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A CHARACTERIZATION OF COMPACT CONNECTED PLANAR LATTICES

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In this paper it is proved that every topological lattice on the two-cell is topologically isomorphic (iseomorphic) to a sublattice of the product lattice $I \times I$. An explicit description of the compact connected sublattices of $I \times I$ containing $(0, 0)$ and $(1, 1)$ is given. These results, together with a theorem of A. D. Wallace, yield a characterization of all compact connected lattices in the plane: each is iseomorphic to a sublattice of $I \times I$.

A topological lattice is a partially ordered space X with the property that every pair of elements a, b of X has a least upper bound, $a \vee b$, and a greatest lower bound, $a \wedge b$, so that the operations $\vee$ and $\wedge$ are continuous. A simple example of a topological lattice is the unit interval I with the usual ordering. The partial order on the n-cell I^n given by $(x_i) \leqq (y_i)$ if and only if $x_i \leqq y_i$ for $i = 1, \cdots, n$ is a lattice ordering, in fact, it is the lattice ordering obtained by regarding I^n as a product lattice. L. W. Anderson and A. D. Wallace have found conditions under which a lattice ordering on the n-cell is the product order. One can also consider the following problem: determine all lattice orderings of the n-cell. It is well known that the usual order is the only lattice order on the interval. In this paper the problem is considered for the two-cell. It is shown that every topological lattice on the two-cell is iseomorphic to a sublattice of the product lattice $I \times I$. This result together with a theorem of A. D. Wallace is used to prove that every compact connected lattice in the plane is iseomorphic to a sublattice of $I \times I$. Finally, an explicit description of the compact connected sublattices of $I \times I$ containing $(0, 0)$ and $(1, 1)$ is given.

1. Lattice orderings of the two-cell. Let L be a topological lattice whose underlying space is homeomorphic to a two-cell. Since L is compact, L has a unique minimum element 0 and a unique maximum element 1. It is known [1] that 0 and 1 lie on the boundary of L and that the boundary arcs T and E determined by 0 and 1 are maximal chains in L and that T and E generate L in the sense that $L = T \vee E = T \wedge E$. In this section we prove that L is iseomorphic to a sublattice of $I \times I$. The proof requires several lemmas.

LEMMA 1. *Let $p, q \in L$. If $(p \wedge T) \cap T = (q \wedge T) \cap T$, then either*

$p \wedge T \subset q \wedge T$ or $q \wedge T \subset p \wedge T$.

Proof. We first assume that $p, q \in E$ and that $p \leqq q$. If $p = 0$, then $p \in q \wedge T$. Suppose $p > 0$ and that $p \notin q \wedge T$. It is well known that $p \vee T$ and $q \wedge T$ are arcs from p to 1 and from q to 0 respectively. Since L is a 2-cell, it must follow that $(p \vee T) \cap (q \wedge T) \neq \square$. Let $z \in (p \vee T) \cap (q \wedge T)$ and let

$$x = \sup \{(q \wedge T) \cap T\} = \sup \{(p \wedge T) \cap T\} .$$

Then $z = p \vee t$ for some $t \in T$. If $t \leqq x$, then by the definition of x, we would have $p \vee t = p$. Hence $t > x$. But now the inequality $t \leqq z \leqq q$ implies that $q \wedge t = t \in q \wedge T$ which contradicts the choice of x.

Now let p and q be arbitrary elements of L and choose $e, f \in E$ such that $p \in e \wedge T$ and $q \in f \wedge T$. This is possible since $E \wedge T = L$. If either of p or q is an element of T, then the lemma is trivial. For suppose $p \in T$. Then

$$p \wedge T = (p \wedge T) \cap T = (q \wedge T) \cap T \subset q \wedge T .$$

We may now assume that $p, q \notin T$. We contend that $(e \wedge T) \cap T = (p \wedge T) \cap T = (f \wedge T) \cap T = (q \wedge T) \cap T$. To establish the first equality, let $t \in (e \wedge T) \cap T$. Then since $e \wedge T$ is a chain and $p, t \in e \wedge T$, either $p \leqq t$ or $t \leqq p$. Suppose $p \leqq t$. Then for some $t_1 \in T, p = e \wedge t_1 = (e \wedge t_1) \wedge t = (e \wedge t) \wedge t_1 = t \wedge t_1 \in T$, which is a contradiction. Therefore $t \leqq p$ and $t \in (p \wedge T) \cap T$. Now suppose $t \in (p \wedge T) \cap T$. Then $t \leqq p \leqq e$ implies that $t \in (e \wedge T) \cap T$. This proves the first equality; the last equality is proved similarly. From the first part of the proof, we conclude that either $e \wedge T \subset f \wedge T$ or $f \wedge T \subset e \wedge T$. Suppose $f \wedge T \subset e \wedge T$. Then $p \wedge T$ and $q \wedge T$ are subchains of $e \wedge T$, so either $p \wedge T \subset q \wedge T$ or $q \wedge T \subset p \wedge T$.

