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NONLINEAR ELLIPTIC CONVOLUTION EQUATIONS OF WIENER-HOPF TYPE IN A BOUNDED REGION

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The existence of a solution of a nonlinear perturbation of an elliptic convolution equation of Wiener-Hopf type in a bounded region G of R^n is proved. More explicitly, let A be an elliptic convolution operator on G of order $\alpha, \alpha > 0$; A_j the principal part of A in a local coordinate system and $\tilde{A}_j(x^j, \xi)$ be the symbol of A_j with a factorization with respect to ξ_n of the form: $\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$ for $x_n^j = 0$. $\tilde{A}_j^+, \tilde{A}_j^-$ are homogeneous of orders $0, \alpha$ in ξ respectively; the first admitting an analytic continuation in $\operatorname{Im} \xi_n > 0$, the second in $\operatorname{Im} \xi_n \leqq 0$. Let T_k, $k = 0, \cdots, [\alpha] - 1$ be bounded linear operators from $H_+^k(G)$ into $L^2(G)$ where $H_+^k(G)$, $k \geqq 0$ are the Sobolev-Slobo detskii spaces of generalized functions.

The purpose of the paper is to prove the solvability of: $Au_+ + \lambda^\alpha u_+ = f(x, T_0u_+, \cdots, T_{[\alpha]-1}u_+)$ on G; u_+ in $H_+^\alpha(G)$ for large $|\lambda|$ and on a ray $\arg \lambda = \theta$ such that $\tilde{A}_j + \lambda^\alpha \neq 0$ for $|\xi| + |\lambda| \neq 0$ and for all j. $f(x, \zeta_0, \cdots, \zeta_{\alpha-1})$ has at most a linear growth in $(\zeta_0, \cdots, \zeta_{\alpha-1})$ and is continuous in all the variables.

Linear elliptic convolution equations in a bounded region for arbitrary α and with symbols having the above type of factorization ($\lambda = 0$) have been considered recently by Visik-Eskin [3]. Those equations are similar to integral equations since no boundary conditions are required.

The notation and terminology are those of Visik-Eskin and are given in § 1. The theorems are proved in § 2.

1. Let s be an arbitrary real number and $H^s(R^n)$ be the Sobolev-Slobodetskii space of (generalized) functions f such that:

$$||f||_s^2 = \int_{E^n} (1 + |\xi|^2)^s |\tilde{f}(\xi)|^2 d\xi < +\infty$$

where $\tilde{f}(\xi)$ is the Fourier transform of f.

We denote by $H^s(R_+^n)$, the space consisting of functions defined on $R_+^n = \{x: x_n > 0\}$ and which are the restrictions to R_+^n of functions in $H^s(R^n)$. Let lf be an extension of f to R^n, then:

$$||f||_s^+ = ||f||_{H^s(R_+^n)} = \inf ||lf||_s .$$

The infimum is taken over all extensions lf of f.

The $\overset{\circ}{H}{}_0^+ = \{f_+; f_+(x) = f(x) \text{ if } x_n > 0, f \in L^2(R^n), f_+(x) = 0 \text{ if } x_n \leqq 0\}$

and similarly for $\overset{\circ}{H}{}_0^-$.

We denote by H_s^+, the space of functions f_+ with f_+ in $\overset{\circ}{H}{}_0^+$ and $f_+ \in H^s(R_+^n)$ on R_+^n.

$\overset{\circ}{H}{}_s^+$ is the subspace of $H^s(R^n)$ consisting of functions with supports in cl (R_+^n). $\tilde{H}_s^+$, $\tilde{H}_s$, $\tilde{\overset{\circ}{H}}{}_s^+$ denote respectively the spaces which are the Fourier images of H_s^+, $H^s(R^n)$, $\overset{\circ}{H}{}_s^+$.

Let $\tilde{f}(\xi)$ be a smooth decreasing (i.e., $|\tilde{f}(\xi)| \leqq M|\xi_n|^{-1-\varepsilon}$ for large $|\xi_n|$ and for some $\varepsilon > 0$) function. The operator Π^+ is defined as:

$$\Pi^+ \tilde{f}(\xi) = \frac{1}{2}\tilde{f}(\xi) + i(2\pi)^{-1}\,\text{v.p.}\int_{-\infty}^{\infty} \tilde{f}(\xi', \eta_n)(\xi_n - \eta_n)^{-1} d\eta_n$$

where $\xi' = (\xi_1, \cdots, \xi_{n-1})$.

