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EXTREME POINTS AND DIMENSION THEORY

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The purpose of this paper is to characterize the topological dimension of a compact metric space X in terms of the extremal structure of the unit ball of the spaces $C(X, R_n)$, where R_n denotes Euclidean n-space with the usual Euclidean norm and $C(X, R_n)$ denotes the space of continuous maps of X into R_n, normed by the sup norm. The main results are that if $n \geqq 2$, the unit ball of $C(X, R_n)$ is always the closed convex hull of its extreme points, and that if the unit ball of $C(X, R_n)$ is actually equal to the convex hull of its extreme points, then the dimension of X is less than n. If n is even, the converse of the second assertion above is shown to be true, and with additional assumptions on X, the converse of the second assertion holds whether n is even or odd.

In the last half of the paper, the corresponding questions for the spaces $C(X, N)$ are studied, where N is an infinite-dimensional strictly convex normed space and $C(X, N)$ is the space of continuous maps of X into N, again with the sup norm. Here it is established that the unit ball of $C(X, N)$ is always the convex hull of its extreme points.

We will be studying spaces $C(X, N)$, where N is either finite-dimensional Euclidean space or an infinite-dimensional strictly convex normed space. If $| \ |$ is the norm on $N, C(X, N)$ is normed by $||f|| = \sup_{x \in X} |f(x)|$. Let U_N denote the (closed) unit ball of $C(X, N)$ and let E_N denote the set of extreme points of U_N; then it is clear that E_N is the set of all continuous maps of X into the surface of the unit ball of N. In case N is n-dimensional Euclidean space, we let U_N be represented by U_n; similarly E_N will be represented by E_n. When no confusion can arise we will sometimes drop the subscript N on U_N and E_N.

It is to be emphasized that all the hypotheses on X are not always needed; we elaborate this in the remarks at the end of the paper.

A theorem in Bade [1] states that U_1 is the closed convex hull of E_1 if and only if X is totally disconnected. Phelps [6] proved that U_2 is always the closed convex hull of E_2; a simpler proof was given by Sine [7]. Related results were obtained by Goodner [2] for the case $n = 1$; here, compactness of X was not assumed.

1. Mappings into Euclidean spaces. We begin with

THEOREM 1. *If $n \geqq 2$, U_n is equal to the closed convex hull of*

E_n.

Proof. Our basic tool is the construction used by Sine in [7], with a suitable modification. By S_{n-1} we will mean the surface of the unit sphere in R_n. If α and β are (small) positive numbers and x_0 is a point of S_{n-1}, let $B(x_0, \alpha) = \{z \in S_{n-1}: |z - x_0| < \alpha\}$ and let $W(x_0, \alpha, \beta)$ equal the convex hull of $(B(x_0, \alpha) \cup \{-\beta x_0\})$. Any set of the form $W(x_0, \alpha, \beta)$ will be called a *wedge*; $-\beta x_0$ will be called the *vertex* of the wedge.

Now let f be in U_n and let $\varepsilon > 0$. Let k be a positive integer such that $(1/k) < \varepsilon$; it is not hard to see that wedges $W_1, \cdots, W_k$ can be chosen so that the wedges W_i are pairwise disjoint outside the set $\{z \in R_n: |z| \leqq \varepsilon\}$. (Choose α_i relatively small in comparison with β_i if $W_i = W(x_i, \alpha_i, \beta_i)$). Let φ_i be the following retraction of the unit ball in R_n onto the unit ball with the (relative) interior of the wedge W_i removed: If z is in W_i, $\varphi_i(z)$ is obtained by projecting z parallel to x_i until it hits the boundary of W_i. If z is not in W_i, $\varphi_i(z) = z$. The number β_i can be chosen $< \varepsilon$; then $|\varphi_i(z)| \leqq \varepsilon$ if $|z| \leqq \varepsilon$.

