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A NOTE ON FUNCTIONS WHICH OPERATE

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Let $\mathfrak{A}$, $\mathscr{B}$ denote two families of functions $a, b\colon X \to Y$. A function $F\colon Z \subseteqq Y \to Y$ is said to operate in $(\mathfrak{A}, \mathscr{B})$ provided that for each $a \in \mathfrak{A}$ with range $(a) \subseteqq Z$ we have $F(a) \in \mathscr{B}$. Let G denote a locally compact Abelian group. In this paper we characterize the functions which operate in two cases:

(i) $\mathfrak{A} = \Phi_r(G) =$ positive definite functions on G with $\phi(e) = r$ and $\mathscr{B} = \Phi_{i.d.,s}(G) =$ infinitely divisible positive definite functions on G with $\phi(e) = s$.

(ii) $\mathfrak{A} = \mathscr{B} = \tilde{\Phi}_1(G) = \operatorname{Log} \Phi_{i.d.,1}(G)$.

The determination of the class of functions that operate in $(\mathfrak{A}, \mathscr{B})$ for other special families may be found in refernces [3]-[8]. Our goal here is to extend the results of [5, 6] and, at the same time, to obtain a new derivation of the results recently announced in [3].

G will denote a locally compact Abelian group and $B^+(G)$ the family of continuous, complex-valued, nonnegative-definite functions on G. Let

$$\Phi_r(G) = \{\phi : \phi \in B^+(G) \text{ and } \phi(e) = r\}^1$$
$$\Phi_{i.d.,r}(G) = \{\phi : \phi \in \Phi_r(G) \text{ and } (\phi)^{1/n} \in B^+(G) \text{ for } n \geqq 1\}$$
$$\tilde{\Phi}_r(G) = \operatorname{Log} \Phi_{i.d.,r}(G) = \{\log \phi : \phi \in \Phi_{i.d.,r}(G)\}\ .$$

In the case where G is the real line $\Phi_1(G)$ is the class of characteristic functions, $\Phi_{i.d.,1}(G)$ the class of characteristic functions corresponding to the infinitely devisible distributions while $\tilde{\Phi}_1(G)$ is the class of logarithms of this latter class whose form is well known since Levy and Khintchine.

THEOREM 1. *If G has elements of arbitrarily high order then F operates on $(\Phi_r(G), \Phi_{i.d.,s}(G))$ if and only if*

$$F(z) = s \exp c(f(z/r) - 1) \qquad (|z| \leqq r)$$

where $c \geqq 0$ and

$$f(z) = \sum_{n,m=0}^{\infty} a_{n,m} z^n \bar{z}^m \qquad (|z| \leqq 1)$$

with

[1] We denote the identity element of G by e.

$$a_{n,m} \geqq 0 \quad and \quad \sum_{n,m=0}^{\infty} a_{n,m} = 1 \, .$$

LEMMA 1. *Let*

$$h(s, t) = \sum_{n,m=0}^{\infty} b_{n,m} s^n t^m \qquad (|s|, |t| \leqq 1)$$

with

$$b_{n,m} \geqq 0 \quad and \quad \sum_{n,m=0}^{\infty} b_{n,m} = 1 \, .$$

Suppose that for each integer k, $k \geqq 1$ we have

$$(h(s, t))^{1/k} = \sum_{n,m=0}^{\infty} b_{n,m}(k) s^n t^m \qquad (|s|, |t| \leqq 1)$$

with

$$b_{n,m}(k) \geqq 0 \quad and \quad \sum_{n,m=0}^{\infty} b_{n,m}(k) = 1 \, .$$

Then

$$h(s, t) = \exp c(g(s, t) - 1)) \qquad (|s|, |t| \leqq 1)$$

where

$$g(s, t) = \sum_{n,m=0}^{\infty} g_{n,m} s^n t^m \qquad (|s|, |t| \leqq 1)$$

with

$$c \geqq 0 \; g_{n,m} \geqq 0 \quad and \quad \sum_{n,m=0}^{\infty} g_{n,m} = 1 \, .$$

Proof of Lemma 1. Since $(h(s, t))^{1/k}$ is to be a generating function with nonnegative coefficients we must have $h(0, 0) = b_{0,0} > 0$. For suitable $\varepsilon > 0$ we then have

$$0 < 1 - h(s, t) < 1 \qquad (0 \leqq s, t \leqq \varepsilon) \, .$$

Thus $k(s, t) = \log\{1 - (1 - h(s, t))\}$ admits an expansion

$$k(s, t) = \sum_{n,m=0}^{\infty} k_{n,m} s^n t^m \qquad (0 \leqq s, t \leqq \varepsilon) \, .$$

