ON THE DERIVATIVE OF CANONICAL PRODUCTS

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In this paper we study an entire function of genus p that has the canonical product form

$$(1.1) \qquad f(z) = \prod_{j=1}^{\infty} [1 - (z/a_j)] \exp [(z/a_j) + (1/2)(z/a_j)^2 + \cdots$$
$$+ (1/p)(z/a_j)^p].$$

For the derivative f' of such a function we develop some results analogous to the theorems of Lucas and Jensen for polynomials, as well as some results in the case that all but one a_j lie on a prescribed set.

The zeros of f are subject to the restrictions that they are simple (so assumed for convenience only) and

$$(1.2) \qquad 0 < |a_1| \leq |a_2| \leq \cdots, \qquad \sum_{j=1}^{\infty} |a_j|^{-p-1} < \infty .$$

By straightforward calculation, we may reduce the logarithmic derivative of f to the form

$$(1.3) \qquad \frac{f'(z)}{f(z)} = \sum_{j=1}^{\infty} \frac{z^p}{a_j^p(z - a_j)} .$$

We thereby see that f' has always a p-fold zero at $z = 0$. In general it has an infinite number of other zeros. Any such zero ζ satisfies the equation

$$(1.4) \qquad \sum_{j=1}^{\infty} \frac{1}{a_j^p(\zeta - a_j)} = 0 .$$

That is, if

$$(1.5) \qquad s_N = \sum_{j=1}^{N} a_j^{-p}(\zeta - a_j)^{-1} ,$$

then $s_\infty = \lim s_N = 0$ for each zero $\zeta \neq 0$ of f'.

Our aim is to locate these zeros ζ, knowing the location of the zeros a_j of f.

For the case that all the a_j are real, Hille [2] establishes the following theorem essentially due to Cesàro [1]: *Let f be a canonical form of genus p whose zeros are real. Then the zeros of f' are also all real and they include one and only one zero between each pair of consecutive a_j of like sign.* In effect this result extends Rolle's Theorem to real canonical products having only real zeros.

In developing analogues to Lucas' and Jensen's theorems we shall be determining the location of the zeros of the derivative for functions (1.1) having at least some nonreal zeros.

2. Real canonical products. Let us assume that f is a real entire function of form (1.1). If a is a real zero of f and if $\zeta = \xi + i\eta$ is any zero of f', the term corresponding to zero a in (1.4) has the imaginary part

$$(2.1) \qquad \mathscr{I}\left[\frac{1}{a^p(\zeta - a)}\right] = -\frac{\eta}{a^p\,|\,\zeta - a\,|^2}\,.$$

If however

$$a = b + ic = Ae^{i\alpha}\,,$$

where $0 < \alpha < \pi$ and $A > 0$, is any nonreal zero of f, then $\bar{a} = b - ic$ is also a zero of f and the pair of terms in (1.4) corresponding to the pair $a, \bar{a}$ has the imaginary part

$$
\begin{aligned}
\mathscr{I}&\left[\frac{1}{a^p(\zeta - a)} + \frac{1}{\bar{a}^p(\zeta - \bar{a})}\right]\\
&= \frac{-(\xi - b)\sin p\alpha - (\eta - c)\cos p\alpha}{A^p[(\xi - b)^2 + (\eta - c)^2]}\\
&\quad + \frac{(\xi - b)\sin p\alpha - (\eta + c)\cos p\alpha}{A^p[(\xi - b)^2 + (\eta + c)^2]}\\
&= -\frac{2\eta\Gamma(\zeta, a)}{A^p\,|\,\zeta - a\,|^2\,|\,\zeta - \bar{a}\,|^2}
\end{aligned}
$$

(2.2)

where

$$\Gamma(\zeta, a) = [(\xi - b)^2 + \eta^2 - c^2]\cos p\alpha + 2c(\xi - b)\sin p\alpha\,.$$

If $\alpha = (\pi/2p)(1 + 2kp)$, $k = 0, 1, \cdots, p - 1$, then

$$(2.3) \qquad \Gamma(\zeta, a) = (-1)^p 2c(\xi - b)$$

and $\Gamma(\zeta, a) = 0$ is the equation of the line through points $a, \bar{a}$. For all other values of α

$$(2.4) \qquad \Gamma(\zeta, a) = \{(\xi - b + c\tan p\alpha)^2 + \eta^2 - c^2\sec^2 p\alpha\}\cos p\alpha$$

so that $\Gamma(\zeta, a) = 0$ is the equation of the circle with center at $(b - c\tan p\alpha)$ and radius $c\sec p\alpha$. This circle goes through points $a, \bar{a}$. In any case, the inequality $\Gamma(\zeta, a) > 0$ defines a circular region.
    We now prove the following theorem.

