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SOME APPLICATIONS OF A PROPERTY OF THE FUNCTOR Ef

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The functors mapping cone, Cf, and its dual, Ef, whose definitions will be recalled below, seem to have been introduced by Puppe and Nomura in 1958 and 1960 respectively. There, various basic properties of these functors were established. Here we shall prove a "3×3 lemma" for the functor Ef (with an obvious dual for Cf). This will be applied in § 3 to the problem of determining a Postnikov system for Ef in terms of f, and to show that any space having a Postnikov decomposition, and whose homotopy is finitely generated, has a decomposition in which the only $K(\pi, n)$'s appearing have π a finitely generated free abelian group.

We remark at the outset, that the basic properties of Ef and Cf, including the above mentioned 3×3 lemma, can be established in a more functorial manner than is attempted here. Since we are mainly interested in applications to Postnikov systems, we have chosen to proceed in as direct a fashion as possible. We shall study the functors Ef and Cf in a suitable abstract category in a later paper.

We shall consider the category of topological spaces with basepoint x_0 in X. (X, Y) will denote the space of (free) maps topologized by the compact open topology. $(X, Y)^{\cdot}$ will denote the space of basepoint preserving maps. We write $X \wedge Y$ for the smash product of X and Y, i.e. $X \times Y$ with $X \times y_0 \cup x_0 \times Y$ collapsed to the basepoint.

The adjointness relations

$$(X \wedge Y, Z)^{\cdot} \approx (X, (Y, Z)^{\cdot})^{\cdot}$$

and

$$(X \times Y, Z) \approx (X, (Y, Z)) ,$$

where $\approx$ means naturally homeomorphic to, are basic for what follows. They hold when, say, Y is locally compact and regular, X is Hausdorff [2]. In view of the importance of these relations to us, we should probably work in the category of quasi spaces [7], where this is universally valid. Due to the relative unfamiliarity of these notions, however, we will stick with spaces, and simply make sure the relations are valid when used.

Let I be the unit interval, and S^1 the circle. Given a space X with basepoint, we shall abbreviate $(I, X)^{\cdot}$, $(S^1, X)^{\cdot}$, $X \wedge I$, and $X \wedge S^1$ by the more usual PX, ΩX, CX, and SX respectively. We shall write $e_X\colon PX \to X$ for the endpoint projection of the space of based paths on

X, and $i_X: X \to CX$ for the injection of X to the base of the cone on X. If $f: X \to Y$ is a basepoint preserving map, let

$$Ef = \{(x, \alpha) \mid x \in X, \alpha \in PY, \text{ and } fx = e_Y\alpha\} ,$$

and denote by $\pi f: Ef \to X$ the map $\pi f(x, \alpha) = x$, and by if: $\Omega Y \to Ef$ the inclusion of the fibre $\pi f^{-1}(x_0)$. Cf is defined by $Cf = CX \cup_f Y$. That is, Cf is the identification space obtained from the disjoint union $CX \cup Y$ by identifying the points $i_X(x)$ and $f(x)$ for x in X. We shall write $jf: Y \to Cf$ for the map taking a point y in Y to its equivalence class in Cf, and $pf: Cf \to SX$ for the map that collapses the image of jf to a point. Then, $\pi f: Ef \to X$ is a fibration (i.e. satisfies the ACHP) with fibre injection $if: \Omega Y \to Ef$, and $jf: Y \to Cf$ is a cofibration (i.e. satisfies the AHEP) with cofibre projection $pf: Cf \to SX$. We shall denote by $\mathscr{E}f$ the sequence

$$Y \xleftarrow{f} X \xleftarrow{\pi f} Ef \xleftarrow{if} \Omega Y \xleftarrow{\Omega f} \Omega X \xleftarrow{\Omega(\pi f)} \Omega(Ef) \longleftarrow \cdots ,$$

and by $\mathscr{C}f$, the sequence

$$X \xrightarrow{f} Y \xrightarrow{jf} Cf \xrightarrow{pf} SX \xrightarrow{Sf} SY \xrightarrow{S(jf)} S(Cf) \longrightarrow \cdots .$$

We call $\mathscr{E}f$ the *Puppe sequence of* f, and $\mathscr{C}f$ the *co-Puppe sequence of* f, despite the fact that $\mathscr{E}f$ was investigated by Nomura [4], and $\mathscr{C}f$ by Puppe [6]. This is in line with recent conventions in category theory, and is explained by the following lemma, which we need later in § 3.

