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A NECESSARY CONDITION FOR d-POLYHEDRALITY

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A graph is d-polyhedral provided it is isomorphic to the graph of a d-dimensional convex polytope. One of the unsolved problems in the field of convex polytopes is to characterize the d-polyhedral graphs for $d > 3$. There are, however, several necessary conditions known for a graph to be d-polyhedral. In this paper we present a new necessary condition which is not implied by the other conditions but which has two of them as corollaries. We also show how this new condition may be useful in solving problems dealing with ambiguity of d-polyhedral graphs.

2. Preliminary remarks. The 3-polyhedral graphs have been shown by Steinitz [8] to be those that are planar and 3-connected. The known necessary conditions for d-polyhedrality, $d > 3$, are:

(i) *A d-polyhedral graph is d-connected* [1].

(ii) *A d-polyhedral graph contains a subgraph homeomorphic to the complete graph of $d + 1$ vertices* [3, 4, 6].

(iii) *The maximum number of components into which a d-polyhedral graph may be separated by removing $n > d$ vertices is equal to the maximum number $\sigma(d, n)$ of facets of a d-dimensional polytope with n vertices* [7].

Let G be a graph homeomorphic to C_d (the complete graph of d vertices). If φ is the homeomorphism then the images under φ of the vertices of C_d will be called the *principal* vertices of G.

3. The main result.

THEOREM 1. *Given a vertex v of a d-polyhedral graph G, let V be the set of vertices joined to v by edges of G. Then V is contained in a $(d-1)$-polyhedral subgraph G' of $G \sim \{v\}$, and G' contains a homeomorphic image of C_d whose principal vertices lie in the set V.*

Proof. We shall make use of the *pulling process* which is discussed in detail in [2]. We say that a d-polytope P' is obtained from a d-polytope P by *pulling* vertex v of P, when we replace the vertex v by a point v' such that the segment $(v, v']$ does not intersect any $(d-1)$-flat determined by vertices of $P \sim \{v\}$, and then take as P' the convex hull of v' and the remaining vertices of P. It follows from the results in [2] that when v has been pulled to v' all $(d-1)$-

dimensional faces of P' incident to v' are pyramids over certain subfacets of P.

We define the *linked complex* of a vertex v in P to be the intersection of the star of v with the antistar of v in the boundary complex of P. It also follows from the results in [2] that if v is pulled to v' then the linked complex remains unchanged.

Let P be a d-polytope $(d > 2)$ and let a vertex v be specified. We pull v to a new vertex v' and then intersect this new polytope P' by a hyperplane, H, which separates v' from the remaining vertices of P'. This intersection is a $(d-1)$-polytope P^* and we shall show that P^* is combinatorially equivalent to the linked complex of v. Let F be any facet (i.e. $(d-1)$-face) of P^*. The facet F is the intersection of H with some pyramidal facet F of P'. Let φ be a function from the collection of facets of P^* to the collection of $(d-1)$-faces of the linked complex of v, which takes F into the facet of $\mathscr{F}$ which misses v'. Since every face of the linked complex of v is a face of some pyramidal face of P' which meets v', the function φ is one-to-one and preserves incidences and is therefore a combinatorial equivalence. For $d \geqq 2$ the graph of a d-polytope is a subset of the boundary complex and thus the graph consisting of the vertices and edges of the linked complex of v is the required $(d-1)$-polyhedral subgraph of $G \sim \{v\}$.

We now prove the second part of our theorem by induction on d. The theorem is obvious for $d = 3$. For $d > 3$ we intersect P with a hyperplane H which separates V from the remaining vertices of P. Let $Q = H \cap P$. The projection from v of P onto the linked complex of v provides a homeomorphism of the graph of Q onto a subgraph of the linked complex of v. Let v' be a vertex of Q; then by the induction hypothesis the set of vertices V' connected to v' contains a set of principal vertices of a homeomorphic image of C_{d-1}. Thus $\{v'\} \cup V'$ contains a set of principal vertices of a homeomorphic image of C_d whence the linked complex contains a graph homeomorphic to C_d. The principal vertices of this graph in the linked complex are joined to v and the theorem is proved.