For any $\tilde{f}$, then the above relation is understood as the result of the closure of the operator Π^+ defined on the set of smooth and decreasing functions.

Π^+ is a bounded mapping from $\tilde{H}_s$ into $\tilde{\overset{\circ}{H}}{}_s^+$ if $0 \leqq s < 1/2$ and is a bounded mapping from $\tilde{H}_s$ into $\tilde{H}_s^+$ if $s \geqq 1/2$.

Set: $\xi_- = \xi_n - i|\xi'|$; $(\xi_- - i)^s$ is analytic for any s if $\text{Im}\,\xi_n \leqq 0$ and:

$$||f||_s^+ = ||\Pi^+(\xi_- - i)^s l\tilde{f}(\xi)||_0$$

where lf is any extension of f to R^n (Cf. [3], p. 93 relation (8.1)).

Let G be a bounded open set of R^n with a smooth boundary. $H^s(G)$ denotes the restriction to G of functions in $H^s(R^n)$ with the norm:

$$||u||_s = \inf ||v||_{H^s(R^n)}; \qquad v = u \text{ on } G .$$

By $H_+^s(G)$, we denote the space of functions f defined on all of R^n, equal to 0 on $R^n/\text{cl}(G)$ and coinciding in cl G with functions in $H^s(G)$.

Definition 1. $\tilde{A}(\xi)$ is in 0_α if and only if:
(i) $\tilde{A}(\xi)$ is a homogeneous function of order α in ξ.
(ii) $\tilde{A}$ is continuous for $\xi \neq 0$.

Definition 2. $\tilde{A}_+(\xi)$ is in 0_α^+ if and only if:
(i) $\tilde{A}_+(\xi)$ is in 0_α.
(ii) $\tilde{A}_+(\xi', \xi_n)$ has an analytic continuation with respect to ξ_n in the half-plane $\text{Im}\,\xi_n > 0$ for each ξ'.

Similar definition for 0_α^-:

Definition 3. $\tilde{A}$ is in E_α if and only if:
(i) $\tilde{A}$ is in 0_α.

(ii) $\tilde{A}(\xi) \neq 0$ for $\xi \neq 0$.

(iii) $\tilde{A}(\xi)$ has, for $\xi' \neq 0$, continuous first order derivatives, bounded if $|\xi| = 1$, $\xi' \neq 0$.

DEFINITION 4. $\tilde{A}(x, \xi', \xi_n)$ is in D_α^0 if and only if:

(i) $\tilde{A}(x, \xi)$ is infinitely differentiable with respect to x and ξ; $\xi \neq 0$.

(ii) $\tilde{A}(x, \xi)$ is in 0_α for x in R^n.

(iii) $a_{k2}(x) = \frac{\partial^k}{(\partial\xi')^k} \tilde{A}(x, 0, -1) = (-1)^k \exp(-i\alpha\pi) \frac{\partial^k}{(\partial\xi')^k} \tilde{A}(x, 0, 1)$

x in R^n; $0 \leqq |k| < \infty$; $k = (k_1, \cdots, k_n)$.

DEFINITION 5. Let A be a bounded linear operator from H_s^+ into $H^{s-\alpha}(R_+^n)$. Then any bounded linear operator T from H_{s-1}^+ into $H^{s-\alpha}(R_+^n)$, (or from H_s^+ into $H^{s-\alpha+1}(R_+^n)$) is called a right (left) smoothing operator with respect to A.

T is a smoothing operator with respect to A if it is both a left ane right smoothing operator.

Let $\tilde{A}(\xi)$ be in 0_α for $\alpha > 0$. For u_+ in H_s^+, $s \geqq 0$, with support in $\mathrm{cl}\,(R_+^n)$, set: $Au_+ = F^{-1}(\tilde{A}(\xi)\tilde{u}_+(\xi))$ where F^{-1} is the inverse Fourier transform. It is well defined in the sense of generalized functions. A is a bounded linear operator from H_s^+ into $H^{s-\alpha}(R^n)$.

