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THE EQUIVALENCE OF GROUP INVARIANT POSITIVE DEFINITE FUNCTIONS

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Let G be a separable locally compact group; ρ, a positive definite function; $M(G)$, the set of all finite Radon measures; and

$$\mathfrak{N}_\rho = \left\{\alpha \in M(G) \mid B_\rho(\alpha, \alpha) \equiv \int_{G\times G}\int \rho(t^{-1}s)\alpha(ds)\alpha(dt) = 0\right\} .$$

Let H_ρ be the Hilbert space obtained by completing $M(G)/\mathfrak{N}_\rho$. Similarly define H_σ as the Hilbert space corresponding to another positive definite function σ. ρ and σ are said to be equivalent (symbolically $\rho \sim \sigma$) if there is an equivalence operator T from H_ρ to H_σ which is induced by the identity operator on $M(G)$; i.e. a linear homeomorphism from H_ρ onto H_σ such that $1 - T^*T$ is Hilbert-Schmidt. Theorem 1 and Theorem 2 give necessary and sufficient conditions for $\rho \sim \sigma$ in terms of the unitary representations of G induced by ρ and σ. We discuss group invariant positive definite functions on $X \times X$ where X is a homogeneous space, and generalize Theorem 1 and 2 accordingly. The notion of equivalence operators comes exactly from Gaussian stochastic processes (cf. J. Feldman [4]). Some statistical applications will be discussed in a separate paper later in the year.

I. Preliminaries. Let H be a separable Hilbert space; $\mathfrak{A}$, a von Neumann algebra of bounded operators in H. It has been proved by von Neumann that H can be decomposed into a direct integral of Hilbert spaces so that $\mathfrak{A}$ is the (central) direct integral decomposition into factors (cf. von Neumann [13]). Let Λ and Λ_1 be two separable metric spaces; μ and μ_1 be two finite regular positive Borel measures on Λ and Λ_1 respectively. Let

$$H = \int_\Lambda^\oplus H(\lambda)\mu(d\lambda)$$

and

$$H_1 = \int_{\Lambda_1}^\oplus H(\lambda_1)\mu_1(d\lambda_1)$$

be two direct integral Hilbert spaces; $\mathfrak{A}$ and $\mathfrak{A}_1$ be two von Neumann algebras which have central decompositions

$$\mathfrak{A} = \int_\Lambda^\oplus \mathfrak{A}(\lambda)\mu(d\lambda) \quad \text{and} \quad \mathfrak{A}_1 = \int_{\Lambda_1}^\oplus \mathfrak{A}_1(\lambda_1)\mu_1(d\lambda_1)$$

in H and H_1 respectively. It can be proved (cf. J. T. Schwartz [11]) that $\mathfrak{A}$ and $\mathfrak{A}_1$ are spatially isomorphic if and only if there exist a pair $\tilde{\Lambda}, \tilde{\Lambda}_1$ of Borel subsets of Λ and Λ_1 respectively such that $\mu(\Lambda - \tilde{\Lambda}) = 0$ and $\mu_1(\Lambda_1 - \tilde{\Lambda}_1) = 0$ and a Borel isomorphism $\varphi: \tilde{\Lambda} \to \tilde{\Lambda}_1$ such that $\mathfrak{A}(\lambda)$ and $\mathfrak{A}(\varphi(\lambda))$ are spatially isomorphic and such that $\mu \circ \varphi^{-1}$ is equivalent to $\mu_1 | \Lambda_1$. Since direct integral decomposition is uniquely determined up to a set of measure zero, it can be assumed for our purpose $\Lambda = \tilde{\Lambda}$ and $\Lambda_1 = \tilde{\Lambda}_1$. Hence the central decomposition is unique by the identification of $\tilde{\Lambda}$ and $\varphi(\tilde{\Lambda})$.

For the general theory, we refer to Dixmier's book (cf. Dixmier [1]).

II. The equivalence of positive definite functions on groups.

Let G be a separable locally compact group; $M(G)$, the set of all finite Radon measures on G.

1. DEFINITION. A continuous function ρ is said to be positive definite on G if for any sequence of g_i, $i = 1, 2, \cdots, n$ and any sequence of complex numbers c_i, $i = 1, 2, \cdots, n$, the following is always satisfied

$$\sum_{i,j=1}^{n} \rho(g_j^{-1} g_i) c_i \bar{c}_j \geqq 0 \, . \tag{2.1}$$

It can be easily verified that ρ is positive definite on G if and only if

$$\int_{G\times G}\!\!\int \rho(g^{-1}h)\mu(dh)\bar{\mu}(dg) \geqq 0 \tag{2.2}$$

for all $\mu \in M(G)$.

