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THE APPROXIMATE SOLUTION OF $y' = F(x, y)$

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In general the exact solution of the differential system

$$y' = F(x, y)\,, \qquad y(0) = 0\,,$$

is either unattainable or is impractical to handle, even though a solution is known to exist and may even be obtained in certain cases. Thus some method of approximation is often employed. After the choice of approximating functions has been made, there still remains the questions of goodness of approximation and, if infinite processes are employed, the question of convergence.

The system described above is restricted in this paper to those cases in which $F(x, y)$ is an analytic function of x and y for $-1 \leqq x \leqq 1$ and all y. Then $F(x, y)$ can be written as a convergent power series

$$F(x, y) = \sum_{i,j=0}^{\infty} a_{ij}x^i y^j\,.$$

By considering a sequence of n-th degree polynomials $\{P_n^k(x)\}$ which are ε-approximate solutions of the truncated system

$$L_k(y) \equiv y' - \sum_{i=0,j=1}^{k} a_{ij}x^i y^j = F(x, 0)\,, \qquad y(0) = 0\,,$$

the solution of the original system can be uniformly approximated by polynomials which satisfy $P_n^k(0) = 0$ and which minimize

$$\| F(x, 0) - L_k[P_n^k(x)] \| = \sup_{0 \leqq x \leqq 1} | F(x, 0) - L_k[P_n^k(x)] |\,.$$

1. Introduction. The differential equation to be considered is the equation

$$y' = F(x, y)\,, \tag{1}$$

where $F(x, y)$ is an analytic function of x and y for $-1 \leqq x \leqq 1$ and all y; that is, $F(x, y)$ has the power series expansion

$$F(x, y) = \sum_{i,j=0}^{\infty} a_{ij}x^i y^j$$

which is valid for $-1 \leqq x \leqq 1$ and all y. Along with (1) we apply the initial condition

$$y(0) = 0\,. \tag{2}$$

It is known that (1) possesses a unique solution satisfying the initial condition [1].

We wish to consider the equation (1) in the form

$$L(y) \equiv y' - \sum_{i=0,j=1}^{\infty} a_{ij}x^i y^j = F(x, 0)\,, \tag{3}$$

where $F(x, 0) = \sum_{i=0}^{\infty} a_{i0}x^i$. Let $R_k(x, y) = F(x, y) - F_k(x, y)$, where $F_k(x, y)$ is defined by the equation

$$F_k(x, y) = \sum_{i,j=0}^{k} a_{ij}x^i y^j\ .$$

Associated with (3) we consider the truncated system in either the form

$$L_k(y) \equiv y' - \sum_{i=0,j=1}^{k} a_{ij}x^i y^j = F(x, 0)\,, \qquad y(0) = 0\,, \tag{3.k_1}$$

or

$$y' - F_k(x, y) = R_k(x, 0)\,, \qquad y(0) = 0\,, \tag{3.k_2}$$

where $R_k(x, 0) = \sum_{i=k+1}^{\infty} a_{i0}x^i$. Note that $R_k(x, 0)$ is not the same as $F(x, 0)$. However, since (3.k_1) and (3.k_2) are the same equations, a solution of one is a solution of the other. Observe also that the conditions assumed for (1) are sufficient to insure solutions for (3.k_1) or (3.k_2).

We wish to consider the *best approximate solution* of (1) by the use of polynomials $P_n^k(x)$ as approximating functions in the sense that

$$\| F(x, 0) - L_k[P_n^k(x)] \| = \sup_{0 \leq x \leq 1} | F(x, 0) - L_k[P_n^k(x)] | \tag{4}$$

is a minimum for each fixed n and k. Here the superscript k in $P_n^k(x)$ represents the k of (3.k_1), and n is the degree of $P_n^k(x)$. That is, we consider the conditions under which a sequence of minimizing polynomials of (4) will converge uniformly to the unique solution y of (1).