Let $\tilde{A}(x, \xi)$ be an element of E_α for each x in $\mathrm{cl}\, G$ and $\tilde{A}(x, \xi)$ be infinitely differentiable with respect to x and ξ. Since G is a bounded set of R^n, we may assume that G is contained in a cube of side $2p$ centered at 0. We extend $\tilde{A}(x, \xi)$ with respect to x to all of R^n by setting $\tilde{A}(x, \xi) = 0$ if $|x| \geqq p - \varepsilon$ for $\varepsilon > 0$. We get a finite function, homogeneous of order α with respect to ξ.

We take the expansion into Fourier series of $\tilde{A}(x, \xi)$:

$$\tilde{A}(x, \xi) = \sum_{k=-\infty}^{\infty} \psi_0(x) \exp[(i\pi kx)/p]\tilde{L}_k(\xi) \; ; \qquad k = (k_1, \cdots, k_n)$$

where:

$$\tilde{L}_k(\xi) = (2p)^{-n} \int_{-p}^{p} \exp[(-i\pi kx)/p]\tilde{A}(x, \xi)dx$$

$\psi_0(x) = 1$ for $|x| \leqq p - \varepsilon$; $\psi_0(x) = 0$ for $|x| \geqq p$; $\psi_0(x) \in C_c^\infty(R^n)$. We have: $|\tilde{L}_k(\xi)| \leqq C|\xi|^\alpha (1 + |k|)^{-M}$ for arbitrary positive M. Let u_+ be in $H_+^s(G)$, we define:

$$Au_+ = \sum_{-\infty}^{\infty} \psi_0(x)[\exp((ikx\pi)/p)]L_k * u_+ \tag{1.1}$$

where $L_k * u_+ = L_k u_+$ is defined as before since $\tilde{L}_k(\xi)$ is independent of x.

Denote by P^+, the restriction operator of functions defined on R^n to G. We consider an elliptic convolution equation of order α, on G of the form:

$$P^+Au_+ = \sum_j P^+\varphi_j A\psi_j u_+ + Tu_+ \tag{1.2}$$

T is a smoothing operator. The φ_j is a finite partition of unity corresponding to a covering N_j of $\operatorname{cl} G$ with $\operatorname{diam}(N_j)$ sufficiently small. The ψ_j are in $C_c^\infty(R^n)$ with $\varphi_j\psi_j = \varphi_j$ and $\operatorname{supp}(\psi_j) \subseteq N_j$.

Suppose $\tilde{A} \in D_\alpha^0$, then the operator $\varphi_j A\psi_j$ taken in local coordinates may be written as:

$$\varphi_j A\psi_j = \varphi_j A_j \psi_j + T_j$$

where A_j is a convolution operator of the form (1.1) and T_j is a smoothing operator (Cf. [3] Appendix 2).

2. The main result of the paper is the following theorem:

THEOREM 1. *Let A be an elliptic convolution operator on G, of order $\alpha > 0$, and of the form* (1.2). *Suppose that:*

(i) $\tilde{A}_j(x^j, \xi) \in E_\alpha \cap D_\alpha^0$.

(ii) $\tilde{A}_j(x^j, \xi)$ *has for $x_n^j = 0$ a factorization of the form:*

$$\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$$

where $\tilde{A}_j^+ \in 0_0^+$; $\tilde{A}_j^- \in 0_\alpha^-$ for all $x^j \in N_j \cap G$.

(iii) *There exists a ray $\arg\lambda = \theta$ such that $\tilde{A}_j(x^j, \xi) + \lambda^\alpha \neq 0$ for $|\xi| + |\lambda| \neq 0$, $x^j \in N_j \cap G$.*

Let $f(x, \zeta_0, \cdots, \zeta_{[\alpha]-1})$ be a function measurable in x on G, continuous in all the other variables. Suppose there exists a positive constant M such that:

$$|f(x, \zeta_0, \cdots, \zeta_{[\alpha]-1})| \leqq M\left\{1 + \sum_{j=0}^{[\alpha]-1} |\zeta_j|\right\}.$$

Let T_k; $k = 0, \cdots, [\alpha] - 1$ be bounded, linear operators from $H_+^k(G)$ into $L^2(G)$. Then for $|\lambda| \geqq \lambda_0 > 0$; $\arg\lambda = \theta$; there exists a solution u in $H_+^\alpha(G)$ of:

$$P^+(A + \lambda^\alpha)u_+ = f(x, T_0u_+, \cdots, T_{[\alpha]-1}u_+) \quad \text{on } G.$$

The solution is unique if f satisfies a Lipschitz condition in $(\zeta_0, \cdots, \zeta_{[\alpha]-1})$.