2. Lemmas.

LEMMA 2.1. *Let X, Y and Z be spaces with basepoint, and let $f: X \to Y$ be a basepoint preserving map. Then the following sequences are naturally equivalent*:

(1) $(Z, \mathscr{C}f)^{\cdot}$ *and* $\mathscr{E}(Z, f)^{\cdot}$ *if Z is locally compact regular and Y is Hausdorff.*

(2) $Z \wedge \mathscr{C}f$ *and* $\mathscr{C}(Z \wedge f)$ *if Z and X are countable CW-complexes or Z is compact.*

(3) $(\mathscr{C}f, Z)^{\cdot}$ *and* $\mathscr{E}(f, Z)^{\cdot}$ *if X is locally compact regular and Z is Hausdorff. That is, each term in one sequence is naturally homeomorphic to the corresponding term in the other, and the homeomorphisms commute with the maps of the sequences.*

The proof is a straightforward application of the adjointness relations on p.1, and will be omitted.

Clearly, Ef is functorial on maps. That is, given a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ Y_1 & \xrightarrow{f_2} & Y_2 \end{array}$$

there is a map

$$(g_1, g_2)\colon Ef_1 \longrightarrow Ef_2$$

given by $(g_1, g_1)(x_1, \alpha_1) = (g_1x_1, g_2\alpha_1)$, and this behaves properly with respect to compositions. With this we may prove the 3×3 lemma, which will be applied in § 3.

LEMMA 2.2. *Let*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ Y_1 & \xrightarrow{f_2} & Y_2 \end{array}$$

be a commutative diagram, and consider the diagram

$$\begin{array}{ccccc} & & Eg_1 & \xrightarrow{(f_1, f_2)} & Eg_2 \\ & & {\scriptstyle \pi g_1}\downarrow & & \downarrow{\scriptstyle \pi g_2} \\ Ef_1 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 \\ {\scriptstyle (g_1, g_2)}\downarrow & & {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ Ef_2 & \longrightarrow & Y_1 & \xrightarrow{f_2} & Y_2 \end{array} .$$

Then in the upper left-hand corner, there is a homeomorphism

$$\varphi\colon E(f_1, f_2) \longrightarrow E(g_1, g_2) ,$$

which makes the following diagram commutative:

$$\begin{array}{ccc} E(f_1, f_2) & \xrightarrow{\varphi} & E(g_1, g_2) \\ & {\scriptstyle (\pi g_1, \pi g_2)}\searrow \quad \swarrow{\scriptstyle \pi(g_1, g_2)} & \\ & Ef_1 & \end{array} .$$

Proof. By definition,
$E(f_1, f_2) = \{(x_1, \alpha_1, \eta) \mid (x_1, \alpha_1) \in Eg_1, \eta \in P(Eg_2)$, and $(f_1, f_2)(x_1, \alpha_1) = \eta(1)\}$. But $\eta\colon I \to Eg_2$ is given by a pair (δ_2, h_2'), where $\delta_2\colon I \to X_2$, and $h_2'\colon I \to PY_2$ satisfy $g_2\delta_2 = e_{Y_2}h_2'$. By adjointness, h_2' corresponds to an $h_2\colon I \wedge$