THEOREM (2.1). *Let f be a real entire function of form (1.1). With each pair of conjugate imaginary zeros $a_j, \bar{a}_j$ of f, associate the circular region Γ_j defined from (2.3) or (2.4) by the inequality $\Gamma(\zeta, a_j) > 0$. Then no nonreal zero of f' lies simultaneously in all the regions Γ_j.*

Proof. If there were a nonreal zero ζ of f' in the upper half plane and also in all the regions Γ_j, the imaginary parts of all the terms in (1.4) would be negative for all real a_j according to (2.1) and would also be negative for all pairs of conjugate imaginary a_j according to (2.2). Thus in (1.5) $\lambda \mathscr{I}(s_h) < 0$, $\lambda \mathscr{I}(s_N - s_h) < 0$ for $\lambda = \pm 1$, $\lambda \eta < 0$, fixed h and all $N > h$ so that $\lambda \mathscr{I}(s_\infty - s_h) \leqq 0$ and $s_\infty \neq 0$ contradicting (1.4). Hence, Theorem (2.1) is valid.

It is clear from the above proof that the following sharper results are also valid.

THEOREM (2.1)'. *Under the hypotheses of Theorem* (2.1) *no nonreal zero of* f' *can lie simultaneously inside some region* Γ_j *and in the closures of the remaining regions* Γ_j. *Likewise no nonreal zero of* f' *can lie simultaneously outside the closure of some region* Γ_j *and outside the remaining regions* Γ_j.

To specify the location of the real zeros $\zeta = \xi + i0$ of f', we compute for each conjugate pair $a, \bar{a}$ of zeros of f

$$\mathscr{R}\left[\frac{1}{a^p(\xi - a)} + \frac{1}{\bar{a}^p(\xi - \bar{a})}\right] = \frac{2L(\xi, a)}{|\xi - a|^2}$$

where

$$(2.5) \qquad L(\xi, a) = (\xi - b)\cos p\alpha - c\sin p\alpha .$$

We thus obtain the following result.

THEOREM (2.2). *Let* f *be a real entire function of form* (1.1). *A real point* $\zeta = \xi$ *is not a zero of* f' *if it satisfies simultaneously the inequalities* $a_j^p(\xi - a_j) > 0$ *for all real* a_j *and* $L(\xi, a_j) > 0$ *for all pairs of conjugate imaginary* a_j, *or, the inequalities* $a_j^p(\xi - a_j) < 0$ *for all real* a_j *and* $L(\xi, a_j) < 0$ *for all pairs of conjugate imaginary* a_j.

In particular, if $p = 0$, then

$$\Gamma(\zeta, a) = (\xi - b)^2 + \eta^2 - c^2$$

so that $\Gamma(\zeta, a) > 0$ is the exterior of the (Jensen) circle which has the line segment joining the pair of conjugate zeros a and $\bar{a}$ as diameter. That is to say, each nonreal zero of f' must lie in at least one Jensen circle. Thus the Jensen Theorem [4] for polynomials carries over without change to entire functions of genus zero, as was shown originally by Walsh [6].

Also, in the case $p = 0$, the inequalities $L(\xi, a) = \xi - b > 0$ (<0)

and $a_j^p(\xi - a_j) \equiv (\xi - a_j) > 0$ (<0) imply that all the zeros of the derivative lie in any strip $-\infty \leq \alpha \leq \mathscr{R}(z) \leq \beta \leq \infty$ that contains all the a_j. In a sense this confirms that Lucas Theorem ([2], p. 22) carries over to entire functions of genus zero as was proved originally by Porter [5].

