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CURVATURE IN HILBERT GEOMETRIES II

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The interior of a closed convex curve C in the Euclidean plane can be given a Hilbert metric, which is preserved by projective mappings. Let p, q be points interior to C and let u, v be the points of intersection of the line pq with C. The Hilbert distance $h(p, q)$ is defined by

$$h(p, q) = \left| \log \frac{d(u, p)d(v, q)}{d(v, p)d(u, q)} \right| ,$$

where $d(x, y)$ denotes Euclidean distance. If C contains at most one line segment then $h(p, q)$ is a proper metric and the metric lines are the open chords of C carried by the Euclidean lines. Following Busemann [1, p. 237], we define the (qualitative) curvature at a point p as positive or negative if there exists a neighborhood U of p such that for every $x, y \in U$ we have

$$2h(\bar{x}, \bar{y}) \geqq h(x, x) \quad \textbf{respectively} \quad 2h(\bar{x}, \bar{y}) \leqq h(x, y) ,$$

where $\bar{x}$ $\bar{y}$ are the Hilbert midpoints of p and x and of p and y respectively.

In an earlier paper [2] we proved that any point p at which the sign of the curvature is determined is a projective center of C; that is, there exists a projective transformation which maps p into an affine center of the image of C. We also stated the conjecture that a Hilbert geometry has no point of positive curvature. It is the purpose of this paper to prove that conjecture.

Let C be centrally symmetric about the origin O. We may further assume that C has vertical lines of support at its points of intersection with the x-axis. Thus we may describe the upper arc of C by $y = y(x)$ and the lower arc by $y = -y(-x)$. Consider the points

$$a = (\varepsilon x, \varepsilon y(x)) , \qquad b = (\varepsilon x, -\varepsilon y(-x)) , \qquad 0 < \varepsilon < 1 .$$

Then

$$(1) \qquad \begin{aligned} h(O, a) &= \log \frac{\sqrt{x^2 + y(x)^2} \cdot (1 + \varepsilon)\sqrt{x^2 + y(x)^2}}{(1 - \varepsilon)\sqrt{x^2 + y(x)^2} \cdot \sqrt{x^2 + y(x)^2}} \\ &= \log \frac{1 + \varepsilon}{1 - \varepsilon} = 2\varepsilon + \frac{2}{3}\varepsilon^3 + O(\varepsilon^5) . \end{aligned}$$

If $\bar{a} = (\lambda x, \lambda y(x))$ is the Hilbert midpoint of O and a, then according to (1) we have

$$2\log\frac{1+\lambda}{1-\lambda}=\log\frac{1+\varepsilon}{1-\varepsilon}$$

so that

$$\lambda=\frac{1-\sqrt{1-\varepsilon^2}}{\varepsilon}=\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^3+O(\varepsilon^5) \tag{2}$$

Now let $h_1=h(a,b)$ so that

$$\begin{aligned}
h_1&=\log\left[\frac{y(\varepsilon x)+\varepsilon y(-x)}{y(\varepsilon x)-\varepsilon y(x)}\cdot\frac{y(-\varepsilon x)+\varepsilon y(x)}{y(-\varepsilon x)-\varepsilon y(-x)}\right]\\
&=\log\frac{1+\varepsilon\dfrac{y(-x)}{y(\varepsilon x)}}{1-\varepsilon\dfrac{y(x)}{y(\varepsilon x)}}\cdot\frac{1+\varepsilon\dfrac{y(x)}{y(-\varepsilon x)}}{1-\varepsilon\dfrac{y(1-x)}{y(-\varepsilon x)}}\\
&=\varepsilon(y(x)+y(-x))\Big(\frac{1}{y(\varepsilon x)}+\frac{1}{y(-\varepsilon x)}\Big)\\
&\quad+\frac{\varepsilon^3}{3}(y^3(x)+y^3(-x))\Big(\frac{1}{y^3(\varepsilon x)}+\frac{1}{y^3(-\varepsilon x)}\Big)+O(\varepsilon^5)\,.
\end{aligned}\tag{3}$$

