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EXTENSIONS OF REGULAR BOREL MEASURES

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This paper is concerned with the extension of regular Borel measures defined on the Borel sets generated by sub-topologies of compact Hausdorff topologies.

Specifically, if X is a nonempty set and τ is a topology on X, the Borel sets of (X, τ) are the members of the smallest σ-ring containing τ. A regular Borel measure is taken to mean a finite-valued measure μ on the Borel sets of (X, τ) with property that

$$\mu(B) = \sup\{\mu(F) \mid F \subseteqq B, F \text{ is closed}\}\ .$$

In this paper, the situation considered is the following: τ is a compact Hausdorff topology on X, and σ is a regular (in the topological sense) sub-topology of τ. The space $C(X, \tau)$ is the (partially ordered) Banach space of all continuous real-valued functions on (X, σ), in the supremum norm. The space $C(X, \sigma)$ is similarly defined. By constructing a one-to-one correspondence between the collection of regular Borel measures on (X, σ) and the collection of positive linear functionals on $C(X, \sigma)$ it is shown that every regular Borel measure on (X, σ) can be extended to a regular Borel measure on (X, τ). This result is used to prove the existence of nonatomic regular Borel measures on compact Hausdorff spaces with perfect sub-sets.

The concept of a "partition space" plays a central role in this development.

DEFINITION 1. Let X be a topological space. A topological space Y is said to be a *partition space of* X if there is an onto function $f: X \rightarrow Y$ such that the topology for X is the smallest topology for which f is continuous.

A partition space is a special kind of quotient space [6: 94]. Every topological space is a partition space of itself, and a partition space of a compact space is compact. It will be important to know when a topological space has a Hausdorff partition space. For a similar result about quotient spaces see [6: 98]. A proof is given here because the notation of the proof is used later on.

THEOREM 2. *A topological space X has a Hausdorff partition space if and only if, for any two points x and y in X, if there is an open set U such that $x \in U$ and $y \notin U$, then there are disjoint neighborhoods of x and y.*

Proof. Let Y be a Hausdorff partition space of X and let $f: X \rightarrow Y$ be the function in Definition 1. Suppose x and y are two points of X, and U is an open set such that $x \in U$ and $y \notin U$. There is an open set E in Y such that $U = f^{-1}(E)$. Then $f(x) \in E$ and $f(y) \notin E$. Thus $f(x) \neq f(y)$, and there are disjoint neighborhoods E_1 of $f(x)$ and E_2 of $f(y)$. Then $f^{-1}(E_1)$ and $f^{-1}(E_2)$ are disjoint neighborhoods of x and y.

Conversely, for each $x \in X$, let N_X be the set all elements $y \in X$ such that, for each open set U, $y \in U$ if and only if $x \in U$.

Let $Y = \{N_X : x \in X\}$. Define a function $f: X \rightarrow Y$ by $x \rightarrow N_X$ for every $x \in X$, and give Y the largest topology for which f is continuous (that is, a subset $B \subset Y$ is open if and only if $f^{-1}(B)$ is open). Then Y is a Hausdorff partition space of X, because f is certainly continuous, and if U is open in X, then $U = \bigcup \{N_X : x \in U\}$ implies $f^{-1}(N_X : x \in U) = U$.

COROLLARY 3. *Every regular topological space has a Hausdorff partition space. In particular, every compact regular space has a compact Hausdorff partition space.*

From now on, the "partition space" of a topological space X will mean the partition space $Y = \{N_X : x \in X\}$ defined in the proof of Theorem 2.

The proof of the following theorem is straight-forward computation and hence will be ommited.

THEOREM 4. *Let Y be the partition space of a topological space X, and B_X and B_Y be the classes of Borel sets in X and Y, respectively. Then*

$$B_X = \{f^{-1}(E) : E \in B_Y\} .$$

If μ and ν are real-valued functions on B_X and B_Y such that $\mu(f^{-1}(E)) = \nu(E)$ for every $E \in B_Y$, then μ is a regular Borel measure on X if and only if ν is a regular Borel measure on Y.

COROLLARY 5. *In the notation of Theorem 4, to every regular Borel measure μ on X assign a unique regular Borel measure ν_μ on Y by means of the formula*

$$\nu_\mu(E) = \mu(f^{-1}(E)), (E \in B_Y) .$$

Then the mapping $\mu \rightarrow \nu_\mu$ is a one-to-one correspondence between the collection of regular Borel measures on X and the collection of regular Borel measures on Y.

THEOREM 6. *Let Y be the partition space of a topological space*

X, and $C(X)$ and $C(Y)$ be the spaces of continuous real-valued functions on X and Y, respectively. Then

$$C(X) = \{g \circ f : g \in C(Y)\} .$$

If $I(g \circ f) = J(g)$ for every $g \in C(Y)$, then I is a positive linear functional on $C(X)$ if and only if J is a positive linear functional on $C(Y)$.