$I \to Y_2$ with $h_2(t, 1) = g_2\delta_2(t)$. Hence,
$E(f_1, f_2) =$ set of quadruples $(x_1, \alpha_1, \delta_2, h_2)$, where $x_1 \in X_1$, $\alpha_1 \in PY_1$, $\delta_2 \in PX_2$, h_2: $I \wedge I \to Y_2$, and $g_1x_1 = \alpha_1(1)$, $f_1x_1 = \delta_2(1)$, $h_2(1, t) = f_2\alpha_1(t)$, and $h_2(t, 1) = g_2\delta_2(t)$.
In this form, $(\pi g_1, \pi g_2)(x_1, \alpha_1, \delta_2, h_2) = (x_1, \delta_2)$. In the same way,
$E(g_1, g_2) =$ set of quadruples $(x_1, \delta_2, \alpha_1, \bar{h}_2)$, where $x_1 \in X_1$, $\delta_2 \in PX_2$, $\alpha_1 \in PY_1$, $\bar{h}_2$: $I \wedge I \to Y_2$, and $f_1x_1 = \delta_2(1)$, $g_1x_1 = \alpha_1(1)$, $\bar{h}_2(1, t) = g_2\delta_2(t)$, and $\bar{h}_2(t, 1) = f_2\alpha_1(t)$.
And again, $\pi(g_1, g_2)(x_1, \delta_2, \alpha_1, \bar{h}_2) = (x_1, \delta_2)$. With μ: $I \wedge I \to I \wedge I$ defined by $\mu(s, t) = (t, s)$, the map

$$\varphi\colon E(f_1, f_2) \longrightarrow E(g_1, g_2)$$

defined by $\varphi(x_1, \alpha_1, \delta_2, h_2) = (x_1, \delta_2, \alpha_1, h_2\mu)$ is the required homeomorphism.

REMARKS. Lemma 2.2 can be extended to homotopy commutative squares as follows: let

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ Y_1 & \xrightarrow{f_2} & Y_2 \end{array}$$

be a homotopy commutative square with homotopy h such that $h(-, 0) = g_2f_1$ and $h(-, 1) = f_2g_1$. As in [4], there are maps (g_1, g_2, h): $Ef_1 \to Ef_2$ and (f_1, f_2, h): $Eg_1 \to Eg_2$ such that in the following diagram, squares (a) and (b) are strictly commutative:

$$\begin{array}{ccccc} & & Eg_1 & \xrightarrow{(f_1, f_2, h)} & Eg_2 \\ & & \downarrow & (b) & \downarrow \\ Ef_1 & \longrightarrow & X_1 & \xrightarrow{\quad f_1 \quad} & X_2 \\ {\scriptstyle (g_1, g_2, h)}\downarrow & (a) & {\scriptstyle g_1}\downarrow & & {\scriptstyle g_2}\downarrow \\ Ef_2 & \longrightarrow & Y_1 & \xrightarrow{\quad f_2 \quad} & Y_2 \end{array}$$

Then the extension says there is a homotopy equivalence

$$\varphi\colon E(g_1, g_2, h) \longrightarrow E(f_1, f_2, h)\ ,$$

which is compatible with the appropriate maps. This is proved by converting g_2 to a fibration, replacing f_1 by a homotopic map giving a strictly commutative diagram, and applying Lemma 2.2. We do not give the proof in detail, since we will need only the commutative case.

3. Applications.

DEFINITION 3.1. Let X be a space with basepoint x_0. A *Postnikov system for* X consists of spaces X_n for $n \geqq 0$ together with maps:

$$p_n : X \longrightarrow X_n$$

and
$$k^n : X_n \longrightarrow K(\pi_{n+1}X, n+2)$$

such that

(1) $p_n : X \to X_n$ induces isomorphisms in homotopy through dimension n, and $\pi_i X_n = 0$ for $i > n$.

(2) $X_0 = x_0$, and $X_{n+1} = Ek^n$ for $n \geqq 0$.

(3) $\pi k^n \cdot p_{n+1} = p_n$.

k^n is called the n^{th} *k-invariant of* X, with respect to the given decomposition.