We now suppose that the curve C is twice differentiable at its points of intersection with the y-axis. We then have

$$y(\varepsilon x)=y(0)+y'(0)\varepsilon x+\frac{1}{2}y''(0)\varepsilon^2x^2+o(\varepsilon^2)\,,$$

so that

$$\begin{aligned}
\frac{1}{y(\varepsilon x)}&=\frac{1}{y(0)}\left[1-\frac{y'(0)}{y(0)}\varepsilon x+\left(\Big(\frac{y'(0)}{y(0)}\Big)^2-\frac{1}{2}\frac{y''(0)}{y(0)}\right)\varepsilon^2x^2\right]+o(\varepsilon^2)\,,\\
\frac{1}{y^3(\varepsilon x)}&=\frac{1}{y^3(0)}+O(\varepsilon)\,.
\end{aligned}\tag{4}$$

Substituting (4) in (3) we have

$$\begin{aligned}
h_1&=\varepsilon(y(x)+y(-x))\frac{2}{y(0)}\left[1+\left(\Big(\frac{y'(0)}{y(0)}\Big)-\frac{1}{2}\frac{y''(0)}{y(0)}\right)\varepsilon^2x^2\right]\\
&\quad+o(\varepsilon^3)+\frac{\varepsilon^3}{3}(y^3(x)+y^3(-x))\frac{2}{y^3(0)}+O(\varepsilon^4)\\
&=\frac{2\varepsilon}{y(0)}(y(x)+y(-x))+\frac{\varepsilon^3}{y(0)}\Big[(y(x))+y(-x))\\
&\quad\left(2\Big(\frac{y'(0)}{y(0)}\Big)^2-\frac{y''(0)}{y(0)}\right)+\frac{2}{3y(0)^2}(y^3(x)+y^3(-x))\Big]+o(\varepsilon^3)\,.
\end{aligned}\tag{5}$$

If $h_2=h(\bar a,\bar b)$, where $\bar b$ is the Hilbert midpoint of O and b, then (5) holds for h_2 if we replace ε by λ. Using (2) we get

$$h_2 = \frac{\varepsilon}{y(0)}(y(x)+y(-x)) + \frac{\varepsilon^3}{8y(0)}(y(x) + y(-x))\Big[2 + 2x^2\Big(\frac{y'(0)}{y(0)}\Big)^2 - x^2\frac{y''(0)}{y(0)} + \frac{2}{3y^2(0)}\frac{y^3(x) + y^3(-x)}{y(x) + y(-x)}\Big] + o(\varepsilon^3) . \tag{6}$$

From (5) and (6) we have

$$(2h_2 - h_1)\frac{y(0)}{\varepsilon^3(y(x) + y(-x))} = \frac{1}{2} - \frac{3}{2}x^2\Big(\frac{y'(0)}{y(0)}\Big)^2 + \frac{3}{4}x^2\frac{y''(0)}{y(0)} - \frac{1}{2y^2(0)}\frac{y^3(x) + y^3(-x)}{y(x) + y(-x)} + o(1) . \tag{7}$$

Thus the origin could be a point of positive curvature only if the right side of (7) is nonnegative for all sufficiently small values of ε, which means that the leading term is nonnegative for all x. For small x we have

$$\frac{y^3(x) + y^3(-x)}{y(x) + y(-x)} = y^2(0)\Big[1 + x^2\Big(3\Big(\frac{y'(0)}{y(0)}\Big)^2 + \frac{y''(0)}{y(0)}\Big)\Big] + o(x^2) ,$$

so that the leading term on the right side of (7) is

$$\frac{1}{4}x^2\frac{y''(0)}{y(0)} - 3x^2\Big(\frac{y'(0)}{y(0)}\Big)^2 + o(x^2) . \tag{8}$$

For small x this term can be nonnegative only if $y'(0) = y''(0) = 0$. Thus if O is a point of positive curvature and C is twice differentiable at $(0, y(0))$ then C has zero curvature at $(0, y(0))$. But C is twice differentiable almost everywhere and any point of such differentiability can be taken as a point of intersection with the y-axis after a suitable affine transformation. Hence C must have zero curvature almost everywhere. This implies that there must be points of infinite curvature (if corner points are included in this category).