COROLLARY 1. *A d-polyhedral graph contains a subgraph homeomorphic to C_{d+1}.*

COROLLARY 2. *A d-polyhedral graph is d-connected.*

Proof. The result is obvious for $d \leqq 2$. Proceeding by induction, we assume the theorem to be true for all dimensions less than d. Suppose G is a d-polyhedral graph which can be separated by $d-1$ or fewer vertices. Let V be a minimal set of vertices separating G.

Let $v \in V$, then there are vertices v_1 and v_2 each joined to v by an edge and lying in different components of the separation. By the above theorem, v_1 and v_2 lie in a $(d-1)$-polyhedral subgraph Q of $G \sim \{v\}$ and thus Q can be separated by removing $d-2$ or fewer vertices which contradicts the induction hypothesis.

The following example shows that (i), (ii) and (iii) do not imply the conditions of Theorem 1.

Let G be the graph obtained from C_7 by removing a circuit of length 3 (Figure I). This graph satisfies (i), (ii) and (iii) for $d = 4$, however, the vertices connected to v_0 do not belong to any 3-polyhedral subgraph of $G \sim \{v_0\}$. To see this, suppose G' were such a subgraph. Then G' must contain each edge incident to v_1, v_2 and v_3, but these are the edges of the complete bipartite graph $K_{3,3}$ which is not planar and thus G' is not 3-polyhedral.

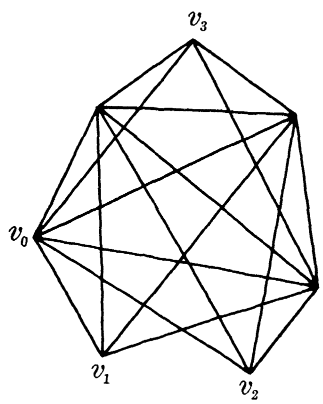

FIGURE I

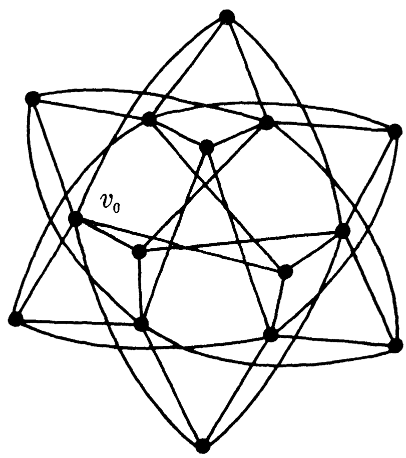

FIGURE II

A more complicated example is the graph in Figure II which satisfies (i), (ii), (iii) and the first part of our theorem for $d = 4$ but there is no 3-polyhedral subgraph containing the set V of vertices attached to v_0 which contains a homeomorphic image of C_4 whose principal vertices lie in V.

A graph may satisfy the conditions of Theorem 1 and (iii) and still not be polyhedral as is shown by the following theorem and example.

The *complete bipartite graph* $K_{a,b}$ is defined to be the graph whose vertices consist of two nonempty sets, A and B with a and b vertices respectively, and two vertices are joined by an edge if and only if one is in A and the other is in B.

THEOREM 2. *The complete bipartite graph $K_{a,b}$ is d-polyhedral only if* $d = 2$, $a = 2$, and $b = 2$; or $d = 1$ $a = 1$, and $b = 1$.

Proof. The theorem is obvious for $d \leqq 2$. Suppose $K_{a,b}$ is 3-polyhedral. The sets A and B must each contain three vertices for otherwise $K_{a,b}$ would not be 3-connected. But if a and b are greater than or equal to three then $K_{a,b}$ contains $K_{3,3}$ and is therefore not planar.

Suppose $K_{a,b}$ is d-polyhedral for $d > 3$. Let G be the graph of some 3-dimensional face F of a d-polytope whose graph is $K_{a,b}$. If two vertices of G are joined by an edge in $K_{a,b}$ then they are joined by an edge in G because F is convex. Therefore, G is a complete bipartite graph which has been shown to be impossible.