2. The decomposition of a positive definite function: Consider the functional B_γ on $M(G) \times M(G)$ defined by

$$B_\gamma(\alpha, \beta) \equiv \int_{G\times G}\!\!\int \gamma(s^{-1}t)\alpha(dt)\bar{\beta}(ds) \tag{2.3}$$

where γ is a positive definite function on G, and $\alpha, \beta \in M(G)$. It is clear that β_γ is sesqui-linear and

$$\mathfrak{N}_\gamma = \{\alpha \in M(G) \mid B_\gamma(\alpha, \alpha) = 0\} \tag{2.4}$$

is invariant under the left action of the group G. Let H_γ be the Hilbert space obtained by completing the quotient space $M(G)/\mathfrak{N}_\gamma$ with the inner product given by (2.3). On $M(G)$, consider the linear transform defined by

$$\tilde{U}_s\alpha(A) = \alpha(s^{-1}A) \tag{2.5}$$

where α is any element in $M(G)$ and A is any measurable subset of G; and s, any element of G. $\mathfrak{N}_\gamma$ is invariant under $\tilde{U}_s$. Let U_s be the unitary transformation on H_γ which is densely defined (on $M(G)/\mathfrak{N}_\gamma$) as the quotient transformation of $\tilde{U}_s$. Then the pair (U, H_γ) forms a unitary representation of G.

Let $\mathscr{U}$ be the von Neumann algebra generated by $\{U_s, s\in G\}$, that is $\{U_s, s\in G\}''$, the double commutant of $\{U_s, s\in G\}$. Let ξ be the element of H_γ corresponding to δ_e, the Dirac point mass at the identity. Then

$$(U_s\xi, \xi) = \int_{G\times G}\!\!\int \gamma(s_1^{-1}t)\delta(s^{-1}t)\delta(s_1)dtds_1 = \gamma(s) \ . \tag{2.6}$$

For any $t\in G$, let

$$\xi_t = U_t\xi \ . \tag{2.7}$$

Since the smallest closed linear manifold containing $\{\xi_t, t\in G\}$ is H_γ, $\mathscr{U}\xi$ is dense in H_γ; so ξ is cyclic. According to the theory of the central decomposition, there are a separable metric space Λ and a Radon measure μ on Λ such that

$$H_\gamma = \int_\Lambda^\oplus H_\gamma(\lambda)\mu(d\lambda) \tag{2.8}$$

and

$$\mathscr{U} = \int_\Lambda^\oplus \mathscr{U}(\lambda)\mu(d\lambda) \tag{2.9}$$

where the decomposition (2.9) is a central decomposition. It is also easy to see that

$$U_s = \int_\Lambda^\oplus U_s(\lambda)\mu(d\lambda) \tag{2.10}$$

where μ-almost all of the $U_s(\lambda)$ are unitary on $H_\gamma(\lambda)$ and that

$$\xi = \int_\Lambda^\oplus \xi(\lambda)\mu(d\lambda) \tag{2.11}$$

where μ-almost all of $\xi(\lambda)$ are cyclic in their respective Hilbert spaces. μ-almost all of functions $s\rightarrow(U_s(\lambda)\xi(\lambda), \xi(\lambda))_\lambda$ are positive definite. So γ becomes an integral of positive definite functions as follows:

$$\gamma(s) = \int_\Lambda \gamma_\lambda(s)\mu(d\lambda) \tag{2.12}$$

where

$$\lambda_\lambda(s) = (U_s(\lambda)\xi(\lambda), \xi(\lambda))_\lambda \ , \tag{2.13}$$

and

$$\gamma_\lambda(e) = (\xi(\lambda), \xi(\lambda))_\lambda = 1 \tag{2.13'}$$

μ-almost all λ.