To prove the theorem, we shall do as in [2]. First, following Visik-Agranovich [4], we establish an *a priori* estimate and show the existence and the uniqueness of a solution of a linear elliptic convolution equation depending on a large parameter in a bounded region. Then we use the Leray-Schauder fixed point theorem to prove Theorem 1.

We have:

THEOREM 2. *Let A be an elliptic convolution operator, of order $\alpha > 0$, of the form* (1.2). *Suppose that all the hypotheses of Theorem* 1 *are satisfied. Let $f \in L^2(G)$; then there exists a unique solution u_+ in $H_+^\alpha(G)$ of*:

$$P^+(A + \lambda^\alpha)u_+ = f \text{ on } G; \; |\lambda| \geqq \lambda_0 > 0 \qquad \arg \lambda = \theta \, .$$

Moreover:

$$\| u_+ \|_\alpha + |\lambda|^\alpha \| u_+ \|_0 \leqq M \| f \|_0$$

where M is independent of λ, u_+.

Proof of Theorem 1. Let v be an element of $H_+^\alpha(G)$ and $0 \leqq t \leqq 1$. Consider the linear elliptic convolution equation:

$$P^+(Au_+ + \lambda^\alpha u_+) = f(x, tT_0 v, \cdots, tT_{[\alpha]-1} v) \, .$$

With the hypotheses of the theorem, $f(x, tT_0 v, \cdots, tT_{[\alpha]-1} v)$ is in L^2(G). It follows from Theorem 2 that there exists a unique solution u_+ in $H_+^\alpha(G)$ of the problem.

Let $\mathscr{A}(t)$ be the nonlinear mapping from $[0, 1] \times H_+^\alpha(G)$ into $H_+^\alpha(G)$ defined by $\mathscr{A}(t)v = u_+$ where u_+ is the unique solution of the above problem.

The theorem is proved if we can show that $\mathscr{A}(1)$ has a fixed point.

PROPOSITION 1. $\mathscr{A}(t)$ is a completely continuous mapping from $[0, 1] \times H_+^\alpha(G)$ into $H_+^\alpha(G)$.

Proof. (i) $\mathscr{A}(t)$ is continuous. Suppose that $t_n \to t$; t_n, t in $[0, 1]$ $v_n \to v$ in $H_+^\alpha(G)$. Set: $u_n = \mathscr{A}(t_n)v_n$. Then from Theorem 2, we get:

$$\begin{aligned} \| u_n - u \|_\alpha \leqq M \| f(\cdot, t_n T_0 v_n, \cdots, t_n T_{[\alpha]-1} v_n) \\ - f(\cdot, tT_0 v, \cdots, tT_{[\alpha]-1} v) \|_0 \, . \end{aligned}$$

It follows from Lemmas 3.1 and 3.2 of [1] that $u_n \to u$ in $H_+^\alpha(G)$.

(ii) $\mathscr{A}(t)$ is compact. Suppose that $\| v_n \|_\alpha \leqq M$. Then from the weak compactness of the unit ball in a Hilbert space and from the generalized Sobolev imbedding theorem, we get:

$$v_{n_j} \to v \text{ weakly in } H_+^\alpha(G) \text{ and strongly in } H_+^{\alpha-\varepsilon}(G); \; 0 < \varepsilon, \alpha - \varepsilon \geqq 0 \, .$$

Applying the argument of the first part, we get the compactness of $\mathscr{A}(t)$.

PROPOSITION 2. $I - \mathscr{A}(0)$ is a homeomorphism of $H_+^\alpha(G)$ into itself. If $v = \mathscr{A}(t)v$, for $0 < t \leqq 1$; then: $\| v \|_\alpha \leqq M$ where M is independent of t.

