, d_{3v_3} of $D_3, \cdots$ closes down on x then, for each positive integer i, $|u_i - v_i| \bmod k_i \leqq 1$. Hence the links t_{u_i} and t_{v_i} of T_i intersect, $i = 1, 2, 3, \cdots$. It follows from condition (c) of Definition 12 that

$$t_{1u_1} \cap t_{2u_2} \cap t_{3u_3} \cap \cdots = t_{1v_1} \cap t_{2v_2} \cap t_{3v_3} \cap \cdots .$$

Thus f is a well defined transformation. To prove that f is a continuous transformation of M onto C, let t be an open set in C and let m be an integer such that t contains three consecutive links

$$t_{mj},\ t_{m(j+1) \bmod k_m},\ t_{m(j+2) \bmod k_m} \quad \text{of} \quad T_m .$$

Now, if $d_{1u_1}, d_{2u_2}, d_{3u_3}, \cdots$ is a sequence of links closing down on a point of $d_{m(j+1) \bmod k_m}$ then one of the links d_{mj}, $d_{m(j+1) \bmod k_m}$ and $d_{m(j+2) \bmod k_m}$ is a member of this sequence. Hence $d_{m(j+1) \bmod k_m} \cap M$ is mapped into the subset $t_{mj} \cup t_{m(j+1) \bmod k_m} \cup t_{m(j+2) \bmod k_m}$ of t under the transformation f. We conclude that f is continuous and the compact set $f(M)$ is everywhere dense in C. This completes the proof.

We now present the second principal theorem of this paper. In the proof of this theorem, Theorem 4.3, it is shown that all p-chainable continua [6] as well as all planar circularly chainable continua are circularly p-chainable. Thus, in particular, we obtain the additional result which is stated as Theorem 4.4. It is observed the Theorem 4.4 can alternatively be obtained from the statement of Theorem 4.3 rather than the proof of Theorem 4.3 in view of the result of Burgess [5, Th. 7] that every indecomposable chainable continuum is circularly chainable and the result of the author [6, Th. 4.1] that every chainable continuum is a continuous image of the pseudo-arc.

THEOREM 4.3. *Every planar circularly chainable continuum is a continuous image of a pseudo-circle.*

Proof. Let C be a planar circularly chainable continuum. Then, from Theorem 9 of [4], C is either a snake-like continuum or C separates the plane into exactly two complementary domains having C as their common boundary. We consider the two cases in turn.

Case 1. C is a snake-like continuum. Then, by Theorems 3.5 and 4.1 of [6] there is a sequence of p-chains $P_1, P_2, P_3, \cdots$ such that for each positive integer i:

(1) The union of the links of P_i is C.

(2) P_{i+1} is a normal refinement of P_i.

(3) The diameter of each link of P_i is less than $1/i$.

(4) The closure of each link of P_{i+1} is a subset of the link of P_i to which it corresponds under the r-pattern of P_{i+1} in P_i.

Now let P_i be represented in the form $P_i(0, n_i)$, $i = 1, 2, 3, \cdots$ and consider the ordered p-chain sum

$$Q_i(0, m_i) = P_i(0, n_i) + P_i(n_i, 0) .$$

Since the first and last links of $Q_i(0, m_i)$ are identical, the p-chain $Q_i(0, m_i)$ is a circular p-chain. Furthermore, by property (1), above, the union of the links of $Q_i(0, m_i)$ is equal to C. Now, since P_{i+1} is a normal refinement of P_i, it follows that the r-pattern of $P_{i+1}(0, n_{i+1})$ in $P_i(0, n_i)$ together with the r-pattern of $P_{i+1}(n_{i+1}, 0)$ in $P_i(n_i, 0)$ determine a monocyclic r-pattern of $Q_{i+1}(0, m_{i+1})$ in $Q_i(0, m_i)$. In addition, the closure of each link of $Q_{i+1}(0, m_{i+1})$ is a subset of the link of $Q_i(0, m_i)$ to which it corresponds under the monocyclic r-pattern of $Q_{i+1}(0, m_{i+1})$ in $Q_i(0, m_i)$. Thus, conditions (a), (b) and (d) of Definition 12 are satisfied by the sequence of circular p-chains

$$Q_1(0, m_1), Q_2(0, m_2), Q_3(0, m_3), \cdots$$

with respect to the continuum C. Finally, condition (c) of Definition 12 for the sequence

$$Q_1(0, m_1), Q_2(0, m_2), Q_3(0, m_3), \cdots$$

is a consequence of property (4), above, for the sequence $P_1, P_2, P_3, \cdots$. Therefore, C is circularly p-chainable and we conclude, by Theorem 4.2, that C is a continuous image of a pseudo-circle.