3. **Zeros in prescribed sectors.** In the sequel we shall use the notation $\mathscr{C}(T)$ for the complement of a set T in the complex plane, $\mathscr{H}(T)$ for the convex hull of T and $\mathscr{S}(T, \nu)$ for the set of points from which T subtends an angle of at least ν. The set $\mathscr{S}(T, \nu)$ is a star shaped domain relative to T and has the properties:

$$T \subset \mathscr{H}(T) = \mathscr{S}(T, \pi) \subset \mathscr{S}(T, \nu_1) \subset \mathscr{S}(T, \nu_2)$$

for $\pi > \nu_1 > \nu_2 \geq 0$. Also, if V is a point set, we shall denote by $[e^{i\omega}V]$ the point set $z = e^{i\omega}z_1$, $z_1 \in V$.

We shall now prove the following theorem.

THEOREM (3.1). *Given the sector* $V_0: 0 \leq \arg z \leq \alpha < \pi/p$, *let* V *be the union of the sectors* $[e^{i2\pi k/p}V_0]$ *for* $k = 0, 1, \cdots, p - 1$. *Let* f, *an entire function of form (1.1), have all its zeros on a set* $T \subset V$. *Then all the zeros of* f' *lie in* $\mathscr{S}(T, \pi - p\alpha)$.

Proof. Let us assume on the contrary that f' has a zero ζ outside $\mathscr{S}(T, \pi - p\alpha)$. Since T subtends an angle ν of less than $\pi - p\alpha$ at ζ, we can associate with ζ a point $\tau \neq \zeta$ such that

$$(3.1) \qquad 0 \leq \arg [\tau - \zeta)/(a_j - \zeta)] \leq \nu < \pi - p\alpha ,$$

for all $a_j \in T$. Let us multiply (1.5) by $e^{ip\alpha}(\tau - \zeta)$ thus obtaining the equation

$$(3.2) \qquad s_N^* = \sum_{j=1}^{N} \frac{e^{ip\alpha}(\tau - \zeta)}{a_j^p(a_j - \zeta)} .$$

If $a_j \in e^{2\pi ki/p}V_0$,

$$(3.3) \qquad \begin{array}{c} 2\pi k \leq \arg a_j^p \leq p\alpha + 2\pi k , \\ -2\pi k \leq \arg (e^{ip\alpha}a_j^{-p}) \leq -2\pi k + p\alpha . \end{array}$$

From (3.1) and (3.3) it follows that we may represent each term in the sum (3.2) as a vector lying in the convex sector

$$(3.4) \qquad \sigma: 0 \leq \arg z \leq \nu + p\alpha < \pi .$$

Hence $s_h^* \in \sigma$, $(s_N^* - s_h^*) \in \sigma$ and $(s_N^* - s_h^*)s_h^* \neq 0$ for fixed h and all $N > h$ so that $(s_\infty^* - s_h^*) \in \sigma$ if $s_\infty^* - s_h^* \neq 0$. In any case $s_\infty^* = s_h^* +$

$(s_\infty^* - s_h^*) \neq 0$ contradicting (1.4).

Having proved Theorem (3.1), let us now derive a number of corollaries.

COROLLARY (3.1). *If all the zeros of an entire function f of genus zero lie in an open half plane, all the zeros of f' lie in the same half plane.*

This corollary is the special case of Theorem (3.1) obtained by setting $p = 0$. It is equivalent to Porter's generalization [5] of Lucas' Theorem that all the zeros of the derivative of an entire function f of genus zero lie in the convex hull of the zeros of f.

COROLLARY (3.2). *Let f be an entire function of form (1.1) having all its zeros in a set T lying on the set of rays $\arg z = 2\pi k/p$, $k = 0, 1, 2, \cdots, p - 1$. Then all the zeros of f' lie in $\mathscr{H}(T)$.*

This corollary is merely the special case $\alpha = 0$ of Theorem (3.1).

COROLLARY (3.3). *Let T be the intersection of the disk $|z| \leq R$ and the sectors V of Theorem (3.1). Let f be an entire function of form (1.1) with all its zeros on T. Then f' has all its zeros on the disk $|z| \leq R \sec (p\alpha/2)$.*

For Corollary (3.3), the function f of form (1.1) necessarily reduces to the product of a finite number of the factors. When $p = 0$, f becomes a polynomial and the corollary becomes one that can be deduced from Lucas' Theorem.