3. DEFINITION. We call the measure μ normalized according to (2.12), (2.13) and (2.13′) the central Radon measure of γ (with respect to

$$\int_\Lambda^\oplus H_\gamma(\lambda)\mu(d\lambda)) .$$

4. The equivalence operator.

a. Definition. Let G be a separable locally compact group. Two positive definite functions ρ and σ are said to be equivalent if the identity operator on $M(G)$ induces an operator T on H_ρ to H_σ such that T is an equalence operator (cf. introduction).

b. Let G be as in definition a; ρ and σ be two positive definite functions on G; H_ρ and H_σ be the corresponding Hilbert spaces as defined in § II. 2. Let (U, H_ρ) and (V, H_σ) be the unitary representations of G induced by ρ and σ respectively. The Dirac point mass δ_e at the identity of G gives rise to cyclic vectors ξ and η in H_ρ and H_σ respectively. Let $\mathscr{U} = \{U_s, s \in G\}''$ and $\mathscr{V} = \{V_s, s \in G\}''$; and

$$\mathscr{U} = \int_\Lambda^\oplus \mathscr{U}(\lambda)\mu(d\lambda) \tag{2.14a}$$

$$\mathscr{V} = \int_{\Lambda_1}^\oplus \mathscr{V}(\lambda_1)\mu_1(d\lambda_1) \tag{2.14b}$$

be their central decompositions.

It has been remarked in § I that if $\mathscr{U}$ and $\mathscr{V}$ are spatially isomorphic, without real loss of generality we may assume Λ and Λ_1 are identical. (2.14) thus can be rewritten as

$$\mathscr{V} = \int_\Lambda^\oplus \mathscr{U}(\lambda)\mathscr{V}(d\lambda) \tag{2.14b'}$$

if $\mathscr{U}$ and $\mathscr{V}$ are spatially isomorphic; that is if U and V are unitarily equivalent.

THEOREM 1. *Let G be a separable locally compact group; ρ and σ two positive definite functions on G. If ρ and σ are equivalent (symbolically $\rho \sim \sigma$), then*

(a) *U and V are unitarily equivalent; and*

(b) *With* (a) *permitting the existence of the central decompositions* (2.14a) *and* (2.14b′), *μ and ν then satisfy the following condi-*

tions, which we shall call the conditions (i), (ii) *and* (iii) *in the rest of the discussion*:

(i) *μ and ν have identical nonatomic parts;*

(ii) *they have the same set of atoms which is countable; and*

(iii) *if $\mathfrak{A}$ is the set of all atoms, then $\mu(a) = \nu(a)$ unless $H_\rho(a)$ is finite dimensional, and*

$$\sum_{a \in \mathfrak{A}} d(a)\left(1 - \frac{\mu(a)}{\nu(a)}\right)^2 < \infty$$

where $d(a)$ is the dimension of $H_\rho(a)$ if $H_\rho(a)$ is finite dimensional and ∞ otherwise.

Proof. We shall divide the proof into four steps:

Step 1. To show: If $\rho \sim \sigma$, then U and V are unitarily equivalent. Let $\xi \in H_\rho$ and $\eta \in H_\sigma$ be the elements corresponding to δ_e so that

$$\rho(s) = (U_s\xi, \xi)_\rho \tag{2.15a}$$

and

$$\sigma(s) = (V_s\eta, \eta)_\sigma \,. \tag{2.15b}$$

It $\rho \sim \sigma$, then it is immediately seen that $\mathfrak{N}_\rho \equiv \mathfrak{N}_\sigma$. Hence $M(G)/\mathfrak{N}_\rho \equiv M(G)/\mathfrak{N}_\sigma$ Let T be the equivalence operator from H_ρ to H_σ induced by the identity operator on $M(G)$. We have for all $s \in G$

$$TU_s = V_sT \,. \tag{2.16}$$

Moreover, the center $\mathscr{Z}_{\mathscr{U}}$ of $\mathscr{U}$ is carried over by T to, the center $\mathscr{Z}_{\mathscr{V}}$ of $\mathscr{V}$. Since T is invertible, T^* is well-defined on all of H_σ. From (2.16)

$$U_sT^* = T^*V_s \tag{2.17}$$

combining (2.16) and (2.17), we obtain

$$(T^*T)U_s = U_s(T^*T) \,. \tag{2.18}$$

Hence T^*T commutes with U_s for all $s \in G$. Consequently T^*T is in the center $\mathscr{Z}_{\mathscr{U}}$ of $\mathscr{U}$; i.e. T^*T is a diagonal operator (cf. Dixmier [1])

$$T^*T = \int_\Lambda^\oplus a(\lambda)I(\lambda)\mu(\lambda) \tag{2.19}$$

where $a(\lambda)$ is a nonnegative function in $L^\infty(\Lambda, \mu)$, and $I(\lambda)$ is the identity operator on $H_\rho(\lambda)$. Let $S = (T^*T)^{1/2}$, and R be the unitary operator: $H_\rho \to H_\sigma$ satisfying

$$T = RS . \tag{2.20}$$

Then, for all $s \in G$.

$$V_s = TU_sT^{-1} = RU_sR^* .$$

Hence (V, H_σ) and (U, H_ρ) are unitarily equivalent.