It is easy to check (using the results mentioned in Remark 2 in §4 of this paper) that $K_{19,19}$ satisfies (iii) for $d = 4$. The graph in Figure III has 37 vertices, is bipartite and is 3-polyhedral. The vertex

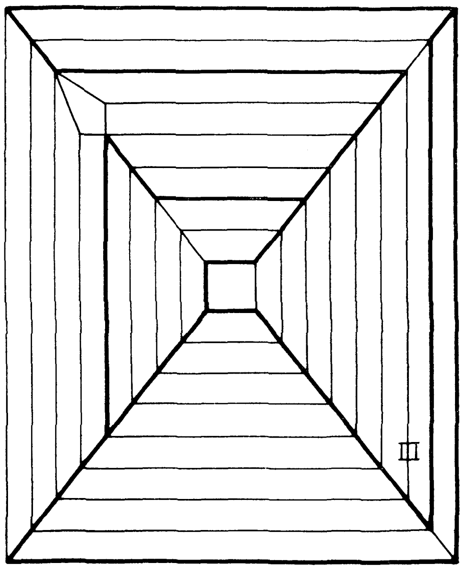

FIGURE III

set with 19 vertices contains the principal vertices of a homeomorphic image of C_4 as is indicated in Figure III. This shows that $K_{19,19}$ satisfies the conditions of Theorem 1, but Theorem 2 shows that it is not 4-polyhedral.

4. Unambiguous graphs. A polyhedral graph G is said to be completely unambiguous provided

(i) G is the graph of only one d-polytope P for some d

(ii) G is not the graph of any e-polytope for $e \neq d$

(iii) G can be realized as the graph of P in only one way. That is, if P_1 and P_2 are two combinatorially equivalent realizations of G and if a set of vertices of G determine a face of P_1 then it determines a face of P_2.

One of the unsolved problems on polyhedral graphs is whether there exist completely unambiguous graphs of d-polytopes for $d \geqq 5$. The main difficulty is in finding graphs which are d-polyhedral for $d \geqq 5$ but are not e-polyhedral for any $e < d$. The only examples of such graphs have been constructed by Klee [7] using (iii) and by Grünbaum [5] using related ideas, but these graphs are ambiguous. An affirmative answer to the following question would provide a completely unambiguous 5-polyhedral graph:

Does there exist a 4-valent 4-polyhedral graph G which is completely unambiguous and which does not contain a 3-polyhedral subgraph using all vertices of G?

If P were a realization of such a 4-polyhedral graph G then a pyramid over P would provide a 5-polytope whose graph G' could not be 4-polyhedral by our theorem. It could not be e-polyhedral for $e > 5$ because it contains 5-valent vertices and is thus not e-connected. There must be at least one 4-dimensional face of any realization of G', which misses the vertex v of $G' \sim G$. Since G is 4-valent it has no 4-polyhedral subgraphs and thus only one 4-dimensional face misses v, and any realization of G' must be a pyramid over a realization of G. Since G has only one realization, G' is completely unambiguous.

5. **Remarks.**

1. It is not known whether all graphs satisfying Theorem 1 also satisfy (iii).

2. The function $\sigma(d, n)$ has been determined for $d \leqq 8$ in which case we have

$$\sigma(d, n) = \binom{n - \left[\frac{d+1}{2}\right]}{n-d} + \binom{n - \left[\frac{d+2}{2}\right]}{n-d} .$$

This equality also holds if $n \leqq d + 3$ or $n > [d/2]^2$, see [3].

3. The graph consisting of C_7 minus a circuit of length three was first used by Grünbaum and Motzkin [5], as an example of a graph which satisfies (i) and (ii) for $d = 4$ but is not 4-polyhedral.

4. It would be of interest to determine the maximum number $C(d)$, such that given any $C(d)$ vertices of a d-polyhedral graph there is a subgraph homeomorphic to C_{d+1} whose principal vertices include these given vertices. Clearly $C(2) = 3$ and it may easily be shown that $C(d) \leqq d$ for $d > 2$. Theorem 1 implies $C(d) \geqq 1$.

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