Proof. Clearly $\{g \circ f : g \in C(Y)\} \subset C(X)$. On the other hand, take $h \in C(X)$. If N_X is a fixed point of Y and $y_1, y_2 \in N_X$, then $h(y_1) = h(y_2)$(otherwise, there are disjoint neighborhoods E_1 and E_2 of $h(y_1)$ and $h(y_2)$, and $h^{-1}(E_1)$ and $h^{-1}(E_2)$ would be disjoint neighborhoods of y_1 and y_2). Thus $g(N_x) = h(x)$ defines a real-valued function g on Y. Clearly $g \in C(Y)$ and $g \circ f = h$. This shows $C(X) = \{g \circ f : g \in C(Y)\}$.

The second part is immediate.

COROLLARY 7. *In the notation of Theorem 6, to every positive linear functional I on $C(X)$ assign a unique positive linear functional J_I on $C(Y)$ by means of the formula*

$$J_I(g) = I(g \circ f), \ (g \in C(Y)) .$$

Then the mapping $I \longrightarrow J_I$ is a one-to-one correspondence between the collection of positive linear functionals on $C(X)$ and the collection of positive linear functionals on $C(Y)$.

Let X be a compact regular space. To every regular Borel measure μ on X assign a unique positive linear functional I_μ on $C(X)$ as follows. If Y is the partition space of X, a regular Borel measure μ on X gives rise (by Corollary 5) to a regular Borel measure ν on Y. Then the formula

$$J_\nu(g) = \int_Y g d\nu \qquad (g \in C(Y))$$

defines a positive linear functional J_ν on $C(Y)$ which (by Corollary 7) defines a positive linear functional I_μ on $C(X)$.

THEOREM 8. *For a compact regular space X, the mapping $\mu \longrightarrow I_\mu$ is a one-to-one correspondence between the collection of regular Borel measures on X and the collection of positive linear functionals on $C(X)$.*

Proof. The Riesz Representation Theorem for compact Hausdorff spaces [4: 177–178] shows that the mapping $\nu \longrightarrow J_\nu$ is a one-to-one correspondence between the collection of regular Borel measures on Y and the collection of positive linear functionals on $C(Y)$. Then Corollaries 5 and 7 complete the proof.

The next theorem (which generalizes a proof in [9]) is the main result of this paper.

THEOREM 9. *Let τ be a compact Hausdorff topology for a set X, and let σ be a regular topology for X such that $\sigma \subset \tau$. Then every regular Borel measure on (X, σ) can be extended to a regular Borel measure on (X, τ).*

Proof. Let μ be a regular Borel measure on (X, σ) and I be the positive linear functional on (X, σ) corresponding to u by the mapping in Theorem 8. By [8, p. 18], I can be extended to a positive linear functional I^* on $C(X, \tau)$. Let μ^* be the regular Borel measure on (X, τ) such that

$$I^*(g) = \int_X g d\mu^*$$

for all $g \in C(X, \tau)$. It is shown that μ^* extends μ.

Let Y be the partition space of (X, σ), and ν be the regular Borel measure on Y defined by $\nu(E) = \mu(f^{-1}(E))$ for every Borel set E in Y.

Let J be the positive linear functional on $C(Y)$ corresponding to ν. Let U be a member of σ. Then there is an open set V in Y such that $f^{-1}(V) = U$. Now, $\mu(U) = \nu(V) =$

$$\begin{aligned} &\sup \{J(h) \mid h \in C(Y), 0 \leqq h \leqq 1, h(y) = 0 \quad \text{if } y \notin V\} \\ &= \sup \{I(k) \mid k \in C(X, \sigma), 0 \geqq k \leqq 1, k(x) = 0 \quad \text{if } x \notin U\} \\ &\leqq \sup \{I^*(k) \mid k \in C(X, \tau), 0 \leqq k \leqq 1, k(x) = 0 \quad \text{if } x \notin U\} \\ &= \mu^*(U) . \end{aligned}$$