2. Preliminary results. In a paper to appear in the Proceedings [3], the authors consider the problem of *best approximation* of the solution of the equation

$$N(y) \equiv y' - \sum_{k=1}^{m} f_k(x) y^k = R(x)\,, \qquad x \in [0, 1]\,, \tag{5}$$

satisfying $y(0) = 0$, by the use of polynomials $P_n(x)$ as approximating functions. The approximation there is a *best approximation* in the sense that the approximating polynomial is required to minimize

$$\| R(x) - N[P_n(x)] \| = \sup_{0 \leq x \leq 1} | R(x) - N[P_n(x)] |\ . \tag{6}$$

From [3] we have the following theorem:

THEOREM 1. *If $y(x)$ is the unique solution of* (5) *which satisfies the initial condition $y(0) = 0$ and which has a continuous first derivative, and if $P_n(x)$ is a polynomial which minimizes* (6) *and is such that $P_n(0) = 0$, then $P_n(x)$ and $P_n'(x)$ converge uniformly throughout* $[0, 1]$ *to $y(x)$ and $y'(x)$ respectively as n increases without bound.*

If we return now to equation $(3.k_1)$, we see that the conditions of Theorem 1 are satisfied, and we know that for each fixed k there exists a sequence of polynomials $P_n^k(x)$ which satisfy $P_n^k(0) = 0$, which minimize (4) for that fixed k, and which uniformly approximate the solution of $(3.k_1)$.

The following definition will be useful in our work to follow [1]. Let $f(x, y)$ be a real-valued continuous function on a domain D in the (x, y)-plane, and consider the equation

$$y' = f(x, y) . \tag{7}$$

An *ε-approximate solution* of (7) on an x interval I is a function $\Phi \in C$ on I such that

(i) $(x, \Phi(x)) \in D,\ x \in I$,

(ii) $\Phi \in C^1$ on I, except possibly for a finite set of points S on I, where Φ' may have simple discontinuities, and

(iii) $|\Phi'(x) - f(x, \Phi(x))| \leqq \varepsilon,\ X \in I - S$.

Denote the set of all ε-approximate solutions of $(3.k_2)$ by S_ε^k. We know that given any $\varepsilon > 0$, there exists a $K(\varepsilon)$ such that

$$\left| \sum_{i,j=k+1}^{\infty} a_{ij} x^i y^j \right| < \varepsilon \tag{8}$$

if $k \geqq K(\varepsilon)$, since the series is convergent, an observation we shall need in the following sequence of lemmas which lead to the proof of the principal result of this paper.

LEMMA 1. *If $y(x)$ is any solution of* (1), *then $y(x)$ is in S_ε^k for all k sufficiently large, say $k \geqq K(\varepsilon)$.*

Proof. If $y(x)$ is a solution of (1), then

$$y' = F(x, y) = F_k(x, y) + R_k(x, y) .$$

Given any $\varepsilon > 0$, then for sufficiently large k, say $k \geqq K(\varepsilon)$, we have that $|R_k(x, y)| \leqq \varepsilon/2$ (and hence $|R_k(x, 0)| \leqq \varepsilon/2$) by (8). Therefore

$$\begin{aligned} |y' - F_k(x, y) - R_k(x, 0)| &= |F(x, y) - F_k(x, y) - R_k(x, 0)| \\ &\leqq |R_k(x, y)| + |R_k(x, 0)| \leqq \varepsilon/2 + \varepsilon/2 = \varepsilon . \end{aligned}$$

But this is precisely the statement of the fact that $y(x)$ is in S_ε^k for

$k \geqq K(\varepsilon)$.

LEMMA 2. *If P_n^k is a minimizing polynomial of* (4), *and if the conditions of Theorem* 1 *are satisfied, then for n sufficiently large, $P_n^k(x)$ is an ε-approximate solution of* $(3.k_2)$.