We now estimate $|z - (1/k) \sum_{i=1}^k \varphi_i(z)|$ for z in the unit ball of R_n. If $|z| \leqq \varepsilon$, then $|\varphi_i(z)| \leqq \varepsilon$ for each i, so

$$\left| z - \frac{1}{k} \sum_{i=1}^{k} \varphi_i(z) \right| \leqq 2\varepsilon ;$$

if $\varepsilon < |z| \leqq 1$, $\varphi_i(z) = z$ for all but at most one i, so

$$\left| z - \frac{1}{k} \sum_{i=1}^{k} \varphi_i(z) \right| \leqq \frac{2}{k} < 2\varepsilon .$$

Hence $||f - (1/k) \sum_{i=1}^k \varphi_i \circ f|| \leqq 2\varepsilon$.

If A is a subset of S_{n-1}, $n \geqq 2$, by a *vector field on A* we will mean a continuous function $\Phi: A \rightarrow S_{n-1}$ such that $\Phi(z)$ is perpendicular to z for all z in A. If n is even, define p on S_{n-1} by

$$p(t_1, t_2, \cdots, t_{n-1}, t_n) = (t_2, -t_1, \cdots, t_n, -t_{n-1}) .$$

Then p is a vector field on S_{n-1}.

If n is odd, $n \geqq 3$, and the complement of A in S_{n-1} contains at least one point, A admits a vector field. We see this as follows: clearly we may assume that the omitted point p_0 is the "north pole" $(0, 0, \cdots, 1)$. If $z \in S_{n-1}$, $z \neq p_0$, we define $P(z)$ to be the stereographic projection of z on the hyperplane $H = \{t_n = 0\}$, where t_n is the n'^{th} coordinate function: $P(z)$ is the intersection of the line through p_0 and z with H. P is one-to-one and bicontinuous from $S_{n-1} \sim \{p_0\}$ onto H. Let T be a translation of H onto itself: $T(y) = y + y_0$, where y_0

is a nonzero element of H. Now let $Q(z) = (P^{-1} \circ T \circ P)(z)$ for $z \in S_{n-1} \sim \{p_0\}$.

For each z in $S_{n-1} \sim \{p_0\}$, $Q(z)$ can be written uniquely as $\lambda z + V(z)$, where λ is a real number and $V(z)$ is an element of R_n which is perpendicular to z. If $V(z) = 0$, then since $|Q(z)| = |z| = 1$, we have $\lambda = \pm 1$. We cannot have that $\lambda = 1$, since $Q(z) \neq z$ (T is fixed-point free); and if the vector y_0 in the definition of T is small enough, $T(y) - y$ is uniformly small, so λ cannot equal -1. Hence $V(z) \neq 0$, so if we define Φ by $\Phi(z) = (V(z)/|V(z)|)$, Φ is the desired vector field. It is not hard to check that P has the properties claimed for it and that V is continuous, whence Φ is continuous.

For each i, let W_i be the wedge associated with φ_i; W_i is the convex hull of v_i and $B(x_i, \alpha_i)$, where v_i is the vertex of W_i. The preceding remarks imply that there is a vector field Φ_i on $S_{n-1} \sim B(x_i, \alpha_i)$. Observe that for each i, $\varphi_i \circ f$ omits the origin and that $\varphi_i(f(x))/|\varphi_i(f(x))|$ is never in $B(x_i, \alpha_i)$; hence we can define g_i and h_i on X by

$$g_i(x) = \varphi_i(f(x)) + (1 - |\varphi_i(f(x))|^2)^{1/2}\Phi_i\left(\frac{\varphi_i(f(x))}{|\varphi_i(f(x))|}\right),$$

$$h_i(x) = \varphi_i(f(x)) - (1 - |\varphi_i(f(x))|^2)^{1/2}\Phi_i\left(\frac{\varphi_i(f(x))}{|\varphi_i(f(x))|}\right).$$

Then g_i and h_i are in E_n and $\varphi_i \circ f = (g_i + h_i/2)$; hence f is approximated within 2ε by a convex combination of elements of E_n. This completes the proof.

Let $\dim X$ denote the dimension of X as defined in Hurewicz and Wallman [3]. We continue with

THEOREM 2. *For $n \geqq 1$, suppose that U_n is equal to the convex hull of E_n. Then* $\dim X < n$.

Proof. By Theorem VI. 4. of Hurewicz and Wallman, it suffices to prove the following: Let A be a closed subset of X. Then if f is a continuous map of A into S_{n-1}, there is an extension of f to a continuous map of X into S_{n-1}.