Step 2. To show: μ and ν have the same set of atoms if $\rho \sim \sigma$. If $\rho \sim \sigma$, by step 1, U and V are unitarily equivalent so that $\mathscr{U}$ and $\mathscr{V}$ are spatially isomorphic. As we remarked in the last part of § I, $\mu \sim \nu$ by the assumption of $\Lambda = \Lambda_1$. Since atoms are points, μ and ν therefore have the same set of atoms. This completes the step 2.

Before we work on step 3, we shall introduce more notations. Let μ_a and μ_c be the atomic and nonatomic parts of μ respectively; ν_a and ν_c be those of ν. Let

$$H_{\rho,c} \equiv \int_\Lambda^\oplus H_\rho(\lambda)\mu_c(d\lambda) \tag{2.21a}$$

and

$$H_{\rho,a} \equiv \int_\Lambda^\oplus H_\rho(\lambda)\mu_a(d\lambda) . \tag{2.21b}$$

Then

$$H_\rho = H_{\rho,c} \oplus H_{\rho,a} \tag{2.22a}$$

$$H_\sigma = H_{\sigma,c} \oplus H_{\sigma,a} . \tag{2.22b}$$

It is easy to see that $T\colon H_{\rho,c} \to H_{\sigma,c}$ and $T\colon H_{\rho,c} \to H_{\sigma,a}$. If T is an equivalence operator, then so are the restrictions $T \mid H_{\rho,c}$ and $T \mid H_{\rho,a}$. It is also true that $\xi = \xi_c \oplus \xi_a$ and $\eta = \eta_a \oplus \eta_c$ where $\xi_a \in H_{\rho,a}$, $\xi_c \in H_{\rho,c}$, $\eta_a \in H_{\sigma,a}$ and $\eta_a \in H_{\sigma,a}$ and $\eta_c \in H_{\sigma,c}$ are cyclic in their respective Hilbert spaces. Moreover,

$$(T \mid H_{\rho,c})\xi_c = \eta_c \tag{2.23a}$$

$$(T \mid H_{\rho,a})\xi_a = \eta_a . \tag{2.23b}$$

So ρ and σ decompose into sum of two positive definite functions respectively:

$$\rho \equiv \rho_c + \rho_a \equiv (U_s\xi_c, \xi_c) + (U_s\xi_a, \xi_a) \tag{2.24a}$$

$$\sigma \equiv \sigma_c + \sigma_a \equiv (V_s\eta_c, \eta_c) + (V_s\eta_a, \eta_a) . \tag{2.24b}$$

Step 3. To show: If $\rho \sim \sigma$, then μ and ν have identical nonatomic parts; i.e. $\mu_c(D) = \nu_c(E)$ for all measurable subsets E of Λ. From above discussion, we may assume μ and ν having only nonatomic parts by passing from the relation $\rho \sim \sigma$ to $\rho_c \sim \sigma_c$. Suppose there exists a measurable subset $\tilde{E}$ such that $\mu(\tilde{E}) \neq \nu(\tilde{E})$. Given a sufficiently small $\varepsilon > 0$, there exists a measurable subset E of $\tilde{E}$ of positive measure such that

$$\left|1 - \frac{d\nu}{d\mu}(\lambda)\right| > \varepsilon$$

for all $\lambda \in E$. Since μ is nonatomic, we can partition E into a disjoint union of infinitely many measurable subsets of positive measures $\{E_1, E_2, \cdots\}$. Since T is an equivalence operator, $\mu(E_i) \neq 0$ implies $\nu(E_i) \neq 0$ for all i. Normalizing

$$\left\{\int_{E_i}^{\oplus} \xi(\lambda)\mu(d\lambda)\right\}$$

we obtain an orthonormal set $\{z_i\}$. Since by (2.19), the definition of central Radon measures and $\mu \sim \nu$,

$$\begin{aligned}\int_A \| x(\lambda) \|^2 d\nu = (T^* Tx, x) &= \int_A a(\lambda) \| x(\lambda) \|^2 d\mu \\ &= \int_A a(\lambda)\left(\frac{d\mu}{d\nu}\right) \| x(\lambda) \|^2 d\nu\end{aligned}$$

for all $x = \int_A^{\oplus} x(\lambda)d\mu \in H_p$, it follows that

$$a(\lambda)\left(\frac{d\mu}{d\nu}\right)(\lambda) = 1 \qquad \nu - \text{a.e.}$$

i.e. $T^* T = \int_A^{\oplus} \{(d\nu)/(d\mu)\} I(x) d\mu$.