For $x \in T$, we define $C_x \subset E$ by $C_x = \{h \in E \mid x = \sup \{(h \wedge T) \cap T\}\}$.

LEMMA 2. *The set C_x is closed for all $x \in T$.*

Proof. We consider the nontrivial case where $C_x \neq \square$. From the continuity of $\wedge$ it follows that the set $\{h \in E \mid x \in h \wedge T\}$ is closed. Let $e' = \inf \{h \in E \mid x \in h \wedge T\}$; then $x \in e' \wedge T$ and $e' \leqq e$ for all $e \in C_x$. If $t \in (e' \wedge T) \cap T$ and $t > x$, then for $e \in C_x$, we would have $t \leqq e' \leqq e$ and hence $t \in (e \wedge T) \cap T$ contradicting the fact that $e \in C_x$. Hence $x = \sup \{(e' \wedge T) \cap T\}$ and $e' \in C_x$.

Let $h_n \in C_x, n = 1, 2, \cdots$, and let $h_n \rightarrow h$. Then $e' \leqq h_n$ for each n and by Lemma 1, we have that $e' \wedge T \subset h_n \wedge T$ for all values of n and therefore $e' \wedge T \subset h \wedge T$. Let $x' = \sup \{(h \wedge T) \cap T\}$. Then

$x' \geqq x$ since $x \in (h \wedge T) \cap T$. We have that $e', x' \in h \wedge T$ and so one of the inequalities $x' \leqq e', e' \leqq x'$ must hold. If $x' \leqq e'$, then $x' \in (e \wedge T) \cap T$ which implies that $x' \leqq x$ and hence $x' = x$ and $h \in C_x$. If $e' \leqq x'$, let $e' = h \wedge t$ for $t \in T$. Then

$$e' = e' \wedge x' = (h \wedge t) \wedge x' = (h \wedge x') \wedge t = x' \wedge t \in T .$$

This involves a contradiction unless $e' = 0$. However, if $e' = 0$, then $x = 0$ and $h_n \wedge T = 0$ for all values of n; hence $n \wedge T = 0$ and $h \in C_x$. This completes the proof of the lemma.

We now define relations $\mathscr{H}$ and $\mathscr{V}$ on T as follows: for $a, b \in T$,

$$a\mathscr{H}b \equiv a \in e \vee T \text{ if and only if } b \in e \vee T \text{ for all } e \in E .$$
$$a\mathscr{V}b \equiv a \in e \wedge T \text{ if and only if } b \in e \wedge T \text{ for all } e \in E .$$

LEMMA 3. *The relations $\mathscr{H}$ and $\mathscr{V}$ are closed congruences on T.*

Proof. It is easy to see that $\mathscr{H}$ and $\mathscr{V}$ are congruences on T. We will show that the relation $\mathscr{V}$ is closed. A dual argument will show that $\mathscr{H}$ is closed.

Let $a_n \rightarrow a$, $b_n \rightarrow b$ with $a_n, b_n \in T$ and $a_n\mathscr{V}b_n$ for each n. Assume that $a \leqq b$. If $h \in e \wedge T$ for $e \in E$, it follows trivially that $a \in e \wedge T$. Suppose $a \in e \wedge T$ for $e \in E$. Let $x = \sup\{(e \wedge T) \cap T\}$; then $a \leqq x$. If $a < x$, then for n sufficiently large, $a_n < x$ and hence $a_n \in e \wedge T$. Since $a_n\mathscr{V}b_n$ we must have $b_n \in e \wedge T$ for n sufficiently large and therefore $b \in e \wedge T$. This gives $a\mathscr{V}b$.