Proof. Since $F_k(x, y) = \sum_{i,j=0}^{k} a_{ij}x^i y^j$, then $F_k(x, y)$ is a continuous function of y. Thus, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|F_k(x, y_k) - F_k(x, P_n^k)| \leqq \varepsilon/2 ,$$

provided that $|y_k - P_n^k| < \delta$, where y_k is the solution of the k-th truncated equation $(3.k_2)$. Since the conditions of Theorem 1 are satisfied,

$$|y_k - P_n^k| < \delta , \quad \text{for} \quad x \in [0, 1] ,$$

for n sufficiently large, say $n > N_1(\varepsilon, k)$. Likewise, by Theorem 1, for $n > N_2(\varepsilon, k)$ we have that

$$|y_k' - (P_n^k)'| \leqq \varepsilon/2 , \quad \text{for} \quad x \in [0, 1] .$$

If we choose $N = \max(N_1, N_2)$, then

$$|F_k(x, y_k) - F_k(x, P_n^k)| \leqq \varepsilon/2 \quad \text{and} \quad |y_k' - (P_n^k)'| \leqq \varepsilon/2$$

for all $x \in [0, 1]$ and for $n \geqq N$. Thus

$$\begin{aligned} &|(P_n^k)' - F_k(x, P_n^k) - R_k(x, 0)| \\ &\quad \leqq |(P_n^k)' - F_k(x, y_k) - R_k(x, 0)| + |F_k(x, y_k) - F_k(x, P_n^k)| \\ &\quad \leqq |(P_n^k)' - y_k'| + |F_k(x, y_k) - F_k(x, P_n^k)| \leqq \varepsilon/2 + \varepsilon/2 = \varepsilon . \end{aligned}$$

Thus $P_n^k(x)$ is an ε-approximate solution of $(3.k_2)$ for n sufficiently large.

LEMMA 3. *If $P_n^k(x)$ is an $\varepsilon/3$ approximate solution of* $(3.k_2)$, *then $P_n^k(x)$ is an ε-approximate solution of* (1), *provided that $k \geqq K(\varepsilon)$.*

Proof. Since $P_n^k(x)$ is an $\varepsilon/3$ approximate solution of $(3.k_2)$, we have that

$$\begin{aligned} &|(P_n^k)' - F(x, P_n^k)| \\ &\quad \leqq |(P_n^k)' - F_k(x, P_n^k) - R_k(x, 0)| + |F_k(x, P_n^k) - F(x, P_n^k)| \\ &\qquad + |R_k(x, 0)| \\ &\quad = |(P_n^k)' - F_k(x, P_n^k) - R_k(x, 0)| + |R_k(x, P_n^k)| + |R_k(x, 0)| \\ &\quad \leqq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon , \end{aligned}$$

since for k sufficiently large, say $k \geqq K(\varepsilon)$,

$$\left| \sum_{i,j=k+1}^{\infty} a_{ij} x^i (P_n^k)^j \right| \leqq \varepsilon/3 \text{ and } \left| \sum_{i=k+1}^{\infty} a_{i0} x^i \right| \leqq \varepsilon/3 .$$

LEMMA 4. *If $P_n^k(x)$ is an $\varepsilon/5$ approximate solution of $(3.k_2)$ where $k \geqq K(\varepsilon)$, then for any j such that $k \geqq j \geqq K(\varepsilon)$, $P_n^k(x)$ is in the class S_ε^j.*