Hence

$$\sum_{i=1}^{\infty} \| (1 - T^* T)z_i \|^2 \geqq \sum_{i=1}^{n} \| (1 - T^* T)z_i \|^2 \geqq n\varepsilon^2 .$$

This estimate increases to infinity as n goes to infinity. This contradicts to the fact that T is Hilbert-Schmidt.

Step 4. To show: If $\rho \sim \sigma$, then

$$\sum_{\lambda \in \mathfrak{A}} d(\lambda)\left(1 - \frac{\nu(\lambda)}{\mu(\lambda)}\right)^2 < \infty .$$

As we remarked in step 3, we may reduce to the case where μ and ν have only atomic parts. The set $\mathfrak{A}$ is at most countable, for μ and ν are finite. Then

$$H_\rho = \bigoplus_{\lambda \in \mathfrak{A}} H_\rho(\lambda) \tag{2.25}$$

and

$$\| x \|^2 = \sum_{\lambda \in \mathfrak{A}} \mu(\lambda) \| x(\lambda) \|_\lambda^2 \tag{2.26}$$

for all $x \in H_\rho$ and $x = \bigoplus_{\lambda \in \mathfrak{A}} x(\lambda)$ where $x(\lambda) \in H_\rho(\lambda)$. Let $\{\varphi_i\}$ be any orthonormal set in H_ρ. If $\rho \sim \sigma$, then

$$T^*T = \bigoplus_{\lambda \in \mathfrak{A}} a(\lambda) I(\lambda) .$$

From a theorem of K. Fan (cf. Fan [3]), it follows that

$$\begin{aligned} \max_{\text{all O.N. } \{\varphi_i\}} \sum_i \| (1 - T^*T)\varphi_i \|^2 &= \max \sum_i | ((1 - T^*T)^2 \varphi_i, \varphi_i) | \\ &= \operatorname{Tr} (1 - T^*T)^2 = \sum_{\Lambda} \operatorname{Tr} (1 - T^*T(\lambda))^2 . \end{aligned}$$

If $H_\rho(\lambda)$ is infinite dimensional, then $\operatorname{Tr} (1 - T^*T(\lambda))^2$ is finite only when $a(\lambda) = 1$. Hence $\mu(\lambda) = \nu(\lambda)$ if $\rho \sim \sigma$ and $H_\rho(\lambda)$ is infinite dimensional. When $H_\rho(\lambda)$ is finite dimensional, then $\operatorname{Tr} (1 - T^*T(\lambda))^2 = d(a)(1 - a(\lambda))^2$. Hence

$$\begin{aligned} \operatorname{Tr} (1 - T^*T)^2 &= \sum d(\lambda)(1 - a(\lambda))^2 \\ &= \sum d(\lambda)\left(1 - \frac{\nu(\lambda)}{\mu(\lambda)}\right)^2 . \end{aligned} \tag{2.27}$$

$1 - T^*T$ is Hilbert-Schmidt, therefore (2.27) is finite. We have proved Theorem 1.

Theorem 1 has a converse which we shall state in the following:

THEOREM 2. *If U and V of the last theorem are unitarily equivalent, and if the corresponding central Radon measures, μ and ν satisfy conditions* (i), (ii) *and* (iii), *then* $\rho \sim \sigma$.

Proof is immediate.

III. Positive definite functions on homogeneous spaces. Let G be a separable locally compact group; H, a closed subgroup of G; X, the space of the right cosets; x_0, be the point of X which corresponds to the identity coset H; and finally let $M(X)$ be the set of all finite Radon measures on X. Then positive definite functions on $X \times X$ can be properly defined in the following way:

1. DEFINITION. A continuous function $\hat{\rho}$ on $X \times X$ is said to be positive definite if for all α in $M(X)$

$$\iint \hat{\rho}(x, y)\alpha(dx)\bar{\alpha}(dy) \geqq 0 \ , \tag{3.1}$$

or alternatively,

$$\sum_{i,j=1}^{n} \hat{\rho}(x_i, x_j)\alpha_i\bar{\alpha}_j \geqq 0 \tag{3.1'}$$

for any sequence of points x_i, $i = 1, 2, \cdots, n$ and any sequence of complex numbers c_i, $i = 1, 2, \cdots, n$.