Proof. The first assertion is trivial.
Suppose that $v = \mathscr{A}(t)v$. It follows from Theorem 2 that:

$$\| v \|_\alpha + |\lambda|^\alpha \| v \|_0 \leqq M \| f(\cdot, tT_0 v, \cdots, tT_{[\alpha]-1} v) \|_0 \\ \leqq M\{1 + \| v \|_{[\alpha]-1}\} .$$

It is well-known that:

$$\| v \|_{[\alpha]-1} \leqq 1/2M \| v \|_\alpha + C \| v \|_0 .$$

Taking $|\lambda|$ sufficiently large, we have: $\| v \|_\alpha \leqq M_2$. $\mathscr{A}(t)$ satisfies the hypotheses of the Leray-Schauder fixed point theorem (the uniform continutiy condition as in [2] is not necessary). So $\mathscr{A}(1)$ has a fixed point, i.e. $\mathscr{A}(1)u_+ = u_+$.

The uniqueness of the solution in the case $f(x, \zeta_0, \cdots, \zeta_{[\alpha]-1})$ satisfies a Lipschitz condition in $(\zeta_0, \cdots, \zeta_{[\alpha]-1})$ follows trivially from the estimate of Theorem 2. We shall not reproduce it.

Proof of Theorem 2. As usual, we consider first the case of the positive half-space R_+^n with the convolution operator A having a constant symbol.

LEMMA 1. *Let $\tilde{A}(\xi)$ be an element of E_α, $(\alpha > 0)$. Suppose that: $\tilde{A}(\xi) = \tilde{A}_+(\xi)\tilde{A}_-(\xi)$ with $\tilde{A}_+(\xi)$ in 0_0^+, $\tilde{A}_-(\xi)$ in 0_α^-. Let P^+ be the restriction operator of functions in R^n to R_+^n and A be the convolution operator with symbol $\tilde{A}(\xi)$. Suppose there exists a ray* $\arg \lambda = \theta$ *such that: $\tilde{A}(\xi) + \lambda^\alpha \neq 0$ for $|\xi| + |\lambda| \neq 0$. If f is in $H^0(R_+^n)$, then there exists a unique solution u in H_α^+ of:*

$$P^+(A + \lambda^\alpha)u_+ = f \text{ on } R_+^n; \ |\lambda| \geqq \lambda_0 > 0 .$$

Moreover:

$$\| u_+ \|_\alpha^+ + |\lambda|^\alpha \| u_+ \|_0^+ \leqq M \| f \|_0^+$$

where M is independent of λ, u_+, f.

Proof. Set $\tilde{A}(\xi, \lambda) = \tilde{A}(\xi) + \lambda^\alpha$. It is homogeneous of order α in (ξ, λ). Since $\tilde{A}(\xi)$ is in E_α, we have the following factorization with respect to ξ_n, which is unique up to a constant multiplier:

$$\tilde{A}(\xi) = \tilde{A}_+(\xi)\tilde{A}_-(\xi)$$

(Cf. Theorem 1.2 of [3], p. 95). The same proof with $\xi_+ = \xi_n + i|\xi'|$ replaced by $\xi_+^\lambda = \xi_n + i(|\lambda| + |\xi'|)$ and ξ_- replaced by:

$$\xi_-^\lambda = \xi_n - i(|\lambda| + |\xi'|)$$

gives:

$$\tilde{A}(\xi, \lambda) = \tilde{A}_+(\xi, \lambda)\tilde{A}_-(\xi, \lambda) .$$

Moreover:

If $\tilde{A}_+(\xi)$ is in 0_0^+, then $\tilde{A}_+(\xi, \lambda)$ is also in O_0 (is homogeneous of order 0 in (ξ, λ)). Similarly for $\tilde{A}_-(\xi, \lambda)$.

Let $lf(x)$ be an extension of f to R^n. Consider:

$$\tilde{u}_+(\xi) = (\tilde{A}_+(\xi, \lambda))^{-1} \prod{}^+ l\tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} .$$

For $|\lambda| \neq 0$, $\tilde{u}_+(\xi)$ has an analytic continuation in $\operatorname{Im} \xi_n > 0$ and:

$$\int |\tilde{u}_+(\xi', \xi_n + i\tau)|^2 \, d\xi' d\xi_n \leqq C ,$$

C is independent of $\tau > 0$. So: $\tilde{u}_+(\xi) \in \tilde{\mathring{H}}_0^+$. (Cf. [3], p. 91).