Hence, let A and f be as above. Using Tietze's theorem, we can extend f to a continuous $\tilde{f}$ from X into the unit ball of R_n. If $\tilde{f}$ is in the convex hull of E_n, there is a probability measure μ defined on the Borel subsets of U_n with $\mu(E_n) = 1$ (μ has finite support, but we do not need this fact) such that $\Psi(\tilde{f}) = \int_{E_n} \Psi(g)d\mu(g)$ for each continuous linear functional Ψ on $C(X, R_n)$. Let $\{x_j\}$ be a sequence dense in A and let $p_j = f(x_j)$. Define continuous linear functionals Ψ_j by

$$\Psi_j(g) = \langle g(x_j), p_j \rangle \quad \text{for } g \text{ in } C(X, R_n) .$$

(Here, $\langle \, , \rangle$ denotes the usual inner product.) Then for each j we have

$$1 = \Psi_j(\tilde{f}) = \int_{E_n} \Psi_j(g) d\mu(g) .$$

If g is in E_n and $g(x_j) \neq p_j$, then $\Psi_j(g) < 1$; since μ is a probability measure it must be the case that

$$\mu\{g \in E_n : g(x_j) \neq p_j\} = 0 .$$

Hence, $\mu(\bigcup_{j=1}^{\infty} \{g \in E_n : g(x_j) \neq p_j\}) = 0$; it follows that there is a g^* in E_n such that $g^*(x_j) = p_j = f(x_j)$ for all j. Since $\{x_j\}$ is dense in A, $g^*(x) = f(x)$ for all x in A. This g^* is the desired extension of f and the proof is complete.

We now show that in case n is even the converse of Theorem 2 holds, and that if $n = 1$, something slightly weaker than the converse of Theorem 2 holds; we also give some related results. Before proceeding, we again note that if n is even, the function p on S_{n-1} defined by

$$p(t_1, t_2, \cdots, t_{n-1}, t_n) = (t_2, -t_1, \cdots, t_n, -t_{n-1})$$

is a continuous map of S_{n-1} into S_{n-1} such that $p(z)$ is perpendicular to z for all z in S_{n-1}.

THEOREM 3. *If n is even and $\dim X < n$, U_n is equal to the convex hull of E_n.*

Proof. The containment one way is trivial. To show that U_n is contained in the convex hull of E_n, it suffices to show that U_n is in the convex hull of those elements of U_n which omit the origin; for if g is an element of U_n which omits the origin we can define f_1 and f_2 in E_n by

$$f_1(x) = g(x) + (1 - |g(x)|^2)^{1/2} p\left(\frac{g(x)}{|g(x)|}\right) ,$$

$$f_2(x) = g(x) - (1 - |g(x)|^2)^{1/2} p\left(\frac{g(x)}{|g(x)|}\right) .$$

Plainly $g = f_1 + f_2/2$.

Hence suppose $\dim X < n$ and that f is in U_n. By Theorem VI. 1. of Hurewicz and Wallman, the origin is an unstable value of f; by Proposition B of the same section in Hurewicz and Wallman, there is a function h_1 in U_n which omits the origin, such that

(1) If $|f(x)| \geqq (1/3)$, then $h_1(x) = f(x)$,
(2) If $|f(x)| < (1/3)$, then $|h_1(x)| < (1/3)$.
Put $h_2 = 2f - h_1$; then h_2 is in U_n.

Suppose $|h_1(x)| > 3\varepsilon > 0$ for all x in X. Using the same results in Hurewicz and Wallman, we can choose g_2 in U_n such that g_2 omits the origin and such that

(3) If $|h_2(x)| \geqq \varepsilon$, then $g_2(x) = h_2(x)$,
(4) If $|h_2(x)| < \varepsilon$, then $|g_2(x)| < \varepsilon$.
Put $g_1 = 2f - g_2$. Now it is easy to check that $\|g_1\| \leqq 1$ and $\|g_2\| \leqq 1$; moreover g_1 omits the origin because $\|g_1 - h_1\| = \|g_2 - h_2\| \leqq 2\varepsilon$. This completes the proof of Theorem 3.