2. DEFINITION. A positive definite function $\hat{\rho}$ is said to be G-invariant if, for any g in G,

$$\hat{\rho}(gx, gy) = \hat{\rho}(x, y) \ . \tag{3.2}$$

3. Group representations and group invariant positive definite function: Let $\hat{\rho}$ be a positive definite function on $X \times X$ which is also group-invariant. Let

$$N_{\hat{\rho}} = \left\{\alpha \in M(X) \mid B_{\rho}(\alpha, \alpha) \equiv \iint \hat{\rho}(x, y)\alpha(dx)\bar{\alpha}(dy) = 0\right\} . \tag{3.3}$$

Then by completing $M(X)/N_{\hat{\rho}}$, we obtain again a Hilbert space $H_{\hat{\rho}}$ with an inner product given by

$$B_{\hat{\rho}}(\alpha, \beta) \equiv \iint \hat{\rho}(x, y)\alpha(dx)\bar{\beta}(dy) \ . \tag{3.4}$$

With the group-invariance property, $N_{\hat{\rho}}$ is invariant under the action of G, i.e., if $\alpha \in N_{\hat{\rho}}$, then the left translates α_g of α also are in $N_{\hat{\rho}}$. This translation gives rise to a unitary transformation $U_g, g \in G$ on $H_{\hat{\rho}}$ in a similar way as in the group case (cf. § II. 2). Moreover, $(U, H_{\hat{\rho}})$ is a unitary representation of G.

3. DEFINITION. We call (U, H_{ρ}) the canonical unitary representation of G associated with $\hat{\rho}$.

4. DEFINITION. Two positive definite functions $\hat{\rho}$ and $\hat{\sigma}$ on $X \times X$ are said to be equivalent if the identity operator on $M(X)$ induces an equivalence operator $\hat{T}$: $H_{\hat{\rho}} \to H_{\hat{\sigma}}$.

5. Relation between positive definite functions on groups and those on homogeneous spaces: By using the same technique as used in § II, we may obtain the necessary and sufficient conditions for $\hat{\rho}$ and $\hat{\sigma}$ to be equivalent. However, we shall establish them through investigation of the relation between the positive definite functions on groups and those on the homogeneous spaces.

Let $(U, H_{\hat{\rho}})$ be the unitary representation of G associated with a G-invariant positive definite function $\hat{\rho}$ on $X \times X$. Let $\xi \in H_{\hat{\rho}}$ be the element corresponding to δ_{x_0}, the Dirac point mass at x_0. Then for any $s \in G$,

$$(3.5) \qquad (U_s\xi, \xi) = \int_{X\times X}\int \hat{\rho}(x, y)\delta_{sx_0}(dx)\delta_{x_0}(dy) = \hat{\rho}(sx_0, x_0) .$$

(3.5) defines a positive definite function σ on the group G which satisfies:

$$(3.6) \qquad \rho(s) = \hat{\rho}(sx_0, x_0) .$$

Since x_0 remains fixed under the action of H, ρ satisfies

$$(3.7) \qquad \rho(s) = \rho(h_1sh_2)$$

for all h_1 and h_2 in H. We thus have the following lemma.

LEMMA 3.1. *For any G-invariant positive definite function $\hat{\rho}$ on $X \times X$, there corresponds a positive definite function ρ on G which is constant on double cosets of H; i.e.,* (3.7) *is satisfied.*

We shall prove the following lemma before we can establish a converse form of Lemma 3.1.

LEMMA 3.2. *Let ρ be a positive definite function on G. Let*

$$(3.8) \qquad K = \{k \in G \mid \rho(k) = \rho(e)\} ,$$

Then K is a closed subgroup of G.

