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CONVOLUTION OPERATORS ON $L^p(G)$ AND PROPERTIES OF LOCALLY COMPACT GROUPS

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A locally compact group G is said to have property (R) if every continuous positive-definite function on G can be approximated uniformly on compact sets by functions of the form $s * \tilde{s}$, $s \in \mathscr{K}(G)$. When μ is a bounded, regular, Borel measure on G, the convolution operator T_μ defined by

$$(T_\mu)(s) = (\mu * s)(x) = \int_G s(y^{-1}x) d\mu(y) , \qquad s \in \mathscr{K}(G) ,$$

can be extended to a bounded operator on $L^p(G)$ whose norm satisfies $\| T_\mu \|_p \leqq \| \mu \|$. In this paper three characterizations of property (R) are given in terms of the norm $\| T_\mu \|_p$, $1 < p < \infty$, for specific operators T_μ. From these characterizations some closely-related, but seemingly weaker properties than (R), are shown to be equivalent to (R). Examples illustrating the results are given also.

If dx denotes left-invariant Haar measure on G and $\mathscr{K}(G)$ the space of continuous, complex-valued functions with compact support on G, the Haar modulus Δ is defined by

$$\int_G s(xa^{-1})dx = \Delta(a)\int_G s(x)dx , \qquad s \in \mathscr{K}(G) .$$

The Haar measure of a set $A \subset G$ is written $m(A)$. The norms on the measure algebra $M(G)$ and on the spaces $L^p(G)$, $1 \leqq p \leqq \infty$, defined with respect to the given Haar measure, will be denoted by $\|(\,.\,)\|$, $\|(\,.\,)\|_p$ respectively. For any space $\mathscr{D}(G)$ of functions or measures on G, the nonnegative elements in $\mathscr{D}(G)$ will be specified by $\mathscr{D}^+(G)$. We set $\tilde{s}(x) = \overline{s(x^{-1})}$, $s(x) = \overline{s(x^{-1})}\Delta(x^{-1})$ when $s \in \mathscr{K}(G)$ and $\mu^*(x) = \overline{\mu(x^{-1})}$ when $\mu \in M(G)$. Since $\mu \rightarrow \mu^*$ is an involution on $M(G)$, a measure μ is called hermitian if $\mu = \mu^*$. Following Godement ([8], see also Dixmier [5] § 13) we say that a measure $\mu \in M(G)$ is of positive type if

$$(1) \qquad \mu(s * \tilde{s}) = \int_G \Big(\int_G \overline{s(x^{-1}y)} s(y) dy\Big) d\mu(x) \geqq 0 ,$$

for all $s \in \mathscr{K}(G)$. When $(\,.\,,\,.\,)$ denotes the usual inner product on $L^2(G)$, inequality (1) can be rewritten as

$$(\mu * s, s) \geqq 0 , \qquad s \in \mathscr{K}(G) ,$$

changing s to $\tilde{s}$, i.e., μ is a positive element in the operator algebra of G. A continuous function ϕ is said to be positive-definite if

$$\phi(s^* * s) = \int_G \int_G \phi(y^{-1}x)\overline{s(y)}s(x)dydx \geqq 0 ,$$

for $s \in \mathscr{K}(G)$, i.e., ϕ is a positive functional on the involutive algebra $L^1(G)$, ([5] p. 256). Note that $s * \tilde{s}$ is positive definite; consequently $s * \tilde{s}(x^{-1}) = s * \tilde{s}(x)$, $|s * \tilde{s}| \leqq s * \tilde{s}(e)$.

The following trivial lemma will be useful.

LEMMA 1. *Let μ be a hermitian measure in $M_+(G)$. Then*

$$\| T_\mu \|_2 = \sup \mu(s * \tilde{s}) , \tag{2}$$

when the supremum is taken over all $s \in \mathscr{K}_+(G)$, $\| s \|_2 = 1$.