Clearly $k_{0,0} < 0$; we want to prove that all of the remaining coefficients $k_{n,m}$ are nonnegative. Assume on the contrary that

$$\{(n, m) : (n, m) \neq (0, 0) \quad \text{and} \quad k_{n,m} < 0\} \neq \phi \, .$$

Let (n_0, m_0) be a minimal element in this set (under the usual partial ordering in the plane). We then write

$$k(s, t) = k_{0,0} + \sum_{\substack{0 \leqq n \leqq n_0 \\ 0 \leqq m \leqq m_0 \\ (n,m) \neq (0,0),(n_0,m_0)}} k_{n,m} s^n t^m + k_{n_0,m_0} s^{n_0} t^{m_0} + r_{n_0,m_0}(s, t) .$$

It is easily seen that the

coefficient of $s^{n_0}t^{m_0}$ in $\exp \dfrac{1}{N} k(s, t) =$

$$\text{coefficient of } s^{n_0}t^{m_0} \text{ in } \exp \frac{1}{N}\left\{k_{0,0} + \sum_{\substack{0 \leqq n \leqq n_0 \\ 0 \leqq m \leqq m_0 \\ (n,m) \neq (0,0),(n_0,m_0)}} k_{n,m} s^n t^m + k_{n_0,m_0} s^{n_0} t^{m_0}\right\} .$$

But this coefficient is of the form

$$\left\{\frac{1}{N} k_{n_0,m_0} + \frac{1}{N^2}\sigma\left(\frac{1}{N}\right)\right\} \exp \frac{1}{N} k_{0,0}$$

where σ is a polynomial. For N sufficiently large this coefficient has the sign of k_{n_0,m_0} which provides a contradiction. Thus $k_{0,0} < 0$ and $k_{n,m} \geqq 0$ $((n, m) \neq (0, 0))$.

Proof of Theorem 1. By setting $\tilde{F}(z) = (1/s)F(rz)$ we may assume that $r = s = 1$. If F operates in $(\Phi_1(G), \Phi_{i.d.,1}(G))$ then $(F)^{1/k}$ operates in $\Phi_1(G)$ for each integer k, $k \geqq 1$. Thus from [5]

$$(F(z))^{1/k} = \sum_{n,m=0}^{\infty} a_{n,m}(k) z^n \overline{z^m} (|z| \leqq 1)$$

with

$$a_{n,m}(k) \geqq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} a_{n,m}(k) = 1 .$$

By virtue of Lemma 1 the proof is complete.

LEMMA 2. *If G has elements of arbitrarily high order then F operates in $\tilde{\Phi}_1(G)$ implies that for any r, $0 < r < \infty$*

$$F(z) = c(r)\left\{\sum_{n,m=0}^{\infty} a_{n,m}(r)(r + z)^n (r + \bar{z})^m - 1\right\}$$

whenever $|z + r| \leqq r$ where $c(r) \geqq 0$, $a_{n,m}(r) \geqq 0$ and

$$\sum_{n,m=0}^{\infty} a_{n,m}(r) r^{n+m} = 1 .$$

Proof. We begin by observing that

$$\Phi_r(G) - r = \{\phi - r : \phi \in \Phi_r(G)\} \subseteqq \tilde{\Phi}_1(G) .$$

Thus if $F_r(z) = F(z - r)$ then $\exp F_r$ operates in $(\Phi_r(G), \Phi_{i.d.,1}(G))$ which proves the lemma by Theorem 1.

THEOREM 2 [3]. *If G has elements of arbitrarily high order then F operates in $\tilde{\Phi}_1(G)$ if and only if*

$$F(z) = -\alpha + \beta z + \gamma \bar{z} + \int_0^\infty \int_0^\infty \{\exp(sz + t\bar{z}) - 1\}\mu(ds, dt) \qquad (*)$$
$$\operatorname{Re} z \leqq 0$$

where

(i) *α, β and γ are real and nonnegative,*

(ii) *μ is a positive measure on $\{(s, t): 0 \leqq s < \infty, 0 \leqq t < \infty\}$ which is bounded (except perhaps at the origin) and for which*

$$\int_0^\infty \int_0^\infty \frac{t + s}{1 + t + s} \mu(ds, dt) < \infty .$$

Proof. Since it is clear that functions of the form (*) operate on $\tilde{\Phi}_1(G)$ it suffices to prove the reverse implication. We begin by noting that if $0 < r < \rho$ then

$$c(r)\left\{\sum_{n,m=0}^{\infty} a_{n,m}(r)(r + z)^n (r + w)^m - 1\right\}$$
$$= c(\rho)\left\{\sum_{n,m=0}^{\infty} a_{n,m}(\rho)(\rho + z)^n (\rho + w)^m - 1\right\}$$

whenever $|z + r| \leqq r$ and $|w + r| \leqq r$, where F admits the expansion

$$F(z) = c(\rho)\left\{\sum_{n,m=0}^{\infty} a_{n,m}(\rho)(\rho + z)^n (\rho + \bar{z})^m - 1\right\}$$
$$|\rho + z| \leqq \rho .$$