We get:

$$\begin{aligned} \| u_+ \|_\alpha^+ &= \| \prod{}^+ (\xi_- - i)^\alpha \tilde{u}_+(\xi) \|_0^+ \\ &\leqq \| (\xi_- - i)^\alpha (\tilde{A}_+(\xi, \lambda))^{-1} \prod{}^+ l\tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_0 . \end{aligned}$$

Since $\tilde{A}_+(\xi, \lambda)$ is homogeneous of order 0 in (ξ, λ), we have:

$$\tilde{A}_+(\xi, \lambda) = \tilde{A}_+(\xi/(|\xi| + |\lambda|), \lambda/(|\xi| + |\lambda|)) .$$

Let $c = \operatorname{Min} |\tilde{A}_+(\xi, \lambda)|$ for $|\xi| + |\lambda| = 1, \arg \lambda = \theta$. Then $c > 0$ and is independent of λ.

So:

$$\begin{aligned} \| u_+ \|_\alpha^+ &\leqq c^{-1} \| (\xi_- - i)^\alpha \prod{}^+ l\tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_0 \\ &\leqq C \| l\tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_\alpha . \end{aligned}$$

We may write:

$$\tilde{A}_-(\xi, \lambda) = (|\xi| + |\lambda|)^\alpha \, \tilde{A}_-(\xi/(|\xi| + |\lambda|), \lambda/(|\xi| + |\lambda|)) .$$

Let $C = \operatorname{Min} |\tilde{A}_-(\xi, \lambda)|$ for $|\xi| + |\lambda| = 1, \arg \lambda = \theta$. Then $C > 0$ and is independent of λ.

We obtain:

$$\| u_+ \|_\alpha^+ \leqq C \| l\tilde{f}(\xi) \|_0 \leqq C_2 \| f \|_0^+ .$$

A similar argument gives:

$$\| u_+ \|_0^+ \leqq C |\lambda|^{-\alpha} \| f \|_0^+ .$$

So:

$$\| u_+ \|_\alpha^+ + |\lambda|^\alpha \| u_+ \|_0^+ \leqq C \| f \|_0^+ .$$

C is independent of λ, f, u_+.

A direct verification shows that u_+ is a solution of the equation. It remains to show that the solution is unique. Let v_+ be an element of H_α^+. Suppose that v_+ is also a solution of the equation. Then as in [3], $\tilde{v}_+(\xi)$, its Fourier transform is given by an expression of the same form as $\tilde{u}_+(\xi)$ with $\widetilde{lf}(\xi)$ replaced by $\widetilde{l_1 f}(\xi)$. $l_1 f$ being an extension of f to R^n.

Set $l_2 f = lf - l_1 f$. Then $l_2 f \in H_0^-$, so $\widetilde{l_2 f} \in \overset{\circ}{\tilde{H}}{}_0^-$. $\widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}$ is analytic in $\operatorname{Im} \xi_n \leqq 0$ for $|\lambda| \neq 0$ and moreover:

$$\int |\widetilde{l_2 f}(\xi', \xi_n + i\tau)|^2 |\tilde{A}_-(\xi', \xi_n + i\tau)|^{-2} d\xi' d\xi_n \leqq C$$

where C is independent of $\tau \leqq 0$.

Hence $\widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}$ is in $\overset{\circ}{\tilde{H}}{}_0^-$ (Cf. [3], p. 91), so:

$$\Pi^+ \widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1})) = 0 .$$

Therefore: $\tilde{A}_+(\xi, \lambda)(\tilde{u}_+(\xi) - \tilde{v}_+(\xi)) = 0$.

But $\tilde{A}_+(\xi, \lambda) \neq 0$ for $|\lambda| \neq 0$, we get $\tilde{u}_+ = \tilde{v}_+$. Q.E.D.