Proof. Certainly $\| T_\mu \|_2 = \sup | \mu(\sigma * \tilde{\sigma}) |$, $\sigma \in \mathscr{K}(G)$, $\| \sigma \|_2 = 1$. Set $s = | \sigma |$. Then $\| s \|_2 = 1$, $| \sigma * \tilde{\sigma} | \leqq s * \tilde{s}$ and

$$| \mu(\sigma * \tilde{\sigma}) | \leqq \int_G | \sigma * \tilde{\sigma} | d\mu \leqq \int_G s * \tilde{s} d\mu = \mu(s * \tilde{s}) ,$$

consequently, (2) holds.

2. In this section we give the principal characterizations of property (R). To every regular Borel measure μ on G there corresponds a convolution operator T_μ defined by

$$(T_\mu)(s) = (\mu * s)(x) = \int_G s(y^{-1}x)d\mu(y) , \qquad s \in \mathscr{K}(G) .$$

If T_μ can be extended to a bounded operator on $L^p(G)$ we say that μ is p-admissible (cf. Leptin [14]); in particular, every bounded measure μ in $M(G)$ is p-admissible and, in this case, the operator norm $\| T_\mu \|_p$ satisfies $\| T_\mu \|_p \leqq \| \mu \|$. Previously, Dieudonne ([3], [4]), Hulanicki ([9]) have shown that there is an interesting relationship between property (R) (or properties equivalent to (R)) and the convolution operators T_μ, $\mu \in M(G)$. On the other hand, if every positive p-admissible measure is necessarily a bounded measure, G is said to be a *K_p-group* (Leptin [14] p. 111).

THEOREM A. *For any $p, 1 < p < \infty$, the following assertions are equivalent;*

(i) *G has property (R),*

(ii) *$\| T_\mu \|_p = \| \mu \|$ for every $\mu \in M_+(G)$,*

(iii) *G is a K_p-group.*

REMARKS. (a) For unimodular groups a result weaker than the equivalence of (i), (ii) has been given by Hulanicki (see [9] Ths. 5.2, 5.3, 5.4). However, in view of the apparent inaccuracies in [9], (cf. remarks [10] p. 99) we shall give an entirely different proof.

(b) The equivalence of (i), (iii) answers negatively a question raised by Leptin ([14] p. 111) concerning the existence of unbounded positive p-admissible measures[1]. The results of Kunze-Stein ([13] p. 52) show that there are positive unbounded p-admissible measures on $SL(R, 2)$.

Proof of Theorem A. (i) $\Rightarrow$ (ii). By convexity it is enough to prove that $\| T_\mu \|_2 = \| \mu \|$ for all $\mu \in M_+(G)$ since $\| T_\mu \|_1 = \| \mu \| = \| T_\mu \|_\infty$ always holds (cf. Wendel [20], Dieudonné [3] p. 284). It is even enough to establish equality when μ has compact support say K. Since G has property (R), for each $\varepsilon > 0$, there exists $s \in \mathscr{K}(G)$ such that

$$\sup_{y \in K} | 1 - (s * \tilde{s})(y) | < \varepsilon , \qquad \| s \|_2 = 1 .$$

Hence

$$| \| \mu \| - | \mu(s * \tilde{s}) | | \leqq \int_K | 1 - s * \tilde{s} | \, d\mu < \varepsilon \| \mu \| .$$

Thus

$$\| \mu \| \geqq \| T_\mu \|_2 \geqq | \mu(s * \tilde{s}) | \geqq (1 - \varepsilon) \| \mu \| ,$$

i.e. $\| T_\mu \|_2 = \| \mu \|$.

(ii) $\Rightarrow$ (iii). Let μ be a nonnegative p-admissible measure and K a compact set in G. If μ_K denotes the restriction of μ to K then, exactly as in the proof of Lemma 1,

$$\| T_{\mu_K} \|_p = \sup_{s,t} \mu_K(s * \tilde{t}) \leqq \sup_{s,t} \mu(s * \tilde{t}) = \| T_\mu \|_p ,$$

where $s, t \in \mathscr{K}_+(G)$, $\| s \|_p, \| t \|_q \leqq 1$. Thus, by property (ii),

$$\| \mu_K \| = \| T_{\mu_K} \|_p \leqq \| T_\mu \|_p < \infty ,$$

for all $K \subset G$. Consequently, $\mu \in M_+(G)$, i.e. G is a K_p-group.

