PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 2, 1968

THE RATIONAL HOMOTOPY OF A WEDGE

ALLAN CLARK AND LARRY SMITH

The rational homotopy of a wedge $X \vee Y$ is given in terms of the rational homotopy of X and Y.

Let X be a pathwise connected and simply connected space with base point e_X, which is a neighborhood deformation retract in X. (See [5].) We shall say that X is a *nicely pointed space*. The *rational homotopy of* X is the connected graded Lie algebra over Q, $\mathscr{L}_*(X)$, defined by setting $\mathscr{L}_n(X) = \pi_{n+1}(X, e_X) \otimes Q$, with the Lie product induced by the Whitehead product on homotopy groups.

The purpose of this note is to show that the functor $\mathscr{L}_*$ preserves coproducts. More precisely we show:

THEOREM 1. *Let X and Y be nicely pointed spaces which are pathwise connected and simply connected and whose rational homotopy has finite type. Then there is a natural isomorphism of graded Lie algebras*

$$\phi(X, Y): \mathscr{L}_*(X \vee Y) \approx \mathscr{L}_*(X) \perp \mathscr{L}_*(Y)$$

where $\perp$ denotes the coproduct in the category of connected graded Lie algebras over Q (defined below).

The result follows easily from the natural isomorphism of $\mathscr{L}_*(X)$ with $H_*(\Omega X; Q)$, the Lie algebra of primitive elements of the Hopf algebra $H_*(\Omega X; Q)$. This isomorphism was discovered by Cartan and Serre; a revised statement [4, page 263] is due to John Moore to whom we are indebted for many useful conversations. Due to this isomorphism we may view $\mathscr{L}_*$ as the composition of four functors: $\mathscr{L}_* = \mathscr{P} H \mathscr{F} \mathscr{C}$ where

1. $\mathscr{C}$ is the functor which assigns to a pathwise and simply connected space the connected differential graded Q-coalgebra formed by its simply connected singular chain complex over Q;
2. $\mathscr{F}$ is the cobar construction;
3. H is the homology functor;
4. $\mathscr{P}$ is the functor which assigns to a connected graded Hopf algebra over Q the associated connected graded Lie algebra of primitive elements.

The idea of the proof is to show that each of the required categories has coproducts preserved by the four functors involved.

This result has long been a part of the folk literature, but to the best of our knowledge no proof appears in print. This result extends and compliments results of Hilton, and Porter on the integral homotopy of a wedge.

1. Coproducts. *A category $\mathscr{C}$ has coproducts* if to every pair of objects A and B of $\mathscr{C}$, there is assigned a diagram in $\mathscr{C}$

$$A \xrightarrow{i_A} A \perp B \xleftarrow{i_B} B$$

with the property that for any morphisms $f: A \to C$ and $g: B \to C$ of $\mathscr{C}$, there is a *unique* morphism $f \perp g: A \perp B \to C$ such that

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & A \perp B & \xleftarrow{i_B} & B \\ & \searrow & \Big\downarrow{\scriptstyle f \perp g} & \swarrow & \\ & & C & & \end{array}$$

is a diagram in $\mathscr{C}$. ("Diagram in $\mathscr{C}$" means a commutative diagram of objects and morphisms of $\mathscr{C}$.)

If $\perp$ is a coproduct on $\mathscr{C}$, then as an immediate consequence of the definition, there are natural $\mathscr{C}$-isomorphisms $A \perp B \approx B \perp A$ and $A \perp (B \perp C) \approx (A \perp B) \perp C$.

EXAMPLE 1. The wedge $\vee$ or one point union is a coproduct on the category of pointed spaces $\mathscr{T}_*$.

In the remaining examples K will be a commutative ring with unit.

EXAMPLE 2. $\mathscr{C}$ = *the category of connected graded K-modules.* For each object A of, we have $A_0 \approx K$. The coproduct is defined by $(A \perp B)_n = A_n \oplus B_n$ for $n > 0$.

EXAMPLE 3. $\mathscr{C}$ = *the category of connected graded K-algebras.*[1] For $A \in \mathscr{C}$ we define a graded K-module $\bar{A}$ by $\bar{A}_n = A_n$ for $n > 0$ and $A_0 = 0$. Then $T(\bar{A}) = K \oplus \sum_{n=1}^{\infty} (\bar{A} \otimes \cdots (n) \cdots \otimes \bar{A})$, *the tensor algebra of* $\bar{A}$, is an object of $\mathscr{C}$ and there is a canonical homomorphism $T(\bar{A}) \to A$ in $\mathscr{C}$, the kernel of which we denote $I(A)$. A coproduct is defined by $A \perp B = T(\bar{A} \oplus \bar{B})/(I(A), I(B))$ where the denominator denotes the ideal of $T(A \oplus \bar{B})$ generated by $I(A)$ and $I(B)$. It is routine to verify that $\perp$ is indeed a coproduct. A simple diagram chase shows that $T(\bar{A}) \perp T(\bar{B}) = T(\bar{A} \oplus \bar{B})$.