For the case $n = 1$, $\dim X = 0$, we have a slightly weaker version of Theorem 3:

THEOREM 4. *If* $\dim X = 0$, *then for every* f *in* U_1 *there is a sequence* $\{h_i\}$ *of elements of* E_1 *such that* $f = \sum_{i=1}^{\infty} (1/2^{i+1})(h_{2i-1} + h_{2i})$, *the convergence being norm convergence.*

We first prove an auxiliary result:

LEMMA 1. *Assume that* $\dim X = 0$ *and that* f *is in* U_1. *Then there are two elements* h_1, h_2 *of* E_1 *such that* $\|f - (1/4)(h_1 + h_2)\| \leqq 1/2$.

Proof. If h_i assumes only the two values ± 1, $h_i = \chi_{A_i} - \chi_{\sim A_i}$, where A_i is an open-and-closed subset of X and χ_T denotes the characteristic function of the set T. If $\|f - (1/4)(h_1 + h_2)\| \leqq 1/2$ we must have that $|f - (1/2)| \leqq 1/2$ on $A_1 \cap A_2$, $|f| \leqq 1/2$ on

$$(A_1 \sim A_2) \cup (A_2 \sim A_1) ,$$

and $|f + (1/2)| \leqq 1/2$ on $(\sim A_1) \cap (\sim A_2)$. Using the zero-dimensionality of X, we can find an open-and-closed set A_1 containing $f^{-1}[1/2, 1]$ and contained in $f^{-1}(0, 1]$; we can then find an open-and-closed subset A_2 containing $f^{-1}[0, 1]$ and contained in $f^{-1}(-(1/2), 1]$. With this choice of A_1 and A_2, $\|f - (1/4)(h_1 + h_2)\| \leqq 1/2$, and this completes the proof of the lemma.

Turning now to the proof of the theorem, we suppose that f is in U_1. By the lemma, there are elements h_1, h_2 of E_1 such that

$$\left\|f - \frac{1}{4}(h_1 + h_2)\right\| \leqq \frac{1}{2} .$$

Assume that elements $h_1, h_2, \cdots, h_{2j-1}, h_{2j}$ of E_1 have been found so that

$$\left\| f - \sum_{i=1}^{j} \frac{1}{2^{i+1}} (h_{2i-1} + h_{2i}) \right\| \leqq \frac{1}{2^j} .$$

Let

$$H_j = f - \sum_{i=1}^{j} \frac{1}{2^{i+1}} (h_{2i-1} + h_{2i}) .$$

Then $\| 2^j H_j \| \leqq 1$; appealing to the lemma again, we find elements h_{2j+1}, h_{2j+2} of E_1 such that

$$\left\| 2^j H_j - \frac{1}{4} (h_{2j+1} + h_{2j+2}) \right\| \leqq \frac{1}{2} ,$$

whence

$$\left\| f - \sum_{i=1}^{j+1} \frac{1}{2^{i+1}} (h_{2i-1} + h_{2i}) \right\| \leqq \frac{1}{2^{j+1}} .$$

This completes the induction step and the proof of the theorem.

We now turn to the case that n is an odd integer, $n \geqq 3$; we would like to prove something like Theorem 3 for such n. The two key elements in the proof of Theorem 3 were the approximation of an f in U_n by a nowhere-vanishing g, and the fact that a nowhere-vanishing g can be written as the midpoint of two elements of E_n. The approximation is always possible, whether n is odd or even, provided $\dim X < n$; but the representation of a nonvanishing g in U_n as the midpoint of two elements of E_n is not always possible, even with $\dim X < n$. For example, if n is odd, let $X = (1/2)S_{n-1}$, the set of points in R_n at distance $1/2$ from the origin. Let f be the identity map of X into the unit ball of R_n. Then if $f = g_1 + g_2/2$, with g_1, g_2 in E_n, it is easy to see that if

$$h(z) = \frac{g_1\left(\frac{z}{2}\right) - \frac{z}{2}}{\left| g_1\left(\frac{z}{2}\right) - \frac{z}{2} \right|}$$

for z in S_{n-1}, h is a vector field on S_{n-1}, which is an impossibility.

We do have the following partial result:

PROPOSITION 1. Suppose that X is a compact metric space such that any two continuous maps of X into S_{n-1} are homotopic in $S_{n-1} (n \geqq 2)$. Then if g is an element of U_n which omits the origin, $g = h_1 + h_2/2$, with h_1, h_2 in E_n.