Proof. Since $P_n^k(x)$ is an $\varepsilon/5$ approximate solution of $(3.k_2)$ we have that

$$|(P_n^k)' - F_k(x, P_n^k) - R_k(x, 0)| \leqq \varepsilon/5 .$$

Also $|R_k(x, P_n^k)| \leqq \varepsilon/5$ and $|R_j(x, P_n^k)| \leqq \varepsilon/5$ by hypothesis. Thus by adding and subtracting $F_k(x, P_n^k)$ and $R_k(x, 0)$ we can write

$$\begin{aligned} &|(P_n^k)' - F_j(x, P_n^k) - R_j(x, 0)| \\ &\quad \leqq |(P_n^k)' - F_k(x, P_n^k) - R_k(x, 0)| \\ &\quad\quad + |F_k(x, P_n^k) - F_j(x, P_n^k) - R_j(x, 0) + R_k(x, 0)| \\ &\quad \leqq |(P_n^k)' - F_k(x, P_n^k) - R_k(x, 0)| + |R_k(x, P_n^k)| + |R_j(x, P_n^k)| \\ &\quad\quad + |R_j(x, 0)| + |R_k(x, 0)| \\ &\quad \leqq \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon , \end{aligned}$$

where use has been made of the fact that

$$|F_k(x, P_n^k) - F_j(x, P_n^k)| = |R_k(x, P_n^k) - R_j(x, P_n^k)| .$$

Hence, P_n^k is an ε-approximate solution of $(3.k_2)$ when $k_2 = j$ and thus belongs to the class S_ε^j.

It is useful to note that any solution of a given differential equation is an ε-approximate solution of that equation for all positive ε, and also if $y(x)$ is an ε-approximate solution of a given differential equation for all $\varepsilon > 0$, then $y(x)$ is a solution of that given differential equation.

3. The approximate solution of the differential equation. We now consider the approximate solution of (1) by a sequence of polynomials $P_n^k(x)$ which minimize (4). Let $\{\varepsilon_k\}$ be a monotone null sequence of positive numbers with $\varepsilon_1 < 1$. For each ε_k there exists a a $K_k(\varepsilon_k)$ such that whenever $h \geqq K_k$,

$$\left| \sum_{i,j=h+1}^{\infty} a_{ij} x^i y^j \right| \leqq \varepsilon_k/3 .$$

From Lemma 3 an $\varepsilon/3$ approximate solution $P_n^k(x)$ of $(3.k_2)$ is an ε-approximate solution of (1) provided that $k \geqq K_k$. Then for ε_1 there exists a K_1 such that

$$P_{n_1}^{K_1},\ P_{n_2}^{K_1+1},\ \cdots,\ P_{n_{j+1}}^{K_1+j},\ \cdots$$

are ε_1-approximate solutions of (1). Here we are denoting by $P_{n_{j+1}}^{K_1+j}$ any $\varepsilon_1/3$ approximate solution of $(3.k_2)$ when $k_2 = K_1 + j$. We also note that $P_{n_{j+1}}^{K_1+j}(0) = 0$; i.e., the polynomials satisfy the boundary condition for all $j = 0, 1, \cdots$.

For ε_2 there exists a K_2' such that

$$P_{n_1}^{K_2'},\ P_{n_2}^{K_2'+1},\ \cdots,\ P_{n_{j+1}}^{K_2'+j},\ \cdots$$

are ε_2-approximate solutions of (1). Choosing $K_2 = \max(K_1, K_2')$ leads to the sequence $P_{n_1}^{K_2}, P_{n_2}^{K_2+1}, \cdots, P_{n_{j+1}}^{K_2+j}, \cdots$, where $P_{n_{j+1}}^{K_2+j}$ is an $\varepsilon_2/3$ approximate solution of $(3.k_2)$ when $k_2 = K_2 + j$, and for $j = 0, 1, \cdots$, and $P_{n_{j+1}}^{K_2+j}(0) = 0$. It should be noted that $P_{n_{j+1}}^{K_2+j}$ is an ε_1- and ε_2-approximate solution of (1).

Define inductively the sequence of solutions

$$P_{n_1}^{K_k},\ P_{n_2}^{K_k+1},\ \cdots,\ P_{n_{j+1}}^{K_k+j},\ \cdots,$$

where $P_{n_{j+1}}^{K_k}$ is an $\varepsilon_k/3$ approximate solution of $(3.k_2)$ for $k_2 = K_k + j$ having the property that $P_{n_{j+1}}^{K_k+j}(0) = 0, j = 0, 1, \cdots$. Again $P_{n_{j+1}}^{K_k+j}$ is an ε_k-approximate solution of (1).