(iii) $\Rightarrow$ (ii). If (ii) is false let μ be a measure in $M_+(G)$ of norm 1 such that $\| T_\mu \|_p = r < 1$. When ν_n denotes the n-fold convolution of μ with itself and T_n the convolution operator on $L^p(G)$ defined by ν_n we have $\| \nu_n \| = 1$, $\| T_n \|_p \leqq r^n$. Now let σ be any function in $\mathscr{K}_+(G)$ with $\int_G \sigma dx = 1$ and set $\nu = (\sum_{n=1}^\infty \nu_n) * \sigma$. We shall prove that

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[1] The referee has kindly informed me that Leptin himself has proved Theorem A in his paper *On locally compact groups with invariant means* (to appear).

ν is an unbounded measure on G for which $\|T_\nu\|_p < (1/1 - r)$ in contradiction to the hypothesis that G is a K_p-group. For arbitrary $s \in \mathscr{K}(G)$,

$$\left|\int\left\{\left(\sum_1^N \nu_n\right) * \sigma\right\} s(x)dx\right| \leqq \left\|\left(\sum_1^N \nu_n\right) * \sigma\right\|_p \cdot \|s\|_q$$
$$\leqq (1/1 - r)\,\|\sigma\|_p \cdot \|s\|_\infty\, m(K)^{1/q}\,, \qquad N \geqq 1\,,$$

where K is the support of s; consequently ν is a continuous linear functional on $\mathscr{K}(G)$. Obviously, r is unbounded, for

$$\sum_{n=1}^N \int (\nu_n * \sigma)dx = N \longrightarrow \infty$$

as $N \to \infty$. On the other hand, for $f \in L^p(G)$,

$$\|\nu * f\|_p \leqq \sum \|\nu_n * \sigma * f\|_p \leqq \|f\|_p/(1 - r)\,,$$

and so ν is a positive unbounded p-admissible measure.

(ii) $\Rightarrow$ (i). If G does not have property (R) there is a measure $\nu \in M(G)$ of positive type for which $\int_G d\nu < 0$, (cf. Darsow [2], Dixmier [5] p. 319). This ν is necessarily hermitian ([5] p. 264) while if $Rl(\nu) = \mu_+ - \mu_-$, $\mu_+, \mu_- \in M_+(G)$ we have

$$\mu_+(s * \tilde{s}) \geqq \mu_-(s * \tilde{s})\,, \qquad s \in \mathscr{K}_+(G)\,,$$
$$\|\mu_+\| = \int d\mu_+ < \int d\mu_- = \|\mu_-\|\,.$$

But μ_+, μ_- are also hermitian; hence, by Lemma 1,

$$\|\mu_+\| = \|T_{\mu_+}\|_p = \|T_{\mu_+}\|_2$$
$$\geqq \|T_{\mu_-}\|_2 = \|T_{\mu_-}\|_p = \|\mu_-\|\,.$$

With this contradiction the proof of Theorem A is complete.

A group G is said to admit an *invariant mean* if there is a positive linear functional $\mathscr{M}$ on $L^\infty(G)$ of norm 1 such that

$$\mathscr{M}(1) = 1\,, \qquad \mathscr{M}(\phi) = \mathscr{M}(\phi_a) = \mathscr{M}({}_a\phi)\,, \qquad a \in G\,,$$

where $\phi_a(x) = \phi(a^{-1}x)$, ${}_a\phi(x) = \phi(xa)$.