We now assume that $a = x$ and let $f = \sup C_x$. This sup exists since C_x is closed by Lemma 2. If $f = 1$, then $a = b = 1$. Suppose $f < 1$ and let $f_m \rightarrow f$ where $f_m \in E, f_m > f$ for $m = 1, 2, \cdots$. Let $y_m = \sup\{(f_m \wedge T) \cap T\}$. Then since $f_m \notin C_x$, $y_m > a$. Thus for fixed m, there exists a positive integer N_m such that if $n \geqq N_m$, then $a_n < y_m$, or $a_n \in f_m \wedge T$. Therefore $b_n \in f_m \wedge T$ for $n \leqq N_m$. We conclude that $b \in f_m \wedge T$ for each positive integer m and hence $b \in f \wedge T$. But $a = \sup\{(f \wedge T) \cap T\}$ and hence $b \leqq a$. Therefore $a = b$.

LEMMA 4. *Let $e \in E$ and let $x = sup\{(e \wedge T) \cap T\}$, $e' = sup\, C_x$, $x' = inf\, V_x$ where V_x denotes the congruence class modulo $\mathscr{V}$ which contains x. Then $\{z \mid x' \leqq z \leqq e'\} \subset e' \wedge T$.*

Proof. If $z \in T$, then $z \leqq x \leqq e'$ implies $z = e' \wedge z \in e' \wedge T$. Suppose $z \notin T$ and let $f \in E$ such that $z \in f \wedge T$. If $f = 0$, then $z = 0 \in e' \wedge T$. Suppose $f > 0$. We have $x' \leqq z \leqq f$ and therefore $x' \in f \wedge T$ and since $x' \mathscr{V} x$, $x \in f \wedge T$. If $t \in (f \wedge T) \cap T$, then $t \in (z \wedge T) \cap T$ since

$z \wedge T \subset f \wedge T$ and $z \notin T$. From the inequality $t \leqq z \leqq e'$ we conclude that $t \in (e' \wedge T) \cap T$ and hence $t \leqq x$. Hence $x = \sup\{(f \wedge T) \cap T\}$ and by Lemma 1 we have $f \wedge T \subset e' \wedge T$ and therefore $z \in e' \wedge T$.

LEMMA 5. *If $e, f \in E$ and $p \in [(f \vee T) \cap (e \wedge T)] \backslash T$, then $\{p\} = (f \vee T) \cap (e \wedge T)$.*

Proof. Suppose $p' \in (f \vee T) \cap (e \wedge T)$. Then either $p' \leqq p$ or $p' \geqq p$ and in either case it is easily seen that $p' \notin T$ since $p \notin T$. Assume that $p' \leqq p$ and let $x = \sup\{(e \wedge T) \cap T\}$. Then since $p, p' \notin T$, $x = \sup\{(p \wedge T) \cap T\} = \sup\{(p' \wedge T) \cap T\}$. Since $p' \leqq p$ on $f \vee T$, we have that $p \in p' \vee T$ so that $p = p' \vee t$ for some $t \in T$ and since $x = \sup\{(p' \wedge T) \cap T\}$, it follows that $t \geqq x$. But $t \leqq p \leqq e$ implies that $t \in (e \wedge T) \cap T$ and so $t \leqq x$. Hence $t = x$ and $p = p' \vee x = p'$.

LEMMA 6. *Let $x \in T$ and let $x' = sup\ Vx$. Then $C_{x'} \neq \square$.*

Proof. The set $\{h \in E \mid x \in h \wedge T\}$ is closed by the continuity of $\wedge$ and is nonempty since $x \in 1 \wedge T$. Let $e = \inf\{h \in E \mid x \in h \wedge T\}$. Then $x \in e \wedge T$ and since $x \mathscr{V} x'$ it follows that $x' \in e \wedge T$. Let $x'' = \sup\{(e \wedge T) \cap T\}$. Then $x'' \leqq x'$. Suppose $h \in E$ and $x \in h \wedge T$. Then $h \geqq e$ by the definition of e and since $x'' \in e \wedge T$ it follows that $x'' \in h \wedge T$. On the other hand, if $x'' \in h \wedge T$ for some $h \in E$, then $x \in h \wedge T$ since $x \leqq x''$. Therefore $x \mathscr{V} x''$ but since $x'' \geqq x'$ and $x' = \sup \mathscr{V} x$, we must have $x'' = x'$. Hence $e \in C_{x'}$.