Other corollaries follow from Theorem (3.1) when we choose T as a half strip, a sector or some other simple convex set. We leave the details to the reader.

4. **All but one zero specified.** We shall now study the location of the zeros of f', the derivative of a function f of form (1.1), when we know the location of all but one zero of f. Let us assume the unknown zero is a_1. If ζ_1 and ζ_2 are any two distinct zeros of $f'(z)$, then from (1.4)

(4.1)
$$\sum_{j=1}^{\infty} \frac{1}{a_j^p(\zeta_k - a_j)} = 0 , \quad k = 1, 2 .$$

From equations (4.1), let us eliminate a_1. We thus obtain the equation

(4.2)
$$\sigma_0^{p+1} + \tau_1 \tau_2 \sigma_1^p = 0$$

where

$$(4.3) \qquad \sigma_k = \sum_{j=2}^{\infty} \frac{1}{a_i^{p-k}(\zeta_1 - a_j)(\zeta_2 - a_j)}, \qquad \tau_k = \sum_{j=2}^{\infty} \frac{1}{a_j^p(\zeta_k - a_j)}.$$

We now prove the following theorem.

THEOREM (4.1). *Let* f, *an entire function of form* (1.1), *have all but one of its zeros on a set* T. *Let* T *be contained in the sector*

$$V: 0 \le \arg z \le \alpha < \pi/p(p + 1)$$

Then f' *has at most one zero in* $\mathscr{C}[\mathscr{S}(T, \nu)]$ *where*
$$\nu = [\pi - p(p + 1)\alpha]/[2(p + 1)].$$

Proof. Let us assume on the contrary that f' has two distinct zeros ζ_1 and ζ_2 in $\mathscr{C}[\mathscr{S}(T, \nu)]$. Since in ζ_1 and ζ_2, the angle subtended by T is less than ν, we may associate with ζ_1 and ζ_2 the two points λ_1, λ_2 with $\lambda_1 \ne \zeta_1$, $\lambda_2 \ne \zeta_2$ such that

$$(4.4) \qquad 0 \le \arg r_{jk} < \nu \quad \text{for} \quad j = 1, 2, \cdots, \quad k = 1, 2.$$

where

$$r_{jk} = [(\lambda_k - \zeta_k)/(a_j - \zeta_k)].$$

Let us multiply (4.2) by $e^{p(p+1)\alpha i}(\lambda_1 - \zeta_1)^{p+1}(\lambda_2 - \zeta_2)^{p+1}$. Equation (4.2) becomes

$$(4.5) \qquad (\sigma_0^*)^{p+1} + \tau_1^* \tau_2^* (\sigma_1^*)^p = 0$$

where

$$\sigma_0^* = e^{p\alpha i}(\lambda_1 - \zeta_1)(\lambda_2 - \zeta_2)\sigma_0$$
$$\sigma_1^* = e^{(p-1)\alpha i}(\lambda_1 - \zeta_1)(\lambda_2 - \zeta_2)\sigma_1$$
$$\tau_1^* = e^{p\alpha i}(\lambda_1 - \zeta_1)\tau_1$$
$$\tau_2^* = e^{p\alpha i}(\lambda_2 - \zeta_2)\tau_2.$$

We shall show that $(\sigma_0^*)^{p+1}$ and $[\tau_1^* \tau_2^* (\sigma_1^*)^p]$ are vectors in the same convex sector and so their sum cannot vanish in contradiction to (4.5).