Before we prove the proposition, we make the following observation (which must be in the literature):

LEMMA 2. *Let X be a compact space and let f, g be two continuous maps of X into S_{n-1}, $n \geqq 2$, such that $\|f - g\| < \sqrt{2}$. Then if there is a continuous g' from X into S_{n-1} such that $g'(x)$ is perpendicular to $g(x)$ for all x in X, there is a continuous f' from X into S_{n-1} such that $f'(x)$ is perpendicular to $f(x)$ for all x in X.*

Proof of the lemma. For each x in X we can write $g'(x)$ uniquely in the form $g''(x) + \lambda(x)f(x)$, where $g''(x)$ is perpendicular to $f(x)$ and $\lambda(x)$ is a scalar between -1 and 1. It is easy to see that g'' is continuous as a function of x. If $g''(y) = 0$ for some y, then $g'(y) = \pm f(y)$; since $g(y)$ is perpendicular to $g'(y)$ we have $|f(y) - g(y)| = \sqrt{2}$, a contradiction. The proof of the lemma is complete if we define $f'(x) = (g''(x)/|g''(x)|)$ for x in X.

Proof of the proposition. Define h on X by $h(x) = (g(x)/|g(x)|)$; then h is a continuous map of X into S_{n-1}. By assumption, there are a constant map k of X into S_{n-1} and a continuous map q of $X \times [0, 1]$ into S_{n-1} such that $q(x, 0) = k(x)$, $q(x, 1) = h(x)$ for all x in X. Clearly there is a continuous map k' of X into S_{n-1} such that $k'(x)$ is perpendicular to $k(x)$ for all x in X. (Simply let k' be another constant map, appropriately chosen.)

Let T be the set of all t in $[0, 1]$ such that there is a continuous map g'_t from X into S_{n-1} with $g'_t(x)$ perpendicular to $q(x, t)$ for all x in X. The set T is nonempty, and by the lemma above, T is open and closed in $[0, 1]$. We conclude that there is a continuous h' of X into S_{n-1} such that $h'(x)$ is perpendicular to $h(x)$ for all x in X.

Now define h_1 and h_2 on X by

$$h_1(x) = g(x) + (1 - |g(x)|^2)^{1/2}h'(x) ,$$
$$h_2(x) = g(x) - (1 - |g(x)|^2)^{1/2}h'(x) .$$

It follows that h_1 and h_2 are in E_n and that $g = h_1 + h_2/2$.

Combining Proposition 1 and the techniques used in the proof of Theorem 3, we obtain the following.

COROLLARY. *If n is an integer $\geqq 3$ and if X is a compact metric space of dimension $< n$ such that any two continuous maps of X into S_{n-1} are homotopic in S_{n-1}, then U_n is the convex hull of E_n.*

In particular, if $\dim X < n$ and X is contractible, then U_n is the convex hull of E_n. Hence if $n \geqq 3$ and $\dim X < n - 1$, U_n is the convex hull of E_n. (Use the cone over X; this has dimension $< n$ and is contractible.)

2. Mappings into infinite-dimensional spaces. We now wish to prove Theorem 3 in the case that the range space N is infinite-dimensional. We assume from here on that X is a compact Hausdorff space (metrizability is no longer assumed) and that N is an infinite-dimensional strictly convex normed space.

THEOREM 5. *Let X and N be as above. Then U_N is the convex hull of E_N.*

We shall prove this in the same way that we proved Theorem 3: every element of U_N can be approximated by an element of U_N which omits the zero vector in N: every element of U_N which omits the origin is the midpoint of two elements of E_N. The first assertion is proved in Proposition 2 below; the second assertion is proved in Proposition 3.

PROPOSITION 2. Let X and N be as above. Then if f is in U_N and ε is a positive number, there is g in U_N such that g omits the origin and $\|f - g\| < \varepsilon$.

