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DEFINING SUBSETS OF E^3 BY CUBES

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This paper is concerned with compact subsets of E^3 which are the intersection of a properly nested sequence of compact 3-manifolds with boundary each of which is the union of a finite collection of pairwise disjoint 3-cells. Such sets are characterized by a property of their complements. Related results are stated in terms of embeddings of compact 0-dimensional sets and upper semicontinuous decompositions of E^3.

Theorem 1 below gives an affirmative answer to a question raised by Štan'ko in [10].

1. Definitions and notation. We use E^3 to denote Euclidean 3-space. In [10], a compact set $K \subset E^3$ is defined to be *cellular-divisible* if there is a sequence $\{M_i\}$ of compact 3-manifolds with boundary such that

(1) if $i = 1, 2, \cdots$, then $M_{i+1} \subset \operatorname{Int} M_i$,

(2) if $i = 1, 2, \cdots$, then M_i is the union of a finite collection of pairwise disjoint topological cubes (3-cells), and

(3) $K = \bigcap_{i=1}^{\infty} M_i$.

We shall use the terminology of [9] and say that such a set is *definable by cubes*. By the approximation theorem for 2-spheres [3] there is no loss of generality in supposing that each M_i in the above definition is polyhedral. If K is a continuum (i.e., compact and connected) and is definable by cubes, then K is said to be *cellular*. If K is compact and 0-dimensional, then K is *tame* (*wild*) if and only if K is (is not) definable by cubes. Tameness in this case is equivalent to the existence of a homeomorphism of E^3 onto itself carrying K into a straight line interval. See [5] or [7].

We use Cl for closure, Bd for boundary, Ext for exterior, and Int for interior. Int may mean "combinatorial interior" or "bounded complementary domain" with context providing the proper interpretation in each case. If K is a subset of E^3 and $\varepsilon > 0$, we use $V(K, \varepsilon)$ to denote the ε-neighborhood of K.

2. Subsets of E^3 which are definable by cubes. The following theorem affirmatively answers question 2 of [10]. An example of Kirkor [8] shows that the hypothesis that J can be separated from K by a 2-sphere cannot be replaced by the weaker hypothesis that J can be shrunk to a point in $E^3 - K$.

THEOREM 1. *Suppose $K \subset E^3$ is compact and fails to separate*

E^3. Then K is definable by cubes if and only if for each polygonal simple closed curve $J \subset E^3 - K$, there is a 2-sphere separating K and J.

Proof. We only consider the "if" part of the proof since the "only if" part is evident. Let J be as above. We first show that there is a 3-cell $C \subset E^3 - K$ such that $J \subset \operatorname{Int} C$. Let S_1 be a 2-sphere separating K and J. By the approximation theorem S_1 may be supposed to be polyhedral, and so if $J \subset \operatorname{Int} S_1$ we may take $C = S_1 \cup \operatorname{Int} S_1$. If $J \subset \operatorname{Ext} S_1$, let S_2 be a polyhedral 2-sphere whose interior contains $S_1 \cup J$. Let α be a polygonal arc from $a \in S_1$ to $b \in S_2$ whose interior fails to intersect $S_1 \cup S_2 \cup J$. Fatten α to a polyhedral 3-cell B whose intersection with $S_1 \cup S_2 \cup J$ is the union of a pair of polyhedral disks $D_1 \subset S_1$ and $D_2 \subset S_2$, and let A denote the annulus $\operatorname{Bd} B - (\operatorname{Int}(D_1 \cup D_2))$. Now let S_3 be the polyhedral 2-sphere $(S_1 - D_1) \cup (S_2 - D_2) \cup A$ and let $C = S_3 \cup \operatorname{Int} S_3$. From this and Lemma 7 of [4] it follows that each polygonal finite graph in $E^3 - K$ lies in the interior of a polyhedral 3-cell in $E^3 - K$.