We now may uniquely define a function $\Psi(z, w)$ in $0 \leqq z < \infty$, $0 \leqq w < \infty$ by

$$\Psi(z, w) = c(r)\left\{1 - \sum_{n,m=0}^{\infty} a_{n,m}(r)(r - z)^n (r - w)^m\right\}$$

provided $0 \leqq w \leqq r$ and $0 \leqq z \leqq r$. We note that

$$\frac{(-1)^{j+k-1}\partial^{j+k}}{\partial^j z \partial^k w} \Psi(z, w) \geqq 0$$
$$0 \leqq w < \infty \qquad 0 \leqq z < \infty$$
$$j, k \geqq 0 \qquad j + k > 0 .$$

It follows from a theorem of Bochner [2, p. 89] that

$$\Psi(z, w) = \alpha + \beta z + \gamma w + \int_0^\infty\int_0^\infty [1 - \exp - (sz + tw)]\mu(ds, dt)$$

where α, β, γ and μ have the desired properties.

We proceed now to give the connection between Theorem 2 and the results announced in [3].

DEFINITION. A continuous complex-valued function defined on a locally compact Abelian group G is said to *negative definite* if

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\{f(x_i) + \overline{f(x_j)} - f(x_i x_j^{-1})\}a_i\bar{a}_j \geqq 0$$

for any complex numbers $\{a_i\}$, any $\{x_i\} \subseteqq G$ and for $n = 1, 2, \cdots$. The class of such functions is denoted by $N(G)$. It was already noticed by Beurling and Deny [1] that $N(G) = -\tilde{\Phi}_1(G)$.[2] We include a brief proof for the reader's convenience.

LEMMA 3. *A continuous, complex-valued, function f on G is negative definitely if and only if* $\exp(-f)$ *is the Fourier transform of an infinitely divisible distribution on* $\hat{G}$.

Proof. (*Necessity*) By Bochner's theorem it suffices to show that $\exp(-(1/n)f)$ is a positive definite function on G for $n = 1, 2, \cdots$. Since $(1/n)f$ is a negative definite function it suffices to check that $\exp(-f)$ is positive definite. Now

$$\begin{aligned}&\sum_{j=1}^{n}\sum_{i=1}^{n}\exp(-f(x_i x_j^{-1}))a_i\bar{a}_j\\ &\quad= \sum_{j=1}^{n}\sum_{i=1}^{n}\exp\{f(x_i) + \overline{f(x_j)} - f(x_i x_j^{-1})\}\\ &\qquad\cdot (a_i \exp(-f(x_i)))\overline{(a_j \exp(-f(x_j)))}\ .\end{aligned}$$

But the matrix

$$\exp(f(x_i) + f(x_j) - f(x_i x_j^{-1}))$$

is the limit of positive linear combinations of "element-wise" products of positive definite matrices. Since such products are again positive definite by Schur's theorem [9] we see that $\exp(-f)$ is indeed positive definite.

(*Sufficiency*) By DeFinetti's theorem and the fact that $N(G)$ is closed under pointwise limits it suffices to show that $1 - \phi \in N(G)$ for $\phi \in \Phi_1(G)$. We must therefore show

[2] Professor C. S. Herz has kindly pointed out that this result was actually first given by I. J. Schoenberg [9], albeit in a different context.

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\{1-\phi(x_i)+1-\phi(x_j)-1+\phi(x_i x_j^{-1})\}a_i\bar{a}_j\\ =\sum_{i=1}^{n}\sum_{j=1}^{n}\phi(x_i x_j^{-1})a_i\overline{a_j}+\left|\sum_{i=1}^{n}a_i\right|^2-2\operatorname{Re}\sum_{i=1}^{n}a_i\sum_{j=1}^{n}\overline{a_j\phi(x_j)}\geqq 0\,.\end{aligned}\tag{**}$$

To prove (**) we first set $\phi(x)=\chi(x)$ where χ is a character of G noting that (**) becomes

$$\left|\sum_{i=1}^{n}a_i\chi(x_i)\right|^2+\left|\sum_{i=1}^{n}a_i\right|^2-2\operatorname{Re}\sum_{i=1}^{n}a_i\sum_{i=1}^{n}\overline{a_i\chi(x_i)}\geqq 0\,.$$

For general ϕ we need only observe that by Bochner's theorem ϕ is in the closure of the convex hull spanned by the characters of G.

It is now clear that F operates on $N(G)$ if and only if $\widetilde{F}$, defined by $\widetilde{F}(z)=-F(-z)$, operates on $\widetilde{\Phi}_1(G)$. Making this transformation Theorem 2 becomes identical with the main theorem of [3].

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