LEMMA 2 (*Følner-Namioka*). *Both the following conditions are necessary and sufficient for G to admit an invariant mean*:

(i) *given any finite set $K = \{a_1, \cdots, a_n\}$ in G and $\varepsilon > 0$, there exists a measurable set A in G such that $0 < m(A) < \infty$ and*

$$m(a_jA \cap A) > (1 - \varepsilon)m(A)\,, \qquad j = 1, 2, \cdots, n\,,$$

(ii) *there is a constant $k, 0 < k < 1$, such that, to each finite* *set* $K = \{a_1, \cdots, a_n\}$ *in* G, *there corresponds a measurable set* A *in* G *with* $0 < m(A) < \infty$ *and*

$$\frac{1}{n}\sum_{j=1}^{n} m(a_j A \cap A) > k \ .$$

For discrete groups these criteria are due to Følner ([7]); for locally compact groups in general, (i) is a combination of the results of Namioka ([15] Th. 3.7) and Dixmier ([6] § 4, 3(a)). The proof of (ii) is a straightforward modification of that given by Følner (see, for instance, Hulanicki ([9] Th. 5.3)).

THEOREM B. *Let* f *be a hermitian function in* $L^1_+(G)$ *nonzero almost everywhere. Then* G *has property* (R) *if and only if*

$$\| T_f \|_p = \int_G f(x)dx$$

for some $1 < p < \infty$.

REMARK. Theorem B gives a partial extension to all locally compact groups of the result of Kesten ([11] p. 150) for countable discrete groups since property (R) is equivalent to the existence of an invariant mean (see Reiter [17], [18]).

Proof of Theorem B. The necessity of the condition follows at once from Theorem A. For the proof of sufficiency we may assume that $p = 2$. Then, by Lemma 1, for any $\varepsilon, \delta > 0$ there exists $s \in \mathscr{K}_+(G)$, $\| s \|_2 = 1$, such that

$$\int_G f(x)dx - \int_G f(x)(s * \tilde{s})(x)dx < \varepsilon\delta \ ,$$

because $0 \leqq s * \tilde{s} \leqq s * \tilde{s}(e) = 1$. Hence, for each compact set K in G,

$$\int_K f(x) \,|\, 1 - (s * \tilde{s})(x) \,|\, dx < \varepsilon\delta \ .$$

If we assume K is of nonzero measure, on the subset K_ε of K on which $| 1 - (s * \tilde{s})(x) | > \varepsilon$, $\int_{K_\varepsilon} f(x)dx < \delta$. Assume for the moment that f is continuous and everywhere nonzero; in this case

$$m(K_\varepsilon) < \delta / \inf_{x \in K} f(x) \ .$$

Consequently, given any compact set $K \subset G, \varepsilon, \delta > 0$ there exists $s \in \mathscr{K}_+(G)$ with $\| s \|_2 = 1$ and a subset K_ε of K such that

$$|1 - s*\tilde{s}(x)| < \varepsilon\ , \qquad x \in K\backslash K_\varepsilon\ , \qquad m(K_\varepsilon) < \delta\ .$$

When $g \in \mathscr{K}_+(G)$ has compact support K we have, therefore,

$$\begin{aligned} |\,\|g\|_1 - |g(s*\tilde{s})|\,| &\leqq \int_G g(x)\,|1 - (s*\tilde{s})(x)|\,dx \\ &\leqq \varepsilon\,\|g\|_1 + \delta\,\|g\|_\infty\ , \end{aligned}$$

i.e., $\|g\|_1 = \|T_g\|_1$. Now let $\mu \in M_+(G), \phi \in \mathscr{K}_+(G)$ be given, where $\|\phi\|_1 = 1$ and μ has compact support. Then, with s, σ arbitrary functions in $\mathscr{K}_+(G)$ satisfying $\|s\|_2 = \|\sigma\|_2 = 1$,

$$\begin{aligned} \|T_\mu\|_2 &= \sup_{s,\sigma} |\mu(s*\tilde{\sigma})| \geqq \sup_{s,\sigma} |\mu(\phi * s*\tilde{\sigma})| \\ &= \sup_{s,\sigma} |(\mu*\phi)(s*\tilde{\sigma})| = \|T_{\mu*\phi}\|_2 = \|\mu*\phi\|_1 = \|\mu\| \end{aligned}$$