Proof. The set $K = f(X)$ is compact, so by a result of Nagumo [4, Th. 2] there are points $x_1, \cdots, x_r$ in the unit ball of N and a continuous map q of K into the convex hull of $\{x_1, \cdots, x_r\}$ such that $|q(z) - z| < \varepsilon/3$ for z in K. If s is the number $1 - (\varepsilon/3)$, $|s \cdot q(z) - z| < 2\varepsilon/3$ for z in K. Now let v be any element of the unit ball of N which is not in the linear span of $\{x_1, \cdots, x_r\}$. Finally if we define g on X by $g(x) = (\varepsilon/3)v + s \cdot q(f(x))$, g is a continuous map of X into the unit ball of N, g omits the origin, and $\|f - g\| < \varepsilon$.

COROLLARY. *Let X and N satisfy the hypotheses of Proposition 2. Let f be an element of U_N. Then for every $\varepsilon > 0$ there is a g in U_N such that g omits the origin, $|g(x)| < \varepsilon$ if $|f(x)| < \varepsilon$, $g(x) = f(x)$ if $|f(x)| \geqq \varepsilon$.*

Proof. The proof of Proposition B § 1 in chapter VI of Hurewicz and Wallman can be used without change, in conjunction with Proposition 2.

Now let N be an infinite-dimensional strictly convex normed space. Let B denote the closed unit ball of N and let S denote the boundary of B. Let X be a compact Hausdorff space and let g be a continuous map of X into $B \sim \{0\}$. We shall show that g is the midpoint of two continuous maps of X into S. To prove this, it is certainly enough to prove the following.

PROPOSITION 3. Let N be an infinite-dimensional strictly convex normed space and let K be a compact subset of the unit ball of N such that K does not contain the origin. Then there are two continuous maps φ_1 and φ_2, defined and continuous on K and assuming values in S, such that for each x in K, $x = \varphi_1(x) + \varphi_2(x)/2$.

Proof. Let K satisfy the hypotheses of the proposition. Then if η is defined on K by $\eta(x) = (x/|x|)$, η is a continuous map of K into S. Since N is infinite-dimensional, S cannot be compact; hence there is a point z in $S \sim (\eta(K) \cup -\eta(K))$. We now define γ on $K \times [0, 2]$ in the following way:

$$\gamma(x, t) = \frac{(1-t)\eta(x) + tz}{|(1-t)\eta(x) + tz|} \qquad \text{for } 0 \leqq t \leqq 1 ;$$

$$\gamma(x, t) = \frac{(2-t)z + (t-1)(-\eta(x))}{|(2-t)z + (t-1)(-\eta(x))|} \qquad \text{for } 1 \leqq t \leqq 2 .$$

(Note that the norms in the denominators are never zero because of the way z was chosen.) It is clear that γ is continuous on $K \times [0, 2]$ and that γ is a map of $K \times [0, 2]$ into S.

Fix x in K; then it is easily verified that $|2x - \gamma(x, 0)| \leqq 1$ and $|2x - \gamma(x, 2)| > 1$. It follows that there is at least one t in $[0, 2]$ such that $|2x - \gamma(x, t)| = 1$.

We assert that there is at most one such t. Since this is an assertion about a two-dimensional subspace of N, our claim is equivalent to the following lemma, in which $(1, 0)$ plays the role of the point $\eta(x)$ and $(0, 1)/|(0, 1)|$ plays the role of the point z:

LEMMA 3. *Let $|\ |$ be any strictly convex norm on the XY-plane. Suppose that $|(1, 0)| = 1$ and that $0 < r \leqq 1$. Then there is at most one point (x_1, y_1) with $y_1 \geqq 0$ such that*

$$|(x_1, y_1)| = |2(r, 0) - (x_1, y_1)| = 1 .$$

Proof. For a contradiction, we may assume there are two such points $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$, with $y_1 > y_2 > 0$. (It is immediate from strict convexity that $y_1 \neq y_2$.) Let $(u, 0)$ denote the point of intersection of the x-axis and the line through q_1 and q_2. Explicitly, $u = (y_1 - y_2)^{-1}(y_1 x_2 - y_2 x_1)$ and

$$q_2 = \lambda q_1 + (1 - \lambda)(u, 0) , \quad \text{where} \quad \lambda = y_2/y_1 \in (0, 1) .$$

We also have

$$q_2 - 2(r, 0) = \lambda[q_1 - 2(r, 0)] + (1 - \lambda)(u - 2r, 0) .$$

We can obviously assume that neither the above-mentioned line nor its translate by $-2(r, 0)$ passes through the origin, so the strict convexity of the norm yields $|(u, 0)| > 1$ and $|(u - 2r, 0)| > 1$. These last two points are at most two units apart (since $0 < r < 1$), so we either have $u - 2r < u < -1$ or $1 < u - 2r < u$. Neither of these is possible (a sketch clarifies this); in the first case, for instance, we would have q_2 in the interior of the triangle defined by $q_2 - 2(r, 0)$, q_1 and the origin, which would imply $|q_2| < 1$. (In the second case, we would get $|q_2 - 2(r, 0)| < 1$.)