Let us start again, as before, but with $(0, y(0))$ taken as a point of infinite curvature. This means that, if $m(x)$ is the slope of a line of support of C at $(x, y(x))$, then

$$\lim_{\varepsilon_1,\varepsilon_2\to 0+} \frac{m(-\varepsilon_1) - m(\varepsilon_2)}{\min\{\varepsilon_1, \varepsilon_2\}} = \infty . \tag{9}$$

We can rewrite (3) as

$$h_1 = \varepsilon(y(x) + y(-x))\Big(\frac{1}{y(\varepsilon x)} + \frac{1}{y(-\varepsilon x)}\Big) + O(\varepsilon^3) \tag{10}$$

and analogously

$$h_2 = \frac{\varepsilon}{2}(y(x) + y(-x))\left(\frac{1}{y\left(\frac{\varepsilon}{2}x\right)} + \frac{1}{y\left(-\frac{\varepsilon}{2}x\right)}\right) + O(\varepsilon^3)\,. \tag{11}$$

Thus

$$\begin{aligned} 2h_2 - h_1 &= \varepsilon(y(x) + y(-x))\left[\left(\frac{1}{y\left(\frac{\varepsilon}{2}x\right)} - \frac{1}{y(\varepsilon x)}\right)\right. \\ &\qquad \left. - \left(\frac{1}{y(-\varepsilon x)} - \frac{1}{y\left(-\frac{\varepsilon}{2}x\right)}\right)\right] + O(\varepsilon^3) \\ &= -2\varepsilon^2 x(y(x) + y(-x))(M(\varepsilon_1 x) - M(-\varepsilon_2 x)) + O(\varepsilon^3)\,, \end{aligned} \tag{12}$$

where $M(x)$ is the slope of a line of support to the curve $y = 1/y(x)$ at the point $(x, 1/y(x))$ and $(\varepsilon/2) < \varepsilon_1, \varepsilon_2 < \varepsilon$. Since the curve C': $y = 1/y(x)$ has infinite positive curvature at $(0, 1/(y(0))$ our comment (9) applied to C' yields

$$+\infty = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{M(\varepsilon_1 x) - M(-\varepsilon_2 x)}{\min\{\varepsilon_1 x, \varepsilon_2 x\}} \leqq 2 \lim_{\varepsilon \to 0} \frac{M(\varepsilon_1 x) - M(-\varepsilon_2 x)}{x\varepsilon}\,.$$

Thus, from (12) we get

$$\lim_{\varepsilon \to 0} \frac{2h_2 - h_1}{\varepsilon^3} = -\infty\,. \tag{13}$$

Hence for sufficiently small ε, we have $2h_2 - h_1 < 0$ and O cannot be a point of positive curvature.

From (7) it can be seen that not every projective center is a point at which the curvature has a sign. For example, if $y'(0) = y''(0) = 0$, then the leading term on the right of (7) is positive for all x. In other words the center O is not a point of curvature if there exist a circumscribed parallelogram that touches C at the midpoints of its sides and C has zero curvature at one of the points of tangency. Even if C is twice differentiable everywhere, it does not seem easy to determine whether the complicated inequality expressed by the right side of (7) and linking three points of C-is always satisfied.

References

1. H. Busemann, *The Geometry of Geodesics*, Academic Press, New York, 1957.
2. Paul Kelly and E. G. Straus, *Curvature in Hilbert geometries*, Pacific J. Math. (1958), 119-125.

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