Set:

$$A_1 u = \sum_{k=-\infty}^{\infty} \psi_0(x) \exp [(ik\pi x)/p] L_k * u$$

$$A_0 u = \sum_{k=-\infty}^{\infty} \psi_0(x_0) \exp [(ik\pi)/p] L_k * u$$

where L_k, ψ_0 are as in § 1.

LEMMA 2. *Let A_1, A_0 be as above and $\psi(x)$ be in $C_c^\infty(R^n)$ with $\psi(x) = 0$ for $|x - x_0| > \delta$; $|\psi(x)| \leqq K$ where K is independent of δ. Then:*

$$\| \psi(A_1 - A_0) u \|_{s-\alpha}^+ \leqq C\delta \| u \|_s^+ + C(\delta) \| u \|_{s-1}^+$$

for all u in H_s^+, $s \geqq 0$.

Proof. Cf. Lemma 4.7 of [3], p. 119.

Proof of Theorem 2 (continued). (1) First, we establish an *a-priori* estimate of the solutions.

Consider:

$$P^+\varphi_j A\psi_j u_+ + \lambda^\alpha P^+(\varphi_j u_+) = P^+(\varphi_j f) - Tu_+$$

where T is a smoothing operator with respect to $\varphi_j A\psi_j$.

It has been shown in [3] (Appendix 2) that in a local coordinates system, the operator $\varphi_j A\psi_j$ becomes: $\varphi_j A_j\psi_j + T_j$ where A_j has for symbol $\tilde{A}_j(x^j, \xi)$ and T_j is a smoothing operator.

So, we have:

$$P^+\varphi_j A_j(\psi_j u_+) + \lambda^\alpha P^+(\varphi_j u_+) = P^+(\varphi_j f) + T_j^2 u_+$$

where T_j^2 is again a smoothing operator.

Let A_{j0} be the convolution operator with symbol $\tilde{A}_j(x_0^j, \xi)$ evaluted at the point x_0^j. We write:

$$P^+\varphi_j A_{j0}(\psi_j u_+) + \lambda^\alpha P^+(\varphi_j u_+) = P^+(\varphi_j f) + T_j^2 u_+ + P^+\varphi_j(A_{j0} - A_j)\psi_j u^+ .$$

Applying Lemma 4.D.1 of [3] (p. 145), we have:

$$P^+\varphi_j A_{j0}(\psi_j u_+) = P^+ A_{j0}(\varphi_j u_+) + T_j^3 u_+$$

where T_j^3 is a smoothing operator.

Therefore:

$$(A_{j0} + \lambda^\alpha)\varphi_j u_+ = \varphi_j f + T_j^4 u_+ + \varphi_j(A_{j0} - A_j)(\psi_j u_+) .$$

The symbols $\tilde{A}_{j0}$ satisfy the hypotheses of Lemma 1. Applying Lemma 1; 2, we obtain:

$$\|\varphi_j u_+\|_\alpha^+ + |\lambda|^\alpha \|\varphi_j u_+\|_0^+ \leqq M\{\|\varphi_j f\|_0^+ + \|u_+\|_0 + 1/2M\|u_+\|_\alpha + \|\psi_j u_+\|_\alpha^+ + \|\varphi_j u_+\|_0^+\}$$

where we have used the well-known inequality:

$$\|u_+\|_{\alpha-1} \leqq \varepsilon\|u_+\|_\alpha + C(\varepsilon)\|u_+\|_0 .$$

On the other hand: $\|\psi_j u_+\|_\alpha^+ \leqq M\|u_+\|_\alpha$. Summing with respect to j, we get:

$$\|u_+\|_\alpha + |\lambda|^\alpha\|u_+\|_0 \leqq M\{\|f\|_0 + 1/2M\|u_+\|_\alpha + \delta\|u_+\|_\alpha + K\|u_+\|_0\} .$$

Taking δ small and $|\lambda|$ sufficiently large, we have:

$$\|u_+\|_\alpha + |\lambda|^\alpha\|u_+\|_0 \leqq M\|f\|_0 .$$

So, if there exists a solution, then the solution is unique.