since $\mu*\phi \in \mathscr{K}_+(G)$. Hence $\|T_\mu\|_2 = \|\mu\|$. The extension of this inequality to all of $M_+(G)$ is immediate. Consequently G has property (R). It remains now to show that f may be assumed continuous and everywhere nonzero. Choose any $\sigma \in \mathscr{K}_+(G)$ with $\int_G \sigma(x)dx = 1$ and let K_1 be the support of σ (we assume K_1 contains the identity e of G). Given any $\varepsilon > 0$ choose $s \in \mathscr{K}_+(G)$ and K_2 a compact set in G such that

$$\int_{G\backslash K_2} f(x)dx < \varepsilon, \qquad |1 - (s*\tilde{s})(x)| < \varepsilon,\ x \in K_1\cdot K_2\backslash K_\varepsilon$$

where $\int_{K_\varepsilon} f(x)dx < \varepsilon$ for some subset K_ε of $K_1\cdot K_2$. Then

$$\begin{aligned} \int_G (\sigma*f)(x)(1 - (s*\tilde{s})(x))dx &= \int_G \sigma(y)\Big\{\int_G f(x)(1 - (s*\tilde{s})(yx)dx\Big\}dy \\ &\leqq \int_G \sigma(y)\Big\{\int_{G\backslash K_2} f(x)dx + \int_{K_2} f(x)(1 - (s*\tilde{s})(yx))dx\Big\}dy \\ &< \int_G \sigma(y)(\varepsilon + \varepsilon\,\|f\|_1 + \varepsilon)dy = \varepsilon(2 + \|f\|_1)\ . \end{aligned}$$

Hence $\|T_{\sigma*f}\|_2 = \|\sigma*f\|_1$; but, obviously $\sigma*f$ is continuous and everywhere nonzero. This completes the proof of Theorem B.

THEOREM C. *Let G be a locally compact group. Then G admits an invariant mean if and only if, for some $p, 1 < p < \infty$, $\|T_\mu\|_p = \|\mu\|$ whenever μ is a discrete measure in $M_+(G)$.*

Proof. If G_d denotes G provided with the discrete topology, the discrete measures in $M_+(G)$ can be identified with $l^1_+(G_d)$. To show that $\|T_\mu\|_p = \|\mu\|$ for some $1 < p < \infty$ and all $\mu \in l^1_+(G_d)$ when G admits an invariant mean, it is enough to prove that $\|T_\mu\|_2 = \|\mu\|$

for all $\mu \in l^1_+(G_d)$ having compact support (note that T_μ is an operator on $L^2(G)$). Let $K = \{a_1, \cdots, a_n\}$ denote the support of any such measure. Then, given $\varepsilon > 0$, there exists a measurable set A in G, $0 < m(A) < \infty$, such that

$$m(a_j A \cap A) > (1 - \varepsilon)m(A) , \qquad j = 1, \cdots, n .$$

Setting $\psi = \chi_A / m(A)^{1/2}$ with χ_A the characteristic function of A we have, therefore,

$$\begin{aligned} &| \, \| \mu \| - | \mu(\psi * \tilde{\psi}) | \, | \\ &\quad \leqq \sum_{j=1}^{n} \mu(a_j) \left| 1 - \frac{m(a_j A \cap A)}{m(A)} \right| < \varepsilon \| \mu \| . \end{aligned}$$

Consequently, $\| T_\mu \|_2 = \| \mu \|$ since $\| \psi \|_2 = 1$. Suppose conversely that $\| T_\mu \|_2 = \| \mu \|$ for all $\mu \in l^1_+(G_d)$, (again by convexity arguments it suffices to consider $p = 2$). Denote by K any finite set $\{a_1, \cdots, a_n\}$ in G and suppose that a_j occurs $w(j)$ times in K; set $C = K \cup K^{-1}$. Then the measure μ in $l^1_+(G_d)$ defined by

$$\mu(x) = \begin{cases} w(j)/2n & x = a_j \ , \quad a_j \neq a_j^{-1} \\ w(j)/2n & x = a_j^{-1} \ , \quad a_j^{-1} \neq a_j \\ w(j)/n & x = a_j \ , \quad a_j = a_j^{-1} \\ 0 & \text{Otherwise} \end{cases}$$

is hermitian. Hence, by Lemma 1, given any $\varepsilon > 0$ there exists $s \in \mathscr{K}_+(G)$, $\| s \|_2 = 1$ such that

$$\| \mu \| - \mu(s * \tilde{s}) < \varepsilon^2/2 ,$$

i.e.,

$$1 - \frac{1}{2n} \sum_{j=1}^{n} \{(s * \tilde{s})(a_j) + (s * \tilde{s})(a_j^{-1})\} < \varepsilon^2/2 .$$