To show the reverse inequality, let $\varepsilon > 0$ and let K be a closed set in (X, τ) such that $K \subset U$ and $\mu^*(U) < \varepsilon + \mu^*(K)$. Since (X, σ) is regular, for each x in k, there is a set $V(x)$ such that $V(x)$ is closed in (X, σ) and $x \in V(x) \subset U$. Since a compact regular space in normal [6: 141], for each $x \in K$, there is a $g_x \in C(X, \sigma)$ such that $CH_{V(x)} \leqq g_x \leqq CH_U$ (CH is the characteristic function). Let $U(x) = \{y \in X \mid g_x(y) + \varepsilon > 1\}$. Then $\{U(x) \mid x \in K\}$ is a family of open subsets $q(X, \tau)$ which covers K, and there are $X_1, \cdots, X_n$ in K such that $\{U(X_i) \mid i = 1, \cdots, n\}$ covers K. Let $g = \max \{g_{x_i} \mid i = 1, \cdots, n\}$. Then $g \in C(X, \sigma)$ and $\mu^*(U) < \varepsilon + \mu^*(K) = \varepsilon + \int_X CH_K d\mu^* \leqq \varepsilon + \int_X (g + \varepsilon) d\mu^* = \varepsilon + \varepsilon\mu^*(X) + I^*(g) \leqq 2\varepsilon + \mu(U)$ (since $0 \leqq g \leqq CH_U$). Thus $\mu^*(U) =$

[1] We wish to thank the referee for pointing out a simplification of the proof.

$\mu(U)$ and by the regularity of μ, μ^* extends μ.

It is now shown how this can be applied to relationships between measures on X and Y and mappings from X and Y. In particular, it is shown that if X, Y are compact Hausdorff spaces and $f: X \to Y$ is a continuous onto map, then each regular Borel measure on Y generates a regular Borel measure on X.

THEOREM 10. *Let X and Y be a compact Hausdorff spaces, f be a continuous function from X onto Y, and ν be a regular Borel measure on Y. Then there is a regular Borel measure μ on X such that*

$$\mu(f^{-1}(E)) = \nu(E)$$

for every Borel set E in Y. Moreover, if ν is nonatomic, then so is μ.

Proof. Let τ denote the topology for X, and let

$$\sigma = \{f^{-1}(U) : U \text{ open in } Y\} .$$

Then σ is a regular topology for X, $\sigma \subset \tau$, $C(X, \sigma)$ is a linear subspace of $C(X, \tau)$, and $C(X, \sigma)$ contains the constant functions. Thus Theorem 9 implies that every regular Borel measure on (X, σ) can be extended to a regular Borel measure on (X, τ).

For every Borel set E in Y, define $\mu_0(f^{-1}(E)) = \nu(E)$. The proof of Theorem 4 shows that μ_0 is a regular Borel measure on (X, σ). Let μ be a regular Borel measure on (X, τ) which extends μ_0. Then $\mu(f^{-1}(E)) = \nu(E)$ for every Borel set E in Y. If ν is nonatomic, then, for every $x \in X$,

$$0 \leqq \mu(x) \leqq \mu(f^{-1}f(x)) = \nu(f(x)) = 0$$

implies $\mu(x) = 0$, and thus μ is nonatomic.

COROLLARY 11. [9]. *If X is a compact Hausdorff space with a nonempty perfect set, then there is a nonzero, nonatomic regular Borel measure on X.*

Proof. There is a continuous map of X onto $[0, 1]$. Thus, in Theorem 10 one can use Lebesgue measure for ν.

COROLLARY 12. *Let X be a compact Haudorff space, $\{P_n\}$ be a disjoint sequence of perfect subsets of X, and $\{a_n\}$ be a sequence of nonnegative real numbers such that $\sum a_n < \infty$. Then there is a nonatomic regular Borel measure μ on X such that $\mu(P_n) = a_n$ for every n.*

Proof. For each n, let ν_n be a nonatomic regular Borel measure on P_n such that $\nu_n(P_n) = a_n$, and define $\mu_n(E) = \nu_n(E \cap P_n)$ for every Borel set E in X. Then $\mu = \sum \mu_n$ is a nonatomic regular Borel measure on X, and $\mu(P_n) = a_n$ for every n.

For the last theorem some additional terminology is needed. Let X and Y be compact Hausdorff spaces. By $M(X)$ is meant the Banach lattice of all regular Borel measures on X under the total variation norm. Of course, $M(X)$ is precisely the Banach space dual of $C(X)$. If ν is a regular Borel measure on Y, by $L^1(\nu)$ is meant the Banach lattice of all ν-integrable functions on Y, under the integral norm.

THEOREM 13. *If there is a continuous map f of X onto Y, then $L^1(\nu)$ is linearly isometric to a subspace of $M(X)$.*

Proof. Let μ be the regular Borel measure associated with ν of Theorem 11. Let N be the normed linear space whose elements are the continuous functions on Y, but whose norm is the integral norm with respect to ν. Then N is dense in $L^1(\nu)$. Define the linear operator $A: N \to M(X)$ by $(Ag)(h) = \int_X h(g \circ f)\, d\mu$ for all $g \in N$, $h \in C(X)$. Now, $\| Ag \| = \int_X | g \circ f |\, d\mu = \| g \|$ and A is an isometry of N into $M(X)$. Since N is dense in $L^1(\nu)$, A can be uniquely extended to an isometry of $L^1(\nu)$ into $M(X)$.

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