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A NOTE ON PARTIALLY ORDERED COMPACTA

J. H. CARRUTH

In this note it is shown that each compact metric space endowed with a closed partial order admits an equivalent radially convex metric. It is also shown that certain functions from subsets of a partially ordered compactum into the Hilbert Cube are extendable to order preserving homeomorphisms on the whole space.

The notion of a radially convex metric on a partially ordered space was introduced by Kock and McAuley in 1964 [1]. A metric d is *radially convex* with respect to a partial order Γ on X if $(x, y) \in \Gamma$, $(y, z) \in \Gamma$, and $y \neq z$ imply $d(x, y) < d(x, z)$. The natural question "Does each partially ordered metric space admit a radially convex metric ?" arises from [1] and [2]. This question is answered in the affirmative for compact spaces. The topological closure of a set $A \subset X$ will be denoted by A^*. The symbol $\square$ will denote the empty set. If Γ is a partial order on X and $A \subset X$ then

$$L(A) = \{x \mid (x, a) \in \Gamma \text{ for some } a \in A\}$$

and

$$M(A) = \{x \mid (a, x) \in \Gamma \text{ for some } a \in A\} .$$

THEOREM 1. *If Γ is a closed partial order on the compact metric space X, then there exists an equivalent metric on X which is radially convex with respect to Γ.*

Proof. Let $\mathscr{V}$ be a countable base for the topology of X and let $\mathscr{U}$ be the collection of finite unions of members of $\mathscr{V}$. Then $\mathscr{U}$ is countable as is

$$\mathscr{B} = \{(U, V) \mid U, V \in \mathscr{U} \text{ and } M(U^*) \cap L(V^*) = \square\} .$$

Let $\{(U_i, V_i) \mid i = 1, 2, \cdots\}$ be an enumeration of $\mathscr{B}$. For each i, the function f_i defined by

$$f_i(z) = \begin{cases} 1 & \text{if } z \in U_i^* \\ 0 & \text{if } z \in V_i^* \end{cases}$$

is continuous and order preserving. Hence, by a result of Nachbin [3], f_i is extendable to a continuous order preserving function g_i from X into $[0, 1]$. Let Φ be the product function Pg_i which takes X into the Hilbert Cube H. Again using results of [3], it is easy to verify that $(x, y) \in (X \times X)\backslash \Gamma$ if and only if there exists an integer i such that $f_i(x) > f_i(y)$. It follows that Φ is an order preserving homeomorphism from X into H whose inverse is also order preserving. Hence, we may assume that X is a subspace of H and that Γ is the natural partial order on H restricted to X. It is obvious that the usual metric on H is radially convex with respect to the natural partial order. It is also obvious that radial convexity is hereditary and the proof is complete.

We note that included in the proof of Theorem 1 is the fact that the Hilbert Cube with the natural partial order is a universal partially ordered compactum in the imbedding sense.

If X is partially ordered by Γ, the element x of X is *maximal* (*minimal*) if $M(x) = \{x\}(L(x) = \{x\})$. The element $\{r_i\}$ of the Hilbert Cube H will be called an *interior point* of H if $0 < r_i < 1$ for each i.

THEOREM 2. *Let Γ be a closed partial order on the compact metric space X. Suppose x is maximal, z is minimal, $x \neq z$, and $Y = \{y_i: i = 1, 2, \cdots, n\}$ is a finite chain (with respect to Γ) of points of X which are neither maximal nor minimal. Let h be a one-to-one order preserving function from Y into H such that $h(y_i)$ is an interior point of H for each i. Then, there exists an order preserving homeomorphism Φ taking X into H whose restriction to Y is h and with the property that $\Phi(x)_i = 1$ and $\Phi(z)_i = 0$ for each integer i.*

Proof. Let $\mathscr{V}$ be a countable base for the topology of X and let $\mathscr{U}$ be the collection of finite union of members of $\mathscr{V}$. Assume that $y_i < y_{i+1}$ for each $i = 1, 2, \cdots, n-1$. Let $\mathscr{B}$ be the collection of $n+2$ tuples $(U, V_1, \cdots, V_n, W)$ satisfying the following properties:

(i) $U, V_1, \cdots, V_n, W \in \mathscr{U}$

(ii) The sets $M(U^*), L(V_1^*) \cap M(V_1^*), \cdots, L(V_n^*) \cap M(V_n^*)$, and $L(W^*)$ are pairwise disjoint.

(iii) $x \in U, y_i \in V_i$ for each i, and $z \in W$.

For each $(U, V_1, \cdots, V_n, W)_i \in \mathscr{B}$, define f_i on

$$U^* \cup (\cup \{V_j^*: j = 1, 2, \cdots, n\}) \cup W^*$$

by

$$f_i(a) = \begin{cases} 1 & \text{if } a \in U^* \\ h(y_j)_i & \text{if } a \in V_j^* \\ 0 & \text{if } a \in W^* . \end{cases}$$

As in the proof of Theorem 1, we extend f_i to g_i taking X into $[0,1]$ and let Φ be the product of the g_i's.

It is not difficult to show that the functions $\{g_i\}$ separate points.

This completes the proof.

Note. The inverse of the homeomorphism Φ in Theorem 2 need not be order preserving. In fact, if $L(x) \neq X$, Φ^{-1} is not order preserving.

The importance of the interiority of the points $h(y_i)$ should not be overlooked. Indeed, if $h(y_i)$ is not an interior point of H for some i, the conclusion of Theorem 2 may fail to hold. The following example exhibits this fact.

EXAMPLE. Let X be the space consisting of the unit circle in the plane along with the arc from $(1, 0)$ to $(2, 0)$. Define Γ by $((a, b), (c, d)) \in \Gamma$ if and only if $a \leqq c$ and $0 \leqq b \cdot d$. Then, Γ is a closed partial order on the compact metric space X. Let $z = (-1, 0)$, $y = (1, 0)$, and $x = (2, 0)$. Now let

$$r_i = \begin{cases} 1 \text{ if } i = 1 \\ 0 \text{ if } i \neq 1 . \end{cases}$$

No order preserving function Φ satisfying $\Phi(z)_i = 0$, $\Phi(y)_i = r_i$, and $\Phi(x)_i = 1$ for each i can be a homeomorphism.

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UNIVERSITY OF TENNESSEE