EXAMPLE 4. $\mathscr{C}$ = *the category of connected graded Lie algebras over Q.* Each $A \in \mathscr{C}$ is a graded K-module with $A_0 = 0$. We set

[1] The word 'algebra' means 'associative algebra with unit'.

$U(A) = T(A)/J$ where J is the ideal generated by all elements $x \otimes y - (-1)^{pq} y \otimes x - [x, y]$ with $x \in A_p$, $y \in A_q$. Then $U(A)$ is a connected graded Q-algebra, called the *universal enveloping algebra of* A. There is a canonical morphism $A \to U(A)$ such that if $f\colon A \to C$ is a map of A into a connected graded Q-algebra, such that $f[x, y] = [fx, fy]$, then there is a unique map of algebras $U(f)\colon U(A) \to C$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \longrightarrow & U(A) \\ & \searrow & \swarrow {\scriptstyle U(f)} \\ & C & \end{array}$$

To form the coproduct $\perp$ in $\mathscr{C}$ we begin by forming $U(A) \perp U(B)$ as in Example 2. We define $A \perp B$ to be the sub Lie algebra of (the associated Lie algebra of) $U(A) \perp U(B)$ generated by the images of A and B. Thus we have a diagram

$$\begin{array}{ccccc} A & \longrightarrow & A \perp B & \longleftarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ U(A) & \longrightarrow & U(A) \perp U(B) & \longleftarrow & U(B) \end{array}$$

It is routine to check the universal property. We note that uniqueness implies $U(A \perp B) \approx U(A) \perp U(B)$ as graded Q-algebras.

EXAMPLE 5. $\mathscr{C}$ = *the category of connected graded Hopf algebras over* K. Since each object of $\mathscr{C}$ is a graded connected K-algebra, we may form the coproduct as in Example 2. Then we need to check that $A \perp B$ is still a Hopf algebra. In the category of graded connected algebras we have the diagram:

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & A \perp B & \xleftarrow{i_B} & B \\ \downarrow{\scriptstyle \Delta_A} & & \downarrow{\scriptstyle \Delta_{A \perp B}} & & \downarrow{\scriptstyle \Delta_B} \\ A \otimes_K A & \xrightarrow[i_A \otimes i_A]{} & (A \perp B) \otimes_K (A \perp B) & \xleftarrow[i_B \otimes i_B]{} & B \otimes_K B \end{array}$$

In other words $\Delta_{A \perp B} = (i_A \otimes i_A)\Delta_A \perp (i_B \otimes i_B)\Delta_B$ is a morphism of graded connected algebras and $A \perp B$ is a Hopf algebra. The required universal property is easily verified.

EXAMPLE 6. $\mathscr{C}$ = *the category of connected differential graded* K*-coalgebras*. The coproduct here is defined as in Example 2 it is only necessary to check that the differential and comultiplication behave well.

EXAMPLE 7. $\mathscr{C}$ = *the category of connected differential graded K-algebras.* The coproduct is defined as in Example 3, and the differential extends naturally.

EXAMPLE 8. $\mathscr{C}$ = *the category of connected differential graded Hopf algebras over K.* The coproduct is defined as in Example 5 and the differential extends naturally.

2. Functors which preserve coproducts. Let $\mathscr{T}_*^1$ denote the category of nicely pointed 1-connected spaces. Let $\mathscr{C}X$ for $X \in \mathscr{T}_*^1$ denote the normalized singular chains of X with all edges at the base point e_X. In other words $\mathscr{C}X = C_N(E_2(X, e_X))$, the normalized chain complex of $E_2(X, e_X)$, the second Eilenberg subcomplex. [3; p. 430.] Then $\mathscr{C}$ is a functor with range the category of 1-connected differential graded coalgebras over $\boldsymbol{Z}$, which we denote C^1DGCO. $\mathscr{C}$ does not preserve coproducts. However there is a diagram in C^1DGCO:

$$\begin{array}{ccccc} \mathscr{C}X & \longrightarrow & \mathscr{C}X \perp \mathscr{C}Y & \longleftarrow & \mathscr{C}Y \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle f \perp g} & \swarrow{\scriptstyle g} & \\ & & \mathscr{C}(X \vee Y) & & \end{array}$$

where f and g are induced by the inclusions into $X \vee Y$. An elementary argument shows that $f \perp g$ induces a homology isomorphism of coalgebras.