To show that K is definable by cubes we need only show that for each open set U containing K there is a finite collection of pairwise disjoint 3-cells in U whose interiors cover K. Let M be a compact polyhedral 3-manifold with boundary such that $K \subset \operatorname{Int} M \subset M \subset U$. Let F be the 1-skeleton of $\operatorname{Bd} M$. By the remark at the end of the preceding paragraph there is a polyhedral 3-cell E such that $F \subset \operatorname{Int} E \subset E \subset E^3 - K$. Using $\operatorname{Bd} E$ and the argument of the preceding paragraph, we construct a polyhedral 2-sphere S such that $K \subset \operatorname{Int} S$ and $F \subset \operatorname{Ext} S$. Now, using S, $\operatorname{Bd} M$, and Lemma 1 below, we obtain pairwise disjoint polyhedral 2-spheres $R_1, R_2, \cdots, R_m$ with pairwise disjoint interiors such that $K \subset \bigcup_{i=1}^m \operatorname{Int} R_i$ and $\operatorname{Bd} M$ lies in each $\operatorname{Ext} R_i$. There is no loss of generality in supposing that K intersects each $\operatorname{Int} R_i$. It then follows that if $i = 1, 2, \cdots$, or m, $R_i \cup \operatorname{Int} R_i \subset \operatorname{Int} M$. Hence $R_1 \cup \operatorname{Int} R_1, R_2 \cup \operatorname{Int} R_2, \cdots, R_m \cup \operatorname{Int} R_m$ is a collection of pairwise disjoint 3-cells lying in U whose interiors cover K, and the proof of Theorem 1 is complete.

LEMMA 1. *Suppose $T_1, T_2, \cdots, T_n$ is a collection of pairwise disjoint polyhedral 2-spheres with pairwise disjoint interiors, K is a compact set that lies in $\bigcup_{i=1}^n \operatorname{Int} T_i$, N is a compact polyhedral 2-manifold (with or without boundary) in $E^3 - K$ whose 1-skeleton lies in each $\operatorname{Ext} T_i$, and $\varepsilon > 0$. Then there is a collection $R_1, R_2, \cdots, R_m$ of pairwise disjoint polyhedral 2-spheres with pairwise disjoint interiors such that*

(1) $K \subset \bigcup_{i=1}^m \operatorname{Int} R_i$,

(2) *N lies in each $\operatorname{Ext} R_i$, and*

(3) $\bigcup_{i=1}^m R_i \subset (\bigcup_{i=1}^n T_i) \cup V(N, \varepsilon)$.

Proof. We may suppose without loss of generality that each T_i is in general position with respect to N so that $(\bigcup_{i=1}^n T_i) \cap N$ is the union of a finite collection of pairwise disjoint polygonal simple closed curves. Consider a 2-simplex Δ of N which intersects $\bigcup_{i=1}^n T_i$ and let J be a component of $(\bigcup_{i=1}^n T_i) \cap \Delta$ with the property that if D is the subdisk of Δ bounded by J, then $D \cap (\bigcup_{i=1}^n T_i) = J$. Suppose J lies on T_j. Thicken D slightly to a polyhedral 3-cell C such that

(1) $C \subset V(D, \varepsilon)$,

(2) $C \cap (\bigcup_{i=1}^n T_i)$ is an annulus A in $T_j \cap \operatorname{Bd} C$,

(3) $C \cap N = D$,

(4) $D \cap \operatorname{Bd} C = J$, and

(5) $K \cap C = \varnothing$.

Let J_1 and J_2 be the boundary components of A and let D_1 and D_2 be the subdisks of T_j bounded by J_1 and J_2 respectively such that if $i = 1$ or 2, then $D_i \cap A = J_i$. Similarly, let D_1' and D_2' be the subdisks of $\operatorname{Bd} C$ bounded by J_1 and J_2 respectively such that if $i = 1$ or 2, then $D_i' \cap A = J_i$. Let $T_{j_1} = D_1 \cup D_1'$ and $T_{j_2} = D_2 \cup D_2'$. We now consider the following two cases.

Case 1. $\operatorname{Int} D \subset \operatorname{Int} T_j$. In this case $T_1, T_2, \cdots, T_{j-1}, T_{j_1}, T_{j_2}, T_{j+1}, \cdots, T_n$ is a collection of pairwise disjoint polyhedral 2-spheres satisfying all of the hypotheses of the Lemma and intersecting N in one less component than $T_1, T_2, \cdots, T_n$.