(2) It remains to show the existence of a solution. From Lemma 1, we know that $P^+(A_{j0} + \lambda^\alpha)$ has an inverse R_{j0}. Let $\hat{R}_{j0}$ be the operator R_{j0} expressed in the global system of coordinates of G. Consider:

$$Rf = \sum_j \varphi_j \hat{R}_{j0}(\psi_j f) .$$

R is a bounded linear mapping from $L^2(G)$ into $H_+^\alpha(G)$.

We show that: $\mathscr{A}Rf = P^+(A + \lambda^\alpha)Rf = f + \mathscr{C}f$ with $\|\mathscr{C}\| \leqq 1/2$.

We have:

$$\mathscr{A}Rf = \sum_j P^+(A + \lambda^\alpha)\varphi_j \psi_j \hat{R}_{j0}(\psi_j f) .$$

Applying Lemma 4.D.1. of [3], we may write:

$$\mathscr{A}Rf = \sum_j P^+\varphi_j(A + \lambda^\alpha)\psi_j \hat{R}_{j0}(\psi_j f) + TRf$$

where T is a smoothing operator.

We express $\varphi_j(A + \lambda^\alpha)\psi_j \hat{R}_{j0}(\psi_j f)$ in local coordinates. We get:

$$\varphi_j(A_{j0} + \lambda^\alpha)\psi_j R_{j0}(\psi_j f) + \varphi_j(A_j - A_{j0})\psi_j R_{j0}(\psi_j f) + T_j^1 R_{j0}(\psi_j f) .$$

Using Lemma 4.D.1 of [3] again, we obtain:

$$\begin{aligned} &\varphi_j(A_{j0} + \lambda^\alpha)R_{j0}(\psi_j f) + \varphi_j(A_j - A_{j0})\psi_j R_{j0}(\psi_j f) + T_j^2 R_{j0}(\psi_j f) \\ &= T_j^2 R_{j0}(\psi_j f) + \varphi_j f + \varphi_j(A_j - A_{j0})\psi_j R_{j0}(\psi_j f) = \varphi_j f + \mathscr{C}_j(\psi_j f) . \end{aligned}$$

The T_j are all smoothing operators.

Applying Lemma 1, we have:

$$\| T_j^2 R_{j0}(\psi_j f) \|_0^+ \leqq C \| R_{j0}(\psi_j f) \|_{\alpha-1}^+ \leqq \varepsilon \| f \|_0 + C |\lambda|^{-\alpha} \| f \|_0 .$$

From Lemmas 1 and 2, we get:

$$\begin{aligned} \| \varphi_j(A_j - A_{j0})\psi_j R_{j0}(\psi_j f) \|_0^+ &\leqq \delta \| \psi_j R_{j0}(\psi_j f) \|_\alpha^+ \\ &\quad + C(\delta) \| \psi_j R_{j0}(\psi_j f) \|_{\alpha-1}^+ \\ &\leqq \delta \| f \|_0 + C(\delta) \| \hat{R}_{j0}(\psi_j f) \|_{\alpha-1} \\ &\leqq \delta \| f \|_0 + \varepsilon C(\delta) \| R_{j0}(\psi_j f) \|_\alpha \\ &\quad + C(\delta)M(\varepsilon) \| \hat{R}_{j0}(\psi_j f) \|_0 \\ &\leqq \{\delta + \varepsilon C(\delta)\} \| f \|_0 \\ &\quad + |\lambda|^{-\alpha} M(\varepsilon) C(\delta) \| f \|_0 . \end{aligned}$$

Taking ε, δ small, $|\lambda|$ large enough, we have:

$$\| \mathscr{C}_j(\psi_j f) \|_0^+ \leqq \frac{1}{4N} \| f \|_0 .$$

We obtain:

$$Rf = f + TRf + \sum_j \hat{\mathscr{C}}_j(\psi_j f) = f + \mathscr{C}f$$

where $\hat{\mathscr{C}}_j$ is the operator $\mathscr{C}_j$ expressed in the global coordinates system of G. We obtain: $\|\mathscr{C}f\|_0 \leqq 1/4\,\|f\|_0 + 1/4\,\|f\|_0$ for large $|\lambda|$.

Hence $\|\mathscr{C}\| \leqq 1/2$; therefore $(I + \mathscr{C})^{-1}$ exists. We define:

$$\mathscr{A}^{-1} = R(I + \mathscr{C})^{-1} .$$

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