For this purpose, let us examine the argument of the j-th term in σ_0^*. Using (4.4), we find

$$(4.6) \qquad 0 \le \arg [e^{\alpha p i} a_j^{-p} r_{j1} r_{j2}] < p\alpha + 2\nu.$$

Since $p\alpha + 2\nu = \pi/(p + 1)$, the j-th term of σ_0^* for each j lies in the same convex sector and so that also σ_0^* lies in the same sector. Hence

(4.7) $$0 \leqq \arg (\sigma_0^*)^{p+1} < p(p+1)\alpha + 2(p+1)\nu = \pi .$$

Similarly, let us examine the argument of the j-th term in σ_1^*.

$$0 \leqq \arg [e^{(p-1)\alpha i} a_j^{-p+1} r_{j_1} r_{j_2}] < (p-1)\alpha + 2\nu .$$

Since $(p-1)\alpha + 2\nu = [\pi/(p+1)] - \alpha$, the j-th term in σ_1^*, for each j, lies in sector

$$0 \leqq \arg z < (p-1)\alpha + 2\nu$$

which therefore, being convex, also contains σ_1^*. Hence,

(4.8) $$0 \leqq \arg (\sigma_1^*)^p < p(p-1)\alpha + 2p\nu .$$

Also, let us examine the argument of the j-th term in τ_k. For $k = 1, 2$ and all j,

$$0 \leqq \arg [e^{p\alpha i} a_j^{-p} r_{jk}] < p\alpha + \nu .$$

Since $p\alpha + \nu = \pi/2(p+1)$, the sector in which the j-th term lies for all j,

$$0 \leqq \arg z < p\alpha + \nu$$

is convex and so also

(4.9) $$0 \leqq \arg \tau_k^* < p\alpha + \nu , \qquad\qquad k = 1, 2 .$$

From (4.8) and (4.9) now follow

(4.10) $$0 \leqq \arg [(\sigma_1^*)^p \tau_1^* \tau_2^*] < p(p+1)\alpha + 2(p+1)\nu < \pi .$$

From (4.7) and (4.10) we observe that the left side of (4.5) does not vanish in contradiction to equation (4.5). Hence, Theorem (4.1) has been established.

Let us specialize Theorem (4.1) to the case that $\alpha = 0$. For T the positive real axis, $\mathscr{S}(T, \pi/2(p+1))$ is the sector

$$|\arg z| \leqq \pi - [\pi/(2p+2)]$$

Hence, $\mathscr{C}[\mathscr{S}(T, \pi/2(p+1))]$ is the sector $|\arg (-z)| < \pi/(2p+2)$. We thus obtain

COROLLARY (4.1). *Let f, an entire function of form (1.1), have only one zero which is not positive real. Then its derivative f' has at most one zero in the sector*

$$|\arg (-z)| < \pi/(2p+2) .$$

Let us note that the function f in Corollary (4.1) is not neces-

sarily real.

In the event that we choose T as the sector V of Theorem (4.1), let us take the angle α so that $\alpha < \pi/[p + 1)(p + 2)]$. For this smaller choice of α than the bound given in Theorem (4.1), we have $\nu > \alpha$. The domain $\mathscr{S}(T, \nu)$ is then the sector

$$-(\pi - \nu) \leqq \arg z \leqq \alpha + \pi - \nu .$$

We thus obtain

COROLLARY (4.2). *Let f, an entire function of form* (1.1), *have all, but one of its zeros in the sector*

$$0 \leqq \arg z \leqq \alpha < \pi/(p + 1)(p + 2) .$$

Then at most one zero of f' lies in the sector

$$\pi - (\nu - \alpha) < \arg z < \pi + \nu$$

where $\nu = [\pi - p(p + 1)\alpha]/2(p + 1)$.

Finally, if we suppose that f has a finite number of zeros, we may derive the following.

COROLLARY (4.3). *Let f an entire function of form* (1.1) *have only a finite number of zeros, of which all except one are in the closed interior of a circle C of radius R drawn inside the sector $0 \leqq \arg z \leqq \alpha < \pi/p(p + 1)$. Then f' has at most zero exterior to the concentric circle C' of radius*

$$R' = R \csc (\nu/2) ,$$

where $\nu = [\pi - p(p + 1)\alpha]/2(p + 1)$.

Corollary (4.3) suggests a previous theorem [3] that if k zeros of an n-th degree polynomial P lie in a disk $|z| \leqq a$ where $1 < k \leqq n$, then at least $k - 1$ zeros of the derivative of P lie in the disk $|z| \leqq a \csc [\pi/2(n - k + 1)]$. In fact, Corollary (4.3) is similar to the case $k = n - 1$ of this theorem.

Again we may state other corollaries of interest by taking T as a half-strip, sector or other configuration drawn interior to sector V. We leave the details to the reader.

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