Case 2. $\operatorname{Int} D \subset \operatorname{Ext} T_j$. In this case either $T_{j_1} \subset \operatorname{Int} T_{j_2}$ or $T_{j_2} \subset \operatorname{Int} T_{j_1}$. We suppose that $T_{j_1} \subset \operatorname{Int} T_{j_2}$. Since the 1-skeleton of N lies in $\operatorname{Ext} T_j$, either $\operatorname{Bd} \Delta \subset \operatorname{Ext} T_{j_2}$ or $\operatorname{Bd} \Delta \subset \operatorname{Int} T_{j_1}$. We shall only consider the case $\operatorname{Bd} \Delta \subset \operatorname{Ext} T_{j_2}$ since the proof in the other case in entirely analogous. Let ab be a polygonal arc from a point $a \in J_1$ to a point $b \in \Delta - D$ such that $ab \cap K = \varnothing$, $ab \subset V(D, \varepsilon)$, $ab \cap C = \{a\}$, and $ab \cap N = \{b\}$. Now let c be a point of $\operatorname{Bd} \Delta$ and let bc be a polygonal arc from b to c lying in $\Delta - D$. Then $ac = ab \cup bc$ is a polygonal arc from $a \in T_{j_1}$ to $c \in \operatorname{Ext} T_{j_2}$. Ordering ac from a to c, let a_1 be the last point of ac lying in T_{j_1} and let b_1 be the first point of ac which follows a_1 and lies in T_{j_2}. Then a_1b_1 is a polygonal arc from $a_1 \in T_{j_1}$ to $a_2 \in T_{j_2}$ which spans the annular region between T_{j_2} and T_{j_1}. Now push a_1b_1 slightly off Δ so that the adjusted arc, which we denote by $a_1'b_1'$, fails to intersect $N \cup K$ and lies in $V(\Delta, \varepsilon)$. Since the 2-spheres $T_1, T_2, \cdots, T_n$ have pairwise disjoint interiors, $a_1'b_1'$ fails to intersect $\bigcup_{i\ 1}^n T_i$ except in its end points. As in the first paragraph of the proof of Theorem 1, we use T_{j_1}, T_{j_2}, and $a_1'b_1'$ to construct a polyhedral 2-sphere T_j' such that $K \cap \operatorname{Int} T_j' = K \cap (\operatorname{Int} T_j)$, $T_j' \cap (\bigcup_{i=1}^n T_i - T_j) = \varnothing$, $T_j' \cap N = (T_{j_1} \cup T_{j_2}) \cap N$, and $\operatorname{Int} T_{j_1} \subset \operatorname{Ext} T_j'$. Now $T_1, T_2, \cdots, T_{j-1}, T_j', T_{j+1}, \cdots, T_n$ is a collection of pairwise disjoint polyhedral

2-spheres satisfying all of the hypotheses of the Lemma and intersecting N in one less component than $T_1, T_2, \cdots, T_n$.

By the above two cases, we may inductively eliminate all components of $(\bigcup_{i=1}^{n} T_i) \cap N$ to obtain a collection of pairwise disjoint polyhedral 2-spheres $R_1, R_2, \cdots, R_m$ satisfying conditions (1)-(3) of the conclusion of Lemma 1.

The following corollary is a special case of Theorem 1 which gives another characterization of tame compact 0-dimensional subsets of E^3. For other characterizations see [5] and [7].

COROLLARY 1. *Suppose K is a compact 0-dimensional subset of E^3. Then K is tame if and only if for each polygonal simple closed curve $J \subset E^3 - K$, there is a 2-sphere separating K and J.*

Bing [2] has given an example of a wild compact 0-dimensional subset K of E^3 and a polygonal simple closed curve $J \subset E^3 - K$ such that, if $p \in K$, there is no 2-sphere in $E^3 - K$ whose interior contains p and whose exterior contains J. This example suggests the following result, which is an improvement on Theorem 1.

THEOREM 2. *Suppose $K \subset E^3$ is compact and fails to separate E^3. Then K is definable by cubes if and only if for each point $p \in K$ and each polygonal simple closed curve $J \subset E^3 - K$, there is a 2-sphere lying in $E^3 - K$ separating p and J.*