We are now prepared to define the iseomorphism from L into $I \times I$. For $p \in L$, define

$$\alpha_1(p) = \sup\{(p \wedge T) \cap T\}$$

and

$$\alpha_2(p) = \inf\{(p \vee T) \cap T\} .$$

Let η_1, η_2, denote the natural maps from T onto $T/\mathscr{V} = T_1$ and $T/\mathscr{H} = T_2$ respectively. Let $\phi_1 = \eta_1 \circ \alpha_1$, $\phi_2 = \eta_2 \circ \alpha_2$ and define

$$\phi : L \rightarrow T_1 \times T_2$$

by

$$\phi = \phi_1 \times \phi_2 .$$

THEOREM 1. *If L is a topological lattice which is homeomorphic to a 2-cell, then L is iseomorphic to a sublattice of $I \times I$.*

Proof. We will show that the map defined above is a one-to-one continuous homomorphism from L into $T_1 \times T_2$. The theorem then follows since $T_1 \times T_2$ is iseomorphic to $I \times I$.

(i) The map ϕ is continuous. We show ϕ_1 is continuous. A dual argument shows that ϕ_2 is continuous.

Let $x \in T_1$ and let $a = \sup \eta_1^{-1}(x)$. Then $C_a \neq \Box$ by Lemma 6. Let $e = \sup C_a$. We claim that $\phi_1^{-1}[0, x] = e \wedge L$. A similar argument shows that $\phi_1^{-1}[x, 1] = a' \vee L$ where $a' = \inf \eta_1^{-1}(x)$. Thus the inverse under ϕ_1 of a subbasic closed set is closed in L and hence α_1 is continuous.

Let $z \in e \wedge L$. Then $b = \sup \{(z \wedge T) \cap T\} \leqq z \leqq e$ and so $b \leqq a$. Then $\phi_1(z) = \eta_1(\alpha_1(z)) = \eta_1(b) \leqq \eta_1(a) = x$. Hence $z \in \phi_1^{-1}[0, x]$. Now let $z \in \phi_1^{-1}[0, x]$, $b = \sup \eta_1^{-1}(\phi_1(z))$, and $f = \sup C_b$. Since $\phi_1(z) \leqq x$, then $b \leqq a$. If $z \in T$ then $z \leqq b \leqq a \leqq e$; thus $z \in e \wedge L$.

Now suppose that $z \notin T$. From the definition of b we have $\eta_1(b) = \eta_1(\alpha_1(z))$ and hence $b \mathscr{V}^{\wedge} \alpha_1(z)$. Therefore $\alpha_1(z) \leqq b$. Let $h \in E$ such that $z \in h \wedge T$. Then since $z \notin T$, it was shown in the proof of Lemma 1 that $\sup \{(z \wedge T) \cap T\} = \sup \{(h \wedge T) \cap T\}$. Therefore $\alpha_1(z) \in h \wedge T$ and since $b \mathscr{V}^{\wedge} \alpha_1(z)$, we have $b \in (h \wedge T) \cap T$ and hence $b \in (z \wedge T) \cap T$. Then by the definition of $\alpha_1(z)$, we have $b \leqq \alpha_1(z)$. Thus $\alpha_1(z) = b$, and $(z \wedge T) \cap T = (f \wedge T) \cap T$. By Lemma 1, $z \wedge T \subset f \wedge T$. Since $b \leqq a$, then $f \leqq e$. Hence $z \leqq f \leqq e$ implies that $z \in e \wedge L$.

(ii) ϕ is one-to-one. Suppose $p, p' \in L$ such that $\phi_i(p) = \phi_i(p')$, $i = 1, 2$. We will show that $p = p'$. We consider three cases.

Case 1. $p, p' \in L \backslash T$. Then since $\phi_1(p) = \eta_1(\alpha_1(p)) = \eta_1(\alpha_1(p')) = \phi_1(p')$, we have that $\alpha_1(p) \mathscr{V}^{\wedge} \alpha_1(p')$. Choose $e, f \in E$ such that $p \in e \wedge T$ and $p' \in f \wedge T$. Then from the proof of Lemma 1, it follows that

$$\sup \{(e \wedge T) \cap T\} = \sup \{(p \wedge T) \cap T\} = \alpha_1(p),$$

and

$$\sup \{(f \wedge T) \cap T\} = \sup \{(p' \wedge T) \cap T\} = \alpha_1(p') .$$

But since $\alpha_1(p) \mathscr{V}^{\wedge} \alpha_1(p')$, we must have $\alpha_1(p') \in (e \wedge T) \cap T$ and $\alpha_1(p) \in (f \wedge T) \cap T$. It now follows that $\alpha_1(p') \leqq \alpha_1(p) \leqq \alpha_1(p')$ and hence $\alpha_1(p) = \alpha_1(p') = \alpha_1(e) = \alpha_1(f)$. Hence by Lemma 1, either $f \wedge T \subset e \wedge T$ or $e \wedge T \subset f \wedge T$. Suppose $f \wedge T \subset e \wedge T$. Then $p, p' \in e \wedge T$. Using a similar argument and the dual of Lemma 1 we obtain $g \in E$ such that $p, p' \in g \vee T$. Since $p, p' \notin T$, we conclude from Lemma 5 that $p = p'$.