The author is indebted to E. H. Brown for the following method, which is a mild variation of the one used in [4], for constructing Postnikov systems. For this construction, all spaces will be assumed to be 1-connected. So, given X 1-connected with basepoint x_0, take $X_1 = x_0$ (same for X_0) and let $p_1 : X \to X_1$ be the unique projection. Assume by induction that we have constructed $p_n : X \to X_n$ such that (1) is satisfied. Consider the sequence $X \xrightarrow{p_n} X_n \xrightarrow{jp_n} Cp_n$. One can show: $\pi_i Cp_n = 0$ for $i < n+2$, and $\pi_{n+2} Cp_n \approx \pi_{n+1} X$. Let $i_n : Cp_n \to K(\pi_{n+1}X, n+2)$ be the fundamental class of Cp_n, constructed by including Cp_n in a space of type $K(\pi_{n+1}X, n+2)$ formed by attaching cells to Cp_n to kill its homotopy in dimensions greater than $n+2$. Let k^n be the composite

$$X_n \xrightarrow{jp_n} Cp_n \xrightarrow{i_n} K(\pi_{n+1}X, n+2) .$$

Let $X_{n+1} = Ek^n$. Then we have

$$\begin{array}{ccccc} & & X_{n+1} & & \\ & & \downarrow & & \\ X & \xrightarrow{p_n} & X_n & \xrightarrow{k^n} & K(\pi_{n+1}X, n+2) . \end{array}$$

Now there is a canonical null-homotopy of $jp_n \cdot p_n$, and hence of $k^n \cdot p_n$. Let $p_{n+1} : X \to X_{n+1}$ be the lifting of p_n given *this* null-homotopy. Clearly, p_{n+1} induces isomorphisms on homotopy through dimension n. A more involved argument shows p_{n+1} also induces an isomorphism in dimension $n+1$, which completes the induction step. The advantage of this construction is the following: given 1-connected spaces with basepoint X and Y, and a map $f : X \to Y$, there are induced maps in the previous construction, which make all possible diagrams strictly commutative. In fact, assuming by induction that we have $f_n: X_n \to Y_n$, we may set $f_{n+1} = (f_n, \bar{f}_n)$, where $\bar{f}_n$ has been constructed so that the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle k_X^n}\downarrow & & \downarrow{\scriptstyle k_Y^n} \\ K(\pi_{n+1}X, n+2) & \xrightarrow{\bar{f}_n} & K(\pi_{n+1}Y, n+2) \end{array}$$

commutes. We leave the details to the reader. The fact that we can get induced maps such that f_{n+1} has the above form for all n is necessary for applications of Lemma 2.2, and is not true, for example, of the induced maps constructed in [3].

Now consider spaces X and Y with given Postnikov systems, and a map $f: X \to Y$, which induces a map on the Postnikov systems of the above form. For example, by the previous construction, any map f where X and Y are 1-connected. We would like to be able to obtain a Postnikov system for Ef by applying E to the maps in the system induced by f. This is not possible in general. However, we can prove the following theorem.

THEOREM 3.1. *If either*
(i) $\pi_n f$ *is a monomorphism for all* n, *or*
(ii) $\pi_n f$ *is an epimorphism for all* n,
then a Postnikov system for Ef *may be obtained by applying* E *to the maps in the system induced by* f.

Proof. In either case we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p_n^X}\downarrow & & \downarrow{\scriptstyle p_n^Y} \\ X_n & \xrightarrow{f_n} & Y_n \end{array}$$

for all $n \geqq 0$. In case (i), put

$$(Ef)_n = Ef_{n+1} \text{ and } p_n^{Ef} = (p_{n+1}^X, p_{n+1}^Y) .$$

In case (ii), put

$$(Ef)_n = Ef_n \text{ and } p_n^{Ef} = (p_n^X, p_n^Y) .$$

Consider the induced commutative ladder in homotopy.

$$\begin{array}{ccccccccc} \pi_{i+1}X & \longrightarrow & \pi_{i+1}Y & \longrightarrow & \pi_i Ef & \longrightarrow & \pi_i X & \longrightarrow & \pi_i Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_{i+1}X_n & \longrightarrow & \pi_{i+1}Y_n & \longrightarrow & \pi_i Ef_n & \longrightarrow & \pi_i X_n & \longrightarrow & \pi_i Y_n \end{array} .$$

An application of the five lemma to this diagram shows that in either case (i) or case (ii), $\pi_i(Ef)_n = 0$ for $i > n$, and p_n^{Ef} induces isomorphisms in homotopy through dimension n. Therefore, in either case, condition (1) in Definition 3.1 is satisfied.