Proof. As in the case of Theorem 1, we consider only the "if" part of the proof. Let J be a polygonal simple closed curve in $E^3 - K$. For each $p \in K$, let S_p be a polyhedral 2-sphere lying in $E^3 - K$ which separates p and J. By the first paragraph of the proof of Theorem 1 there is no loss of generality in supposing that $p \in \operatorname{Int} S_p$ and $J \subset \operatorname{Ext} S_p$. By compactness of K there is a finite collection $S_1, S_2, \cdots, S_n$ of such 2-spheres such that $K \subset \bigcup_{i=1}^{n} \operatorname{Int} S_i$. Now by Lemma 2 below, applied to S_1 and S_2, there is a finite collection $S_1', S_2', \cdots, S_m'$ of pairwise disjoint polyhedral 2-spheres with pairwise disjoint interiors such that $K \cap (\bigcup_{i=1}^{2} \operatorname{Int} S_i) \subset \bigcup_{i=1}^{m} \operatorname{Int} S_i'$ and J lies in each $\operatorname{Ext} S_i'$. Another application of Lemma 2 to the collection $S_1', S_2', \cdots, S_m'$ and S_3 yields a finite collection $S_1'', S_2'', \cdots, S_k''$ of pairwise disjoint polyhedral 2-spheres with pairwise disjoint interiors such that $K \cap (\bigcup_{i=1}^{3} \operatorname{Int} S_i) \subset \bigcup_{i=1}^{k} \operatorname{Int} S_i''$ and J lies in each $\operatorname{Ext} S_i''$. Continuing in this manner we finally obtain a collection $R_1, R_2, \cdots, R_j$ of pairwise disjoint polyhedral 2-spheres with pairwise disjoint interiors such that $K \subset \bigcup_{i=1}^{j} \operatorname{Int} R_i$ and J lies in each $\operatorname{Ext} R_i$. Running polygonal arcs lying in $E^3 - J$ between various members of the collection $R_1, R_2, \cdots, R_j$ and using (once again) the idea of the first paragraph of the proof of Theorem 1, we construct a polyhedral 2-sphere S such that $K \subset \operatorname{Int} S$ and $J \subset \operatorname{Ext} S$. By Theorem 1, K is definable by cubes.

LEMMA 2. *Suppose $T_1, T_2, \cdots, T_n$ is a collection of pairwise disjoint polyhedral 2-spheres with pairwise disjoint interiors, K is a compact set that lies in $\bigcup_{i=1}^n \operatorname{Int} T_i$, N is a polyhedral 2-sphere in $E^3 - K$, L is a compact set that lies in* Ext N *and each* Ext T_i, *and $\varepsilon > 0$. Then there is a collection $R_1, R_2, \cdots, R_m$ of pairwise disjoint polyhedral 2-spheres with pairwise disjoint interiors such that*

(1) $K \subset \bigcup_{i=1}^m \operatorname{Int} R_i$,

(2) $N \cap (\bigcup_{i=1}^m R_i) = \emptyset$,

(3) *L lies in each* Ext R_i, *and*

(4) $\bigcup_{i=1}^m R_i \subset (\bigcup_{i=1}^n T_i) \cup V(N, \varepsilon)$.

Proof. The proof of Lemma 2 only differs slightly from that of Lemma 1. There is no difficulty in carrying over the proof through Case 1, so we begin here at Case 2, where the notation has been carried over directly.

First consider the case where $T_{j_1} \cap N = \emptyset = T_{j_2} \cap N$. In this case $T_{j_1} \subset \operatorname{Int} N$ and $T_{j_2} \subset \operatorname{Ext} N$. The cube C has been constructed so as to miss L, so $L \subset \operatorname{Ext} T_{j_2}$. We then obtain a collection of pairwise disjoint polyhedral 2-spheres satisfying the conclusions of Lemma 2 by replacing T_j by T_{j_2} and throwing out any T_i's lying in $\operatorname{Int} T_{j_2}$.

Now suppose $T_{j_2} \cap N \neq \emptyset$. The proof in the case $T_{j_1} \cap N \neq \emptyset$ is analogous. Let ab be a polygonal arc from $a \in J_1$ to $b \in N - D$ such that $ab \cap (K \cup L) = \emptyset$, $ab \subset V(D, \varepsilon)$, $ab \cap C = \{a\}$, and $ab \cap N = \{b\}$. Now let c be a point of $T_{j_2} \cap N$ and complete the proof as in the proof of Lemma 1.

COROLLARY 2. *Suppose K is a compact 0-dimensional subset of E^3. Then K is tame if and only if for each point $p \in K$ and each polygonal simple closed curve $J \subset E^3 - K$, there is a 2-sphere lying in $E^3 - K$ separating p from J.*

The following theorem is a slight improvement of Theorem 4 of [10] and will be proved here using Theorem 1.

THEOREM 3. *Suppose $L \subset K$ are compact subsets of E^3 such that K is definable by cubes, L fails to separate E^3, and $K - L$ is at most 1-dimensional. Then L is definable by cubes.*

Proof. Let J be a polygonal simple closed curve in $E^3 - L$. By Lemma 3 below there is a homeomorphism h of E^3 onto itself which is fixed on L and moves J onto a polygonal simple closed curve in $E^3 - K$. By Theorem 1 there is a 2-sphere S in $E^3 - K$ separating K and $h(J)$. Then $h^{-1}(S)$ is a 2-sphere in $E^3 - L$ separating L and J. By Theorem 1 L is definable by cubes.