Case 2. $p, p' \in T$. Assume $p \leqq p'$. If $p' = 1$, then $p' \in 1 \vee T$ and $p' \mathscr{H} p$ implies that $p \in 1 \vee T$ and so $p = 1$. Suppose $p' < 1$ and let $f = \sup \{h \in E \mid p \in h \vee T\}$. Then $f < 1$. Let $f_n \rightarrow f$, where $f_n \in E$ and $f_n > f$ for all n. Then $p \notin f_n \vee T$ and hence $p' \notin f_n \vee T$ for all n. Therefore if $f_n \vee p \in T$, then $f_n \vee p > p'$, and if $f_n \vee p \notin T$ then

$p = (f_n \vee p) \wedge p \in (f_n \vee p) \wedge T$ and hence $p' \in (f_n \vee p) \wedge T$ since $p \mathscr{Y}^c p'$ and $f_n \vee p \notin T$. So $f_n \vee p \geqq p'$ for all n. Therefore, by the continuity of $\vee$, $p = f \vee p \leqq p'$. Then $p = p'$.

Case 3. $p \notin T, p' \in T$. Choose $e, f \in E$ such that

$$p \in (e \wedge T) \cap (f \vee T) .$$

Then since $p \notin T, \{p\} = (e \wedge T) \cap (f \vee T)$ by Lemma 5. Since $\phi_1(p) = \phi_1(p')$, we have $\sup\{(p \wedge T) \cap T\} \mathscr{Y}^c p'$ from which follows $p' \in p \wedge T \cap e \wedge T$. Similarly, $p' \in f \vee T$, contradicting Lemma 5.

(iii) ϕ is a homomorphism. We will show that ϕ_1 is a homomorphism with respect to $\vee$, Similar arguments will show that ϕ_1 is a homomorphism with respect to $\wedge$ and that ϕ_2 is a homomorphism with respect to $\vee$ and $\wedge$.

Let $p, p' \in L; x = \alpha_1(p) = \sup\{(p \wedge T) \cap T\}$,

$$x' = \alpha_1(p') = \sup\{(p' \vee T) \cap T\} ,$$

and

$$z = \alpha_1(p \vee p') = \sup\{((p \vee p') \wedge T) \cap T\} .$$

Assume that $x \leqq x'$. Then $x \vee x' = x'$ and $\eta_1(x \vee x') = \eta_1(x) \vee \eta_1(x') = \eta_1(x')$. Then $\phi_1(p) \vee \phi_1(p') = \eta_1(x) \vee \eta_1(x') = \eta_1(x')$. We will show that $\phi_1(p \vee p') = \eta_1(z) = \eta_1(x')$, i.e., $z \mathscr{Y}^c x'$.

We have that $x' \leqq p' \leqq p \vee p'$, so $x' \in ((p \vee p') \wedge T) \cap T$ and hence $x' \leqq z$. If $z \in e \wedge T$ for $e \in E$, then clearly $x' \in e \wedge T$. Now suppose $x' \in e \wedge T, e \in E$. We consider two cases.

Case 1. $p' \notin E$. We may assume that $e = \inf\{h \in E \mid x' \in h \wedge T\}$. If $p' \in T$, then $p' = x' \in e \wedge T$. If $p' \notin T$, then choose $g \in E$ such that $p' \in g \wedge T$. Then $x' \leqq p' \leqq g$ implies that $x' \in g \wedge T$ and hence $e \leqq g$.

From Lemma 6, $e = \sup C_{x'}$. But the proof of Lemma 1 gives

$$x' = \sup\{(p' \wedge T) \cap T\} = \sup\{(g \wedge T) \cap T\} ,$$

and therefore $g \leqq e$. Hence $g = e$ and $p' \leqq e$.