Set $\sigma = s^2$. Then

$$\begin{aligned} \sum_{j=1}^{n} (\| \sigma - \sigma_{a_j} \|_1)^2 &\leqq 4 \sum_{j=1}^{n} (\| s - s_{a_j} \|_2)^2 \\ &= 8 \sum_{j=1}^{n} | 1 - (s * \tilde{s})(a_j) | < 4n\varepsilon^2 , \end{aligned}$$

since $(s * \tilde{s})(a_j) = (s * \tilde{s})(a_j^{-1}) \leqq$ when $s \in \mathscr{K}_+(G)$. Thus

$$\frac{1}{n} \sum_{j=1}^{n} \| \sigma - \sigma_{a_j} \|_1 \leqq \frac{1}{n} (4n\varepsilon^2)^{1/2} n^{1/2} = 2\varepsilon .$$

If, for $\lambda \geqq 0$, $E_\lambda = \{x \in G : \sigma(x) \geqq \lambda\}$ and χ_λ is the characteristic function of E_λ, we can repeat the proof of Hulanicki ([10] p. 98) to obtain

$$\frac{1}{2n}\|\sigma - \sigma_{a_j}\|_1 = \frac{1}{2n}\sum_{j=1}^{n}\int_0^\infty m(a_jE_\lambda \Delta E_\lambda)d\lambda$$
$$= \int_0^\infty m(E_\lambda)\Big\{\frac{1}{2n}\sum_{j=1}^{n}\frac{m(a_jE_\lambda \Delta E_\lambda)}{m(E_\lambda)}\Big\}d\lambda < \varepsilon .$$

Since

$$\int_0^\infty m(E_\lambda)d\lambda = \int_G \sigma(x)dx = 1$$

there exists E_λ, $m(E_\lambda) \neq 0$, such that

$$\frac{1}{2n}\sum_{j=1}^{n}\frac{m(a_jE_\lambda \Delta E_\lambda)}{m(E_\lambda)} < \varepsilon .$$

Consequently,

$$\frac{1}{n}\sum_{j=1}^{n} m(a_jE_\lambda \cap E_\lambda) > (1-\varepsilon)m(E_\lambda) ,$$

i.e., G admits an invariant mean (Lemma 2).

DEFINITION. *For given C, $0 < C < 1$, a locally compact group G is said to have property $R(C)$, resp. $R_d(C)$, if, given any compact set $K \subset G$, resp. finite set $K = \{a_1, \cdots, a_n\} \subset G$, there exists $s \in \mathscr{K}(G)$ with $\| s \|_2 = 1$ such that*

$$\sup_{x \in K} | 1 - (s * \tilde{s})(x) | < C ,$$

respectively

$$\sup_{1 \le j \le n} | 1 - (s * \tilde{s})(a_j) | < C .$$

Thus, if G has property $R(C)$ for all $0 < C < 1$ it has property (R), (cf. Dixmier [5] p. 319).