Proof. Since $\rho(e)$ is positive and finite, it can be assumed that $\rho(e) = 1$. According to the theory of group representations (cf. Naimark, [10] Godement [5, 6]) there exists a vector ξ in some Hilbert space L such that

$$(3.9) \qquad \rho(s) = (U_s\xi, \xi)$$

where $(\cdot, \cdot)$ is the inner product in the Hilbert space L and (U, L) is a unitary representation of the group. Let $k \in K$. Then it can be proved that

$$(3.10) \qquad U_k\xi = \xi .$$

So for any $h, k \in K$

$$U_h\xi = U_k\xi = \xi .$$

We have $U_{kh}\xi = U_kU_h\xi = U_k\xi = \xi$; i.e., if $h, k \in K$, then $hk \in K$. If

$h \in K$, $\rho(h) = \bar{\rho}(h^{-1}) = 1 = \rho(h^{-1})$. Hence $h^{-1} \in K$ if $h \in K$. Moreover, since ρ is continuous, we see that K is closed.

LEMMA 3.3. *For any positive definite function ρ on G, there is a largest closed subgroup K such that ρ is constant on double cosets of K. As a consequence, it gives rise to a group-invariant positive definite function $\hat{\rho}$ on $(G/K) \times (G/K)$.*

Proof. Let $\{e\}$ be the subgroup consisting of only the identity of G. Then it is clear that ρ is constant on double cosets of $\{e\}$. Let

$$H = \{h \mid \rho(gh) = \rho(hg) = \rho(g) \quad \text{for all} \quad g \in G\}$$
$$K = \{k \mid \rho(k) = \rho(e) = 1\} .$$

If $h \in H$, choosing $g = e$, then $\rho(h) = \rho(e)$. So $H \subset K$. We now prove that ρ is constant on double cosets of K. Let $k \in K$, then as in Lemma 3.2, $U_k\xi = \xi$ and

$$\begin{aligned}\rho(gk) &= (U_g U_k \xi, \xi) = (U_g \xi, \xi) = \rho(g) = (U_g \xi, U_{k^{-1}}\xi) \\ &= (U_k U_g \xi, \xi) = \rho(kg) ;\end{aligned}$$

i.e., $\rho(g) = \rho(KgK)$.

So K is the largest closed subgroup such that ρ is constant on double cosets of K. (3.6) defines a G-invariant positive definite function as it can be easily verified. Combining the preceding three lemmas, we have the following theorem.

THEOREM 3. *Let G be a separable locally compact group. Each positive-definite function ρ on G gives rise by* (3.6) *to a G-invariant positive definite function $\hat{\rho}$ on $(G/H) \times (G/H)$ where H is the set of all x in G such that $\rho(x) = \rho(e)$. Conversely, to any G-invariant positive definite function $\hat{\rho}$ on $(G/H) \times (G/H)$ there corresponds by* (3.6) *a positive definite function ρ on G such that $H = \{x: \rho(x) = \rho(e)\}$.*

5. Let G and H be the same as before; τ, the canonical mapping from G to $G/H = X$. Then a subset E of X is measurable if and only if $\tau^{-1}(E)$ is measurable in G (cf. Mackey [11]). According to the theory of decomposition of measure (cf. Halmos [7, 8], von Neumann [12], Dieudonné [2], Mackey [9]), for any finite Radon measure α on G, there is a measure $\hat{\alpha}$ in X such that for all measurable $E \subset X$

$$\hat{\alpha}(E) = \alpha(\tau^{-1}(E)) \tag{3.11}$$

and

$$\int_X f(x) \int_G g(s) d\alpha_x(s) d\hat{\alpha}(x) = \int_G f(\tau(s)) g(s) d\alpha(s) \tag{3.12}$$

where α_x is a finite Radon measure on G which only lives on the coset x; f, a function in $L^1(X, \tilde{\alpha})$; and g, a bounded measurable function on G. Conversely, $\tilde{\alpha}$ certainly defines a measure on G. If ρ is a positive definite function on G which is constant on H, then $\rho(t^{-1}s) = \hat{\rho}(sx_0, tx_0) = \hat{\rho}(x, y)$. By generalizing (3.11) and (3.12) to the two dimensional product measures (cf. Mackey [9]), it follows that

$$\int_{G\times G}\!\!\int \rho(t^{-1}s)\alpha(ds)\bar{\alpha}(dt) = \int_{X\times X}\!\!\int \hat{\rho}(x, y)\tilde{\alpha}(dx)\tilde{\alpha}(dy)\int_{G\times G}\!\!\int \alpha_x(ds')\alpha_y(dt') . \tag{3.13}$$