Now consider the diagram

$$\begin{array}{ccccc} Ef_n & \longrightarrow & X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle (k_X^n,\, k_Y^n)}\downarrow & & {\scriptstyle k_X^n}\downarrow & & {\scriptstyle k_Y^n}\downarrow \\ E\bar{f}_n & \longrightarrow & K(\pi_{n+1}X, n+2) & \xrightarrow{\bar{f}_n} & K(\pi_{n+1}Y, n+2) \end{array} .$$

From the bottom row we obtain the following exact sequence in homotopy.

$$\pi_{i+1}K(\pi_{n+1}X, n+2) \xrightarrow{\pi_{i+1}\bar{f}_n} \pi_{i+1}K(\pi_{n+1}Y, n+2) \longrightarrow \pi_i E\bar{f}_n$$
$$\longrightarrow \pi_i K(\pi_{n+1}X, n+2) \xrightarrow{\pi_i \bar{f}_n} \pi_i K(\pi_{n+1}X, n+2) \longrightarrow \cdots .$$

Thus, for $i \neq n+1$ or $n+2$, $\pi_i E\bar{f}_n = 0$, and

$$\pi_{n+1}E\bar{f}_n \approx \operatorname{cok} \pi_{n+2}\bar{f}_n \approx \operatorname{cok} \pi_{n+1}f$$
$$\pi_{n+2}E\bar{f}_n \approx \ker \pi_{n+2}\bar{f}_n \approx \ker \pi_{n+1}f .$$

Now in case (i), $\pi_i Ef \approx \operatorname{cok} \pi_{i+1}f$ for all i, and in case (ii), $\pi_i Ef \approx \ker \pi_i f$ for all i. Thus in case (i), $E\bar{f}_n$ is a space of type $K(\pi_n Ef, n+1)$, and in case (ii), $E\bar{f}_n$ is a space of type $K(\pi_{n+1}Ef, n+2)$. So for (i), let k_{Ef}^{n-1} be the map (k_X^n, k_Y^n), and for (ii) let k_{Ef}^n be the same map. Then condition (2) in both cases follows from Lemma 2.2 applied to the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle k_X^n}\downarrow & & \downarrow{\scriptstyle k_Y^n} \\ K(\pi_{n+1}X, n+2) & \xrightarrow{\bar{f}_n} & K(\pi_{n+1}Y, n+2) \end{array} .$$

Condition (3) in either case follows from the functorial nature of E.

Naturally, given a map $f: X \to Y$ it is rare that f satisfies the conditions of Theorem 3.1. Nevertheless, some examples of interest do emerge.

EXAMPLES. (1) Consider the functor $\Omega X = (S^1, X)^{\cdot}$. This is Ei, where $i : x_0 \longrightarrow X$ is the inclusion of the basepoint. The induced map is just the inclusion of the basepoint at every stage. $\pi_n i$ is a monomorphism for all n, so that Theorem 3.1 says that a Postnikov system and k-invariants for ΩX are obtained by applying Ω to the spaces and maps in a system for X, as is well known.

(2) Let X be a space with a Postnikov decomposition. For any $n \geqq 1$, an n-connective fibre space over X is obtained by applying E to the map $p_n : X \longrightarrow X_n$, in the given system for X. Using the obvious decomposition for X_n, the induced map on Postnikov systems is just given by the maps in the system for X. $\pi_j p_n$ is clearly an epimorphism for each j. Thus, by Theorem 3.1, a Postnikov system for the n-connective fibre space over X is given by taking n-connective fibre spaces over each piece of the Postnikov system for X. This is relatively clear in any event, and is intended only to be used in our main example, which is the following one.