We will show that $p \leqq e$ also. If $p \in T$, then $p = x \leqq x' \leqq e$. Suppose $p \notin T$ and let $f = \inf\{h \in E \mid p \in h < T\}$. Then since $p \notin T$, $\sup\{(f \wedge T) \cap T\} = \sup\{(p \wedge T) \cap T\} = x \leqq x'$ and hence $f \leqq e$. Then the inequality $p \leqq f \leqq e$ gives the desired conclusion.

We now have $p' \leqq e, p \leqq e$; hence $p \vee p' \leqq e$. Since $p' \in e \wedge T$, the inequality $p' \leqq p \vee p' \leqq e$ and Lemma 4 gives $p \vee p' \in e \wedge T$. Hence $z \in e \wedge T$. This concludes the proof for Case 1.

Case 2. $p' \in E$. If $p' \leqq p$, then $p \vee p' = p$ implies $x = z$. But then $x \leqq x' \leqq z$ implies $x' = z$ and so $z \in e \wedge T$.

If $p' \notin p \wedge L$ then since

$$x = \sup \{(p \wedge T) \cap T\} \leqq x' = \sup \{(p' \wedge T) \cap T\} ,$$

the proof of the continuity of ϕ_1 shows that $p \in p' \wedge L$. Hence $p \vee p' = p'$ and again we conclude that $z = x'$. This concludes the proof that ϕ_1 is a homomorphism with respect to $\vee$, and the proof of Theorem 1.

2. Compact connected lattices in the plane. In [4] Wallace proved that a compact connected lattice L which is imbeddable in the plane is a cyclic chain (in the sense of Whyburn [5]) and that each true cyclic element is a convex sublattice and is homeomorphic to a 2-cell. Thus by Theorem 1, each true cyclic element is iseomorphic to a sublattice of $I \times I$. Let Δ denote the diagonal thread in $I \times I$. Label the true cyclic elements of L, $\{C_i\}_{i=1}^{\infty}$. Denote the 0 and 1 of C_i by a_i and b_i respectively. Let T be any maximal chain from 0 to 1 in L, and let h be an iseomorphism from T onto Δ, the diagonal in $I \times I$. Then the "square" in $I \times I$ with upper right hand vertex $h(b_i)$ and lower left hand vertex $h(a_i)$ is a sublattice of $I \times I$ which is iseomorphic to $I \times I$. Hence C_i may be imbedded in this sublattice as in Theorem 1. In this manner an iseomorphism of L into $I \times I$ is determined. Thus we have proven:

THEOREM 2. *Every compact connected lattice in the plane is iseomorphic to a sublattice of $I \times I$.*

Finally we state an explicit description of the compact connected sublattices of $I \times I$ containing $(0, 0)$ and $(1, 1)$.

THEOREM 3. *Let f and g be functions from I into I satisfying*
(i) *f, g are nondecreasing, $f(0) = 0, g(1) = 1$,*
(ii) *$f(x) \leqq g(x)$ for all $x \in I$,*
(iii) *f is continuous from the left and g is continuous from the right.*
Then the set $L = \{(x, y) : f(x) \leqq y \leqq g(x)\}$ is a compact connected sublattice of $I \times I$ containing $(0, 0)$ and $(1, 1)$. Conversely, if L is a compact connected sublattice of $I \times I$ containing $(0, 0)$ and $(1, 1)$ then there exist functions f and g satisfying i-iii *such that*

$$L = \{(x, y) : f(x) \leqq y \leqq g(x)\} .$$

Proof. The proof is straightforward and will be omitted. The functions f and g alluded to in the second part are defined as follows:

$$g(x) = \sup \{L \cap (\{x\} \times I)\} \qquad \text{for } x \in I$$
$$f(x) = \inf \{L \cap (\{x\} \times I)\} \qquad \text{for } x \in I .$$

3. Comments. Edmondson has given an example of a topological lattice on a 3-cell which is nonmodular; hence this lattice is not a sublattice of $I \times I \times I$ [2]. This shows that the higher dimensional analogous of Theorem 1 are false.

This the result of this paper does not hold if the term "lattice" be replaced by "semilattice" is a consequence of the results of D. R. Brown, [1], regarding semilattice structures on the two-cell.

Wallace has conjectured that every 2-dimensional compact connected lattice with no cutpoints is a two-cell. A related conjecture is that every 2-dimensional compact connected lattice can be imbedded in the plane. If this were true, the words "in the plane" in the statement of Theorem 2 could be replaced by "2-dimensional."

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