Hence $\alpha \in \mathfrak{N}_\rho \subset M(G)$ if $\alpha_x(G) = 0$. If we let $x_\alpha \in H_\rho$ be the element corresponding to α in $M(G)$ and $x_{\hat{\alpha}} \in H_{\hat{\rho}}$ be the element corresponding to $\alpha_x(G)\tilde{\alpha}$ in $M(X)$, we conclude from the above discussion that φ: $H_\rho \to H_{\hat{\rho}}$ satisfying $\varphi(x_\alpha) = x_{\hat{\alpha}}$ is a spatial isomorphism which commutes with the transformation induced by translations. There is a similar mapping ψ: $H_\sigma \to H_{\hat{\sigma}}$ satisfying $\psi(y_\alpha) = y_{\hat{\alpha}}$ where $y_\alpha \in H_\sigma$, $y_{\hat{\alpha}} \in H_{\hat{\sigma}}$ are the corresponding elements of α and $\alpha_x(G)\tilde{\alpha}$ respectively. Furthermore, if T: $H_\rho \to H_\sigma$ and $\tilde{T}$: $H_{\hat{\rho}} \to H_{\hat{\sigma}}$ are the mapping induced by the identity mappings on $M(G)$ and $M(X)$ respectively, then it is clear that

$$Tx_\alpha = y_\alpha \tag{3.14}$$

$$\tilde{T}x_{\hat{\alpha}} = y_{\hat{\alpha}} \tag{3.15}$$

and the following diagram commutes

$$\begin{tikzcd} H_\rho \arrow[r, "T"] \arrow[d, "\varphi"] & H_\sigma \arrow[d, "\psi"] \\ H_\rho \arrow[r, "\tilde{T}"] & H_{\hat{\sigma}} \end{tikzcd}$$

Hence $\tilde{T}$ is a linear homeomorphism if and only if so is T. If $\tilde{T}$ is such, the similar commutative diagram for $\tilde{T}^*$, T^* holds

$$\begin{tikzcd} H_\rho \arrow[r, "T^*"] \arrow[d, "\varphi"] & H_\sigma \arrow[d, "\psi"] \\ H_{\hat{\rho}} \arrow[r, "\tilde{T}^*"] & H_{\hat{\sigma}} \end{tikzcd}$$

Therefore,

$$\varphi T^* T = \tilde{T}^*\psi T = \tilde{T}^*\tilde{T}\varphi \tag{3.16}$$

and

$$\| (I - T^*T)x \|^2 = \| \varphi(I - T^*T)x \|^2 = \| (I - \tilde{T}^*\tilde{T})\varphi x \|^2 . \tag{3.17}$$

We immediately conclude that $\tilde{T}$ is an equivalence operator if and only if so is T, and arrive the following theorem.

THEOREM 4. *Two group-invariant positive definite functions $\hat{\rho}$ and $\hat{\sigma}$ on $X = G/H$ are equivalent if and only if the corresponding positive definite functions ρ and σ on G are equivalent.*

6. Let X be a separable metric space; G, a locally compact transformation group of X. Let $M(X)$ be the set of all finite Radon measures on X. The positive definite functions and group invariance property can be similarly defined as in Definitions III.1 and III.2. Suppose that X has dense orbits. Let $x_0 \in X$ such that $X_0 = Gx_0$ is dense in X. Then $X_0 = G/H_0$, where $H_0 = \{h \in G \mid hx_0 = x_0\}$. We now embed G/H_0 in X.

COROLLARY 4.1. *Let X, G be as above; and suppose that X has dense orbits. Then two G-invariant positive definite functions $\hat{\rho}$ and $\hat{\sigma}$ are equivalent if and only if the conditions in Theorem* 1 *are satisfied.*

Proof. The elements ξ in H_ρ and η in H_σ corresponding to the Dirac point mass at x_0 are cyclic, by the continuity of positive definite functions. Hence applying Theorem 4 and Theorem 1, we assert the corollary.

COROLLARY 4.2. *Let X be a separable metric space; G, a locally compact group acting ergodically on X. Suppose that the ergodic measure μ satisfies $\mu(0) > 0$ for any open set $0 \subset X$. Then $\hat{\rho} \sim \hat{\sigma}$ if and only if* (a) *and* (b) *of Theorem* 1 *are satisfied.*

Proof. It is known that if G acts ergodically on X and satisfies the above hypothesis, then X has dense orbits; in fact

$$\mu\{x \mid \bar{G}\bar{x} \neq X\} = 0 .$$

Hence applying Corollary 4.1, we prove the corollary.

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