LEMMA 3. *Under the hypotheses of Theorem* 3, *if J is a polygonal simple closed curve in $E^3 - L$, then there is a homeomorphism h of E^3 onto itself which is fixed on L and moves J onto a polygonal simple closed curve in $E^3 - K$.*

Proof. Since $K - L$ is at most 1-dimensional, there is no problem in moving the vertices of J into $E^3 - K$. We suppose that this has been done. We now show how to move J into $E^3 - K$ moving one simplex at a time.

Let I be a 1-simplex of J with end points a and b. Then I spans a polyhedral solid cylinder C with bases D_1 and D_2 such that

(1) $a \in \operatorname{Int} D_1$ and $b \in \operatorname{Int} D_2$,

(2) $(D_1 \cup D_2) \cap J = \{a, b\}$,

(3) $C \cap J = I$,

(4) I is an unknotted chord of C, and

(5) $C \cap L = \emptyset$.

Denote the annulus $\operatorname{Bd} C - (\operatorname{Int}(D_1 \cup D_2))$ by A. We now show that no component of $K \cap A$ separates $\operatorname{Bd} D_1$ from $\operatorname{Bd} D_2$ in A.

Since $C \cap K$ is at most 1-dimensional, there is an arc α_1 from a to b such that $\operatorname{Int} \alpha_1 \subset \operatorname{Int} C$ and $K \cap \alpha_1 = \emptyset$. Similarly there is an arc α_2 from a to b such that $\operatorname{Int} \alpha_2 \subset \operatorname{Ext} C$ and $K \cap \alpha_2 = \emptyset$. Now let N be a component of $K \cap A$, and suppose N separates $\operatorname{Bd} D_1$ from $\operatorname{Bd} D_2$ in A. Since K is definable by cubes there is a 3-cell E such that $N \subset \operatorname{Int} E$ and $\alpha_1 \cup \alpha_2 \subset \operatorname{Ext} E$. Using the fact that N separates $\operatorname{Bd} D_1$ from $\operatorname{Bd} D_2$ in A, one can construct a simple closed curve J' in $A \cap \operatorname{Int} E$ such that J' separates $\operatorname{Bd} D_1$ from $\operatorname{Bd} D_2$ in A. Since $J' \subset \operatorname{Int} E$ and $\alpha_1 \cup \alpha_2 \subset \operatorname{Ext} E$, J' can be shrunk to a point in $E^3 - (\alpha_1 \cup \alpha_2)$. But this is a contradiction, since J' and $\alpha_1 \cup \alpha_2$ are linked. Hence, no component of $K \cap A$ separates $\operatorname{Bd} D_1$ from $\operatorname{Bd} D_2$ in A.

By the above paragraph, there is a polygonal arc I' from a to b in $\operatorname{Bd} C - K$. Since I is an unknotted chord of C and $C \cap L = \emptyset$, I can be pushed onto I' by a space homeomorphism without moving points of L or $J - I$. Adjusting each 1-simplex of J in turn, we move J into $E^3 - K$.

The following two results are special cases of Theorem 3.

COROLLARY 3. *Every compact* 0-*dimensional subset of a cellular* 1-*dimensional continuum in E^3 is tame.*

COROLLARY 4. *If M is a continuum and A is a 1-dimensional set such that $A \cup M$ is cellular, then M is cellular.*

The following result is an application of Theorem 3 and the result of [6]. Here we use G to denote a monotone upper semicontinuous decomposition of E^3, H to denote the union of the nondegenerate elements of G and P to denote the natural map from E^3 onto the quotient space E^3/G. For definitions see [1].

COROLLARY 5. *Using the above notation, suppose that $P(\mathrm{Cl}\, H)$ is a compact 0-dimensional subset of E^3/G and that there is a 1-dimensional set $A \subset E^3$ such that $A \cup \mathrm{Cl}\, H$ is cellular. Then G is a cellular decomposition and E^3/G is homeomorphic to E^3.*

3. **Remarks.** In Theorem 3 it is necessary that $K - L$ be at most 1-dimensional. One can embed, for example, a noncellular arc in a cellular, and in fact polyhedral, book with one page. Every compact 0-dimensional subset of a 2-dimensional polyhedron is tame, but there are wild compact 0-dimensional sets which lie on a 2-dimensional cellular continuum.

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