THEOREM D. *Let G be a locally compact group. Then the following assertions are equivalent:*
(i) *G has property (R),*
(ii) *G has property $R(C)$ for some $0 < C < 1$,*
(iii) *G has property $R_d(C)$ for some $0 < C < 1$.*

Proof. Obviously (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). To show that (iii) $\Rightarrow$ (i) it is enough to prove that, when G has property $R_d(C)$ for some $0 < C < 1$, then $\| T_\mu \|_2 = \| \mu \|$ for every $\mu \in l^1_+(G_d)$. Since then, by Theorem C, G admits an invariant mean; consequently it will also have property (R) (cf. Reiter [17], [18]). Let μ be an element of $l^1_+(G_d)$ having compact support say $K = \{a_1, \cdots, a_n\}$. By $R_d(C)$ there exists $s \in \mathscr{K}(G)$, $\| s \|_2 = 1$, such that

$$| \| \mu \| - | \mu(s * \tilde{s}) | | \leqq \sum \mu(a_j) | 1 - (s * \tilde{s})(a_j) | \\ \leqq C \| \mu \| .$$

Thus $\| T_\mu \|_2 \geqq (1 - C) \| \mu \|$ for any $\mu \in l^1_+(G_d)$ having compact support. But, if $\| T_\mu \|_2 = r \| \mu \|$, $r < 1$, for sufficiently large n

$$(1 - C) \| \nu_n \| = (1 - C) \| \mu \|^n \\ \leqq \| T_n \|_2 \leqq (\| T_\mu \|_2)^n = r^n \| \mu \|^n < (1 - C) \| \mu \|^n$$

where ν_n denotes the n-fold convolution product of μ with itself and $T_n = T_{\nu_n}$. This is an obvious contradiction. Thus $\| T_\mu \|_2 = \| \mu \|$ for all $\mu \in l^1_+(G_d)$ and so G has property (R).

3. By way of illustration we shall consider two groups:

(i) free group G_∞ with generators a_n, $n = 1, 2, \cdots$, each of order 2,

(ii) $G = SL(R, 2)$.

3(i). Let G_n be the free group generated by $a_j, j = 1, \cdots, n$. Darsow ([2]) has shown that, for any $s \in \mathscr{K}_+(G_n)$, $\| s \|_2 = 1$,

$$\sup_{1 \leqq j \leqq n} | 1 - (s * \tilde{s})(a_j) | > [1 - (2/n)(n - 1)^{1/2}] . \tag{3}$$

Consequently, G_∞ fails to have property $R(C)$ for any $0 < C < 1$ (note that the restriction to G_n of an $s \in \mathscr{K}_+(G_\infty)$, $\| s \|_2 = 1$, cannot decrease (3)). Repeating the proof of Darsow ([2] p. 452) we can show that for any such s

$$\sum_{j=1}^{n} (s * \tilde{s})(a_j) \leqq \sum_{j=1}^{n} t_j^{1/2}(1 - t_j)^{1/2}$$

for some n-tuple $(t_1, \cdots, t_n)$, $0 \leqq t_j \leqq 1$, $t_1 + t_2 + \cdots + t_n \leqq 1$. An elementary argument using Lagrange's Multipliers shows that

$$\sum_{j=1}^{n} (s * \tilde{s})(a_j) \leqq n(1/n)^{1/2}(1 - 1/n)^{1/2} \\ = (n - 1)^{1/2} \tag{4}$$

whenever $s \in \mathscr{K}_+(G_\infty)$, $\| s \|_2 = 1$. Now the characteristic function of the subset $(a_1, \cdots, a_n)$ of G_∞ is a hermitian measure μ_n in $M_+(G_\infty)$ of norm n. But, by (4), as an operator on $L^2(G_n)$,

$$\| T_{\mu_n} \|_2 \leqq (n - 1)^{1/2} .$$

All the above calculations again hold when G_n is regarded as a subgroup of G_∞. Consider the measure

$$\mu = \sum_{n=1}^{\infty} (1/n^2)\mu_n .$$

Then $\mu \notin M_+(G_\infty)$, but $\| T_\mu \|_2 \leqq \sum_{n=1}^{\infty} (1/n^2)(n-1)^{1/2} < \infty$, i.e., μ is a positive, unbounded, 2-admissible measure.