Case 2. C is a planar circularly chainable continuum which separates the plane into exactly two complementary domains. In this case we note by Theorem 7 of [4] that it may be assumed without loss in generality that for each positive number ε, C can be irreducibly covered by a planar circular chain each of whose links is an open circular disk of diameter less than ε. Hence, there is a sequence of chains $D_1, D_2, D_3, \cdots$ in the plane such that for each positive integer i:

(a) The union of the links $(d_{i0}, d_{i1}, \cdots, d_{ik_i})$ of D_i contains C.

(b) There is a cyclic r-pattern f_i of D_{i+1} in D_i.

(c) Each link of D_i is an open circular disk of diameter less than $1/i$.

(d) The closure of each link of D_{i+1} is a subset of the link of D_i to which it corresponds under f_i.

Then, since each link of each circular chain of the sequence $D_1, D_2, D_3, \cdots$ is a connected open set, it follows from the proof of Theorem 3 of [4] that for each positive integer i, the absolute value of the winding number of f_i is equal to at most 1. We show that for all but a finite number of values of i, $i = 1, 2, 3, \cdots$, the absolute value of the winding number of f_i is equal to 1. For suppose that there is an increasing sequence of positive integers $h_1, h_2, h_3, \cdots$ such that for each positive integer i, f_{h_i} has winding number zero. Then, by Theorem 3.1, $D_{h_1}, D_{h_2}, D_{h_3}, \cdots$ is a subsequence of $D_1, D_2, D_3, \cdots$ with the property that $D_{h_{i+1}}$ has a cyclic r-pattern with winding number zero in D_{h_i} for $i = 1, 2, 3, \cdots$. It follows that the sequence of open sets obtained by successively intersecting the links of $D_{h_{i+1}}$ with the set which is the union of the links of D_{h_i}, $i = 1, 2, 3, \cdots$, is a linear chain. Thus we obtain the conclusion that C is a snake-like continuum, which contradicts the fact that C separates the plane. In view of the foregoing contradiction we conclude that there is a positive integer n such that for each positive integer j, f_{n+j} is a monocyclic r-pattern. Now let $D_{n+j} \cap C$ denote the circular p-chain

$$(d_{n+j0} \cap C, d_{n+j1} \cap , C \cdots, d_{n+jk_{n+j}} \cap C), j = 1, 2, 3, \cdots .$$

Then the sequence of circular p-chains $D_{n+1} \cap C, D_{n+2} \cap C, D_{n+3} \cap C, \cdots$ satisfies each of the conditions of Definition 12 with respect to the continuum C. Therefore, by Theorem 4.2, C is a continuous image of a pseudo-circle.

THEOREM 4.4. *Every snake-like continuum is a continuous image of a pseudo-circle.*

Proof. This is a consequence of the argument given in Case 1 of the proof of Theorem 4.3.

The referee has noted that Theorem 3.3 together with the remark following this theorem suggest the question of whether or not there exists a result similar to Theorem 3.3 in the case that the winding numbers of the cyclic r-patterns are arbitrary positive integers. He has also suggested that it be mentioned that it is not known whether or not every pseudo-circle is a continuous image of the pseudo-arc, and it is not known whether or not every solenoid is a continuous image of a pseudo-circle. Solutions to each of these problems will be included in a subsequent paper to be presented by this author.

Added in proof. Professor F. Burton Jones has mentioned to this author that James T. Rogers Jr, one of his Ph. D. students, has recently obtained some of the results of this paper by independent investigation.

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