(3) Let m be a positive integer, and let $S^1 \xrightarrow{m} S^1$ be a map of degree m. Consider the co-Puppe sequence of m.

$$\mathscr{C}m : S^1 \xrightarrow{m} S^1 \longrightarrow L(m, 2) \longrightarrow S^2 \xrightarrow{m} S^2 \longrightarrow \cdots .$$

Here $L(m, 2) = Cm$ and is a Moore space with one nonvanishing integral cohomology group $\mathbf{Z}_m$ in dimension 2. Let BSU be the classifying space for the infinite special unitary group. By Lemma 2.1 we have

$$(\mathscr{C}m, \mathrm{BSU})^{\cdot} \approx \mathscr{E}(m, \mathrm{BSU})^{\cdot} .$$

In particular, in the diagram

$$\begin{array}{ccc} (S^2, \mathrm{BSU})^{\cdot} \longrightarrow (L(m, 2), \mathrm{BSU})^{\cdot} & & \\ \downarrow & & \\ (S^1, \mathrm{BSU})^{\cdot} & \xrightarrow{(m,1)^{\cdot}} & (S^1, \mathrm{BSU})^{\cdot} , \end{array}$$

$(L(m, 2), \mathrm{BSU})^{\cdot} \approx E(m, 1)^{\cdot}$. To simplify notation, let $(L(m, 2), \mathrm{BSU})^{\cdot} = BU_m$. Then BU_m is the classifying space for complex K-theory mod m. The cohomology of these spaces for $m = p$ a prime is computed in [8]. Now a Postnikov system for BU—the classifying space for the infinite unitary group—is determined in [5]. BSU is a 2-connective fibre space over BU, so by (2) we know a Postnikov system for BSU. In [1], the homotopy groups of $(S^1, \mathrm{BSU})^{\cdot}$ are determined. They are $\mathbf{Z}$ in odd dimensions (except 0 in dimension 1) and 0 in even dimensions. $\pi_i(m, 1)$ is multiplication by m, and hence is a monomorphism for each i. Clearly, the induced maps are given by $(m, 1)^{\cdot}$ on each piece of the Postnikov system for $(S^1, \mathrm{BSU})^{\cdot}$, since such a system is given by applying $(S^1, -)^{\cdot}$ to a Postnikov decomposition of BSU by (1). Thus, by Theorem 3.1 we finally obtain a Postnikov system and k-invariants for BU_m by applying the functor $(L(m, 2), -)^{\bullet}$ to the decomposition of BSU. Further information on these Postnikov systems may be found in [8].

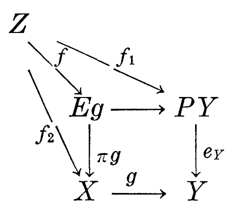
Another type of application of Lemma 2.2 is the following. Let X and Y be spaces with basepoint, and let $g\colon X \to Y$ be a basepoint preserving map. Let Z be another space, and $f\colon Z \to Eg$ a map. Consider the fibration $\pi f\colon Ef \to Z$. Then we may decompose πf into a composite of two fibrations each induced by a map into a space involving only X or Y. Before doing this, we remark that in a homotopy commutative diagram with homotopy h,

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\varphi_2} \\ Y_1 & \xrightarrow{g} & Y_2 \end{array}$$

if φ_1 and φ_2 are homotopy equivalences, so is

$$(\varphi_1, \varphi_2, h) : Ef \longrightarrow Eg \qquad\qquad ([4]\ \S\,4.2)\,.$$

Now, since Eg is a pullback, f is determined by two maps $f_1\colon Z \to PY$ and $f_2\colon Z \to X$ making the diagram



commutative. In the following diagram, let $\equiv$ denote homotopy equivalent with, and let $\sim$ in a square mean that square is homotopy commutative. All other squares will be strictly commutative. Since PY is contractible, the signs $\equiv$ follow from the previous remark about homotopy equivalences. The equivalence $E(f_1, g) \approx E(f_2, e_Y)$ follows from Lemma 2.2.