3(ii). The group $SL(R, 2)$ contains a discrete subgroup H isomorphic to the free group $G_{a,b}$ on two generators a, b (see, for example, [1]). Furthermore, $G = SL(R, 2)$ possesses a fundamental domain F measurable with respect to Haar measure on G (cf. [16], [19]) such that

$$\int_G s(x)dx = \sum_{\xi \in H} \int_F s(\xi x)dx , \qquad s \in \mathscr{K}(G) .$$

Following Reiter ([16] p. 2883) we set

$$s_H(\xi) = \int_F s(\xi x)dx , \qquad \xi \in H ,$$

whenever $s \in \mathscr{K}(G)$. Now, for fixed $y \in H$, when $s \in \mathscr{K}_+(G)$, $\| s \|_2 = 1$ and $\sigma = s^2$, we have

$$\begin{aligned}\sum_{\xi \in H} | \sigma_H(\xi) - \sigma_H(\eta^{-1}\xi) | &= \sum_{\xi \in H} \left| \int_F (\sigma(\xi x) - \sigma(\eta^{-1}\xi x))dx \right| \\ &\leqq \int_G | \sigma(x) - \sigma(\eta^{-1}x) | \, dx \leqq \| s + s_\eta \|_2 \cdot \| s - s_\eta \|_2 \\ &\leqq 2^{3/2} | 1 - (s * \tilde{s})(\eta) |^{1/2} ;\end{aligned}$$

clearly $\sum_{\xi \in H} \sigma_H(\xi) = 1$. Denote by M the subset of H which can be identified with $\{a, a^2, \cdots, a^n, b, b^2, \cdots, b^n\}$ in $G_{a,b}$. Then, if N denotes all words in $G_{a,b}$ starting with b and $P = G_{a,b} \backslash N$

$$1 \geqq \sum_{m=0}^{n} \sum_{\xi \in N} \sigma_H(a^m \xi) > (n+1) \sum_{\xi \in N} \sigma_H(\xi) - n\varepsilon$$

$$1 \geqq \sum_{m=0}^{n} \sum_{\xi \in P} \sigma_H(b^m \xi) > (n+1) \sum_{\xi \in P} \sigma_H(\xi) - n\varepsilon$$

where $\varepsilon = \sup_{\eta \in M} \sum_{\xi \in H} | \sigma_H(\xi) - \sigma_H(\eta^{-1}\xi) |$, (see Yoshizawa [12] p. 57). Hence $\varepsilon > (n-1)/2n$. But then

$$\sup_\eta | 1 - (s * \tilde{s})(\eta) | \geqq \frac{1}{8}\left(\frac{n-1}{2n}\right)^2 .$$

This inequality persists for arbitrary $s \in \mathscr{K}(G)$ with $\| s \|_2 = 1$ (cf. Darsow [2] p. 453), consequently $SL(R, 2)$ does not have $R_d(C)$ for any $0 < C < 1/32$.

If μ denotes the characteristic function of the set $M \cup M^{-1}$ in H (so that μ is a discrete measure in $M_+(SL(R, 2))$) then

$$(\|\mu\| - \|T_\mu\|_2) = \inf\left[2\sum_{m=1}^{n}(2 - (s*\tilde{s})(a^m) - (s*\tilde{s})(b^m))\right]$$

the infinium being taken over all $s \in \mathscr{K}_+(G)$ with $\|s\|_2 = 1$. Hence

$$\frac{1}{2}(\|\mu\| - \|T_\mu\|_2) \geqq \frac{1}{8}\left(\sum_{\eta\in M}\left|\sum_{\xi\in H}\left|\sigma_H(\xi) - \sigma_H(\eta^{-1}\xi)\right|\right|^2\right).$$

With only a simple modification of the argument of Yoshizawa we see that

$$\sum_{\eta\leqq M}\sum_{\xi\in H}|\sigma_H(\xi) - \sigma_H(\eta^{-1}\xi)| > (n-1)/2 .$$

Thus

$$4(\|\mu\| - \|T_\mu\|_2) \geqq (1/2n)[(n-1)/2]^2 ,$$

i.e., $\|\mu\| = 4n$, but,

$$\|T_\mu\|_2 \leqq \{4n - (n-1)^2/32n\} .$$

Hence $\|T_\mu\|_2 < \|\mu\|$.

For more definitive results in the contex of free groups one should consult Dieudonné ([4]), Kesten ([12]).

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