Consider the sequence

$$P_{n_1}^{K_1},\ P_{n_2}^{K_2+1},\ P_{n_3}^{K_3+2},\ \cdots,\ P_{n_j}^{K_j+(j-1)},\ \cdots\ .$$

Denote this sequence by $\{Y_i\}$. We observe that each Y_i is at least an ε_i-approximate solution of (1), $i = 1, 2, \cdots$.

At this point we need a theorem given by Coddington and Levinson [1].

THEOREM 2. *Suppose that $f(x, y)$ is continuous in some domain D of the (x, y)-plane and further that $f(x, y)$ satisfies a Lipschitz condition with respect to y in D with Lipschitz constant K. Let Φ_1 and Φ_2 be ε_1- and ε_2-approximate solutions of (7) of class C^1, at least piecewise on $[a, b]$, satisfying*

$$|\Phi_1(x_0) - \Phi_2(x_0)| \leqq \delta$$

for some x_0 such that $a \leqq x_0 \leqq b$, where δ is a nonnegative constant. If $\varepsilon = \varepsilon_1 + \varepsilon_2$, then for all $x \in [a, b]$

$$|\Phi_1(x) - \Phi_2(x)| \leqq \delta e^{K|x-x_0|} + \frac{\varepsilon}{K}(e^{K|x-x_0|} - 1)\ .$$

By a theorem given by Coppel [2] and stated in [3], we know that the solutions y_k of $(3.k_2)$ are uniformly bounded on $[0, 1]$. Hence the polynomials of the set $\{Y_i\}$ are uniformly bounded, and there exists a uniform Lipschitz constant K for (1) for x in $[0, 1]$.

Hence if i^* and j^* are chosen so large that ε_{i^*} and ε_{j^*} are both less than $\varepsilon/2C$ for arbitrary small $\varepsilon > 0$, where $C = (|e^K - 1|)/K$, then

$$\begin{aligned} | Y_{i^*}(x) - Y_{j^*}(x) | &\leqq 0 \cdot e^{K|x-0|} + \frac{\varepsilon_{i^*} + \varepsilon_{j^*}}{K}(e^{K|x-0|} - 1) \\ &< \frac{\varepsilon}{C}\left(\frac{|e^K - 1|}{K}\right) = \varepsilon , \qquad x \in [0, 1] . \end{aligned}$$

Thus, the sequence $\{Y_i(x)\}$ is a Cauchy sequence and hence converges uniformly on $[0, 1]$. Therefore there exists a continuous limit function $Y(x)$ on this interval such that $Y_i(x) \to Y(x)$ as $i \to \infty$ uniformly on $[0, 1]$. Recalling the fact that each member of $\{Y_i\}$ is at least an ε_i-approximate solution of (1), it is clear that $Y(x)$ is a solution of (1) satisfying $Y(0) = 0$, since it is an ε-approximate solution of (1) for all $\varepsilon > 0$. Applying the uniqueness of solutions of (1), it is necessary that $Y(x) = y(x)$.

Thus we have shown the existence of a sequence of polynomials, $\{Y_i(x)\}$, such that $Y_i(0) = 0$ and each $Y_i(x)$ is a minimizing polynomial of (4). Furthermore, this sequence converges uniformly to the solution of (1) as i increases without bound. We state these results in the following theorem.

THEOREM 3. *If $y(x)$ is the unique solution of* (1) *having the value $y(0) = 0$, then $y(x)$ may be uniformly approximated on* $[0, 1]$ *by a sequence of polynomials $\{P_n^k(x)\}$ which are ε-approximate solutions of* $(3.k_2)$ *which satisfy $P_n^k(0) = 0$ and which minimize condition* (4).

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