Continuing with the proof of the theorem, we let $t(x)$ be the unique point in $[0, 2]$ such that $|2x - \gamma(x, t(x))| = 1$. We now claim that t is continuous on K. If not, there are a point x_0 in K and a sequence $\{x_j\}$ converging to x_0 such that $|t(x_j) - t(x_0)| > \varepsilon > 0$ for all j. Taking a subsequence, if necessary, we may assume that $\{t(x_j)\}$ converges to $t_0 \neq t(x_0)$. Using the continuity of γ we find that

$$|2x_0 - \gamma(x_0, t_0)| = \lim_j |2x_j - \gamma(x_j, t(x_j))| = 1 ;$$

this contradicts the uniqueness of $t(x_0)$ and the continuity of t is established. It is now clear how φ_1 and φ_2 are to be defined on K:

$$\varphi_1(x) = \gamma(x, t(x)) ,$$
$$\varphi_2(x) = 2x - \gamma(x, t(x)) .$$

This completes the proof of the proposition.

Observe that a much simpler proof is available if N is complex linear. Indeed, if N is complex linear and if x is in the unit ball B of N, $x \neq 0$, define φ_1 and φ_2 by

$$\varphi_1(x) = (1 + (|x|^{-2} - 1)^{1/2} i) \cdot x ,$$
$$\varphi_2(x) = (1 - (|x|^{-2} - 1)^{1/2} i) \cdot x .$$

The modulus of each of the coefficients of x in the above expressions is $|x|^{-1}$, so it follows that for x in $B \sim \{0\}$, $|\varphi_1(x)| = |\varphi_2(x)| = 1$. Plainly, $x = \varphi_1(x) + \varphi_2(x)/2$, and it is equally clear that φ_1 and φ_2 are continuous on $B \sim \{0\}$.

Combining the above proposition, the Corollary to Proposition 2, and the techniques of Theorem 3, we obtain Theorem 5.

We conclude with a question: what are necessary and sufficient conditions on the compact metric space X so that U_n is equal to the convex hull of E_n, if n is an odd integer $\geqq 3$?

Author's note. Since this paper was written, the results have been improved on in several ways. Professor Joram Lindenstrauss has communicated a proof that the conclusion of Theorem 1 holds for the case of $C(X, N)$, where N is any finite-dimensional real vector space, normed in such a way that the extreme points of the unit ball of N form an arcwise connected set. In the proof of Theorem 3 compactness of X appears essential ($|h_1(x)| > 3\varepsilon > 0$ for all x in X), but Professor James L. Cornette has shown that compactness is unnecessary by modifying h_1 slightly. A similar device is used by Professor John Cantwell in a paper to appear in the AMS *Proceedings*; in this paper Cantwell establishes the converse of our Theorem 2 for odd $n, n \geqq 3$, without any additional hypotheses on X. (He shows that for odd $n, n \geqq 3$, each element of U_n is in the convex hull of eight elements of E_n if $\dim X < n$.) For $n = 1$ our Theorem 4 appears best possible, since convex combinations of elements of E_1 assume only finitely many values and there are certainly zero-dimensional compact metric spaces admitting a continuous real-valued function which assumes infinitely many values.

Note that the proof of Theorem 1 shows that the theorem is really a statement about the normed space of all bounded continuous functions from a Hausdorff space X into $R_n, n \geqq 2$. Finally, we remark that the proof of Theorem 2 would have been simpler if $\tilde{f}$ had been written explicitly as a convex combination of elements of E_n; the point here is that the weak form of "representability" of $\tilde{f}$ used in the proof is enough to give the conclusion.

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