$$\begin{array}{ccccccc}
 & & Ef & \xrightarrow{\ \pi f\ } & Z & \xrightarrow{\ f\ } & Eg \\
 & & \equiv & & \equiv & \sim & \downarrow{\scriptstyle 1} \\
Ek \equiv E(f_1, g) & \approx & E(f_2, e_Y) & \longrightarrow & Ef_1 & \xrightarrow{(f_2, e_Y)} & Eg \\
{\scriptstyle \pi k}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Ef_2 & \xrightarrow{\ 1\ } & Ef_2 & \xrightarrow{\ \pi f_2\ } & Z & \xrightarrow{\ f_2\ } & X \\
{\scriptstyle k}\downarrow & \sim & {\scriptstyle (f_1, g)}\downarrow & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g} \\
\Omega Y & \equiv & Ee_Y & \longrightarrow & PY & \xrightarrow{\ e_Y\ } & Y
\end{array}\;.$$

In this diagram, k is, of course, (f_1, g) up to the given homotopy equivalence. Thus, up to homotopy equivalence, we have replaced the fibration

$$Ef \xrightarrow{\pi f} Z \quad \text{by} \quad Ek \xrightarrow{\pi k} Ef_2 \xrightarrow{\pi f_2} Z \, .$$

This has the following obvious application to Postnikov systems. Let X be a space with a given Postnikov system, and assume the homotopy groups of X are finitely generated. Consider a piece of the Postnikov system of X at stage $n \geqq 1$.

$$\begin{array}{ll} X_{n+1} & \\ \downarrow{\scriptstyle \pi k^n} & \\ X_n \xrightarrow{k^n} & K(\pi_{n+1}X, n+2) \, . \end{array}$$

Now decompose $\pi_{n+1}X$ into a direct sum of copies of $\mathbf{Z}$ and cyclic groups $\mathbf{Z}_{r_i}$ by the fundamental theorem of abelian groups. For each copy of $\mathbf{Z}$, take a map

$$* \longrightarrow K(\mathbf{Z}, n+3) \, .$$

For each $\mathbf{Z}_{r_i}$, take a map

$$K(\mathbf{Z}, n+3) \xrightarrow{r_i} K(\mathbf{Z}, n+3)$$

induced from multiplication by r_i. The product of these is a map

$$g\colon K(\bigoplus_c \mathbf{Z}, n+3) \longrightarrow K(\bigoplus_a \mathbf{Z}, n+3)$$

where $\bigoplus_c \mathbf{Z}$ and $\bigoplus_a \mathbf{Z}$ denote, respectively, a direct sum of as many copies of $\mathbf{Z}$ as there are finite cyclic groups in the decomposition of $\pi_{n+1}X$, and a direct sum of as may copies of $\mathbf{Z}$ as there are summands of all types in the decomposition of $\pi_{n+1}X$. Then, $K(\pi_{n+1}X, n+2) \equiv Eg$, and the previous result applies. Namely, consider the diagram

$$\begin{array}{lll} X_{n+1} & & \\ \downarrow{\scriptstyle \pi k^n} & & \\ X_n \xrightarrow{k^n} & K(\pi_{n+1}X, n+2) & \\ \quad \searrow{\scriptstyle f} & \downarrow{\scriptstyle \pi g} & \\ & K(\bigoplus_c \mathbf{Z}, n+3) \xrightarrow{g} & K(\bigoplus_a \mathbf{Z}, n+3) \, , \end{array}$$

where we set $f = \pi g \cdot k^n$. Then, by the previous discussion, there is a map

$$k\colon Ef \longrightarrow K(\bigoplus_a \mathbf{Z}, n+2)$$

so that up to homotopy equivalence πk^n can be factored as the composite

$$Ek \xrightarrow{\pi k} Ef \xrightarrow{\pi f} X_n .$$

Thus we have proved,

PROPOSITION 3.1. Let X be a 1-connected space with basepoint whose homotopy is finitely generated. Then X has a decomposition into a tower of induced fibre spaces in which the only $K(\pi, n)$'s occurring have π a finitely generated free abelian group.

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