The cobar construction $\mathscr{F}$ is a functor with domain C^1DGCO and range C^0DGAl, the category of connected differential graded algebras. We want to show that

2.1. PROPOSITION. *$\mathscr{F}$ preserves coproducts.*

Proof. Let C_1 and C_2 belong to C^1DGCO. Then $\mathscr{F}$ induces maps $\mathscr{F}(C_i) \to \mathscr{F}(C_1 \perp C_2)$. Consequently we have in C^0DGAl:

$$\begin{array}{ccccc} \mathscr{F}(C_1) & \longrightarrow & \mathscr{F}(C_1) \perp \mathscr{F}(C_2) & \longleftarrow & \mathscr{F}(C_2) \\ & \searrow & \downarrow \phi & \swarrow & \\ & & \mathscr{F}(C_1 \perp C_2) & & \end{array}$$

Let $\#$ denote the functors which forget the differentials in various categories. Then $\mathscr{F}(C)_\# = T(C_\#)$ so that

$$\phi_\#\colon T(\bar{C}_{1\#}) \perp T(\bar{C}_{2\#}) \longrightarrow T(\bar{C}_{1\#} \oplus \bar{C}_{2\#})$$

is an isomorphism. Since $\#$ is faithful, ϕ is an isomorphism.

Next we restrict our attention to algebras over the rational field

Q and consider the homology functor $H_*: C^0DGAl/Q \to C^0GAl/Q$, the category of connected graded Q-algebras.

2.2. PROPOSITION. $H_*(A \perp B) \approx H_*(A) \perp H_*(B)$.

Proof. We can readily construct a diagram in C^0DGAl/Q

$$\begin{array}{ccc} & H(A) & \\ \swarrow & & \searrow \\ H(A) \perp H(B) & \xrightarrow{\phi} & H(A \perp B)\, . \\ \nwarrow & & \nearrow \\ & H(B) & \end{array}$$

The additive isomorphisms $A \perp B = (\bar{A} \oplus \bar{B}) \oplus ((\bar{A} \otimes \bar{B}) \oplus (\bar{B} \otimes \bar{A})) + \cdots$ and

$$H(A) \perp H(B) = (\bar{H}(\bar{A}) \oplus \bar{H}(\bar{B})) \oplus ((\bar{H}(\bar{A}) \otimes \bar{H}(\bar{B})) \oplus (\bar{H}(\bar{B}) \otimes H(A))) + \cdots$$

together with the Kunneth Theorem implies that ϕ is an isomorphism.

3. Proof of Theorem 1. In the notation above we have isomorphisms of graded Q-algebras

$$H_*(\mathscr{F}\mathscr{C}X) \perp H_*(\mathscr{F}\mathscr{C}Y) \xrightarrow{\approx} H_*(\mathscr{F}\mathscr{C}X \perp \mathscr{F}\mathscr{C}Y) \xrightarrow{\approx} H_*(\mathscr{F}(\mathscr{C}X \perp \mathscr{C}Y)) \xrightarrow{\approx} H_*(\mathscr{F}\mathscr{C}(X \vee Y))\, .$$

By a theorem of Adams, for any pathwise and simply connected space Z, there is a natural isomorphism of algebras, $H_*(\Omega Z; Q) \to H_*(\mathscr{F}\mathscr{C}Z)$. Consequently the morphism of Hopf algebras

$$H_*(\Omega X; Q) \perp H_*(\Omega Y; Q) \longrightarrow H_*(\Omega(X \vee Y); Q)$$

is an isomorphism of algebras, and hence of Hopf algebras. Moore's statement says $H_*(\Omega X; Q) = U(\mathscr{L}_*(X))$ so we have

$$U(\mathscr{L}_*(X) \perp \mathscr{L}_*(Y)) \approx U(\mathscr{L}_*(X)) \perp U(\mathscr{L}_*(Y)) \approx U(\mathscr{L}_*(X \vee Y))$$

and since PU is the identity, $\mathscr{L}_*(X) \perp \mathscr{L}_*(Y) \approx \mathscr{L}_*(X \vee Y)$.

REMARK. It is apparent from the above argument and the theorem of Adams that

$$H_*(\Omega X; k) \perp H_*(\Omega Y; k) \longrightarrow H_*(\Omega(X \vee Y); k)$$

is an isomorphism of Hopf algebras for any field k.

This has been proved by Berstein in [2] by slightly differer methods.

REMARK. The calculation of the Poincaré Series of the coprodu(of two Lie algebras is a difficult number theoretic problem involvin Witt numbers.

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Received November 8, 1966. The first author was supported by a National Scien(Foundation Grant GP-3446. The second author was partially supported by a Nation Science Foundation Grant GP-3946.

BROWN UNIVERSITY AND
PRINCETON UNIVERSITY