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BASES IN HILBERT SPACE

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A sequence (x_i) of elements of a Hilbert space, $\mathscr{H}$, is a *basis* for $\mathscr{H}$ if every $h \in \mathscr{H}$ has a unique, norm-convergent expansion of the form $h = \sum a_i x_i$, where (a_i) is a sequence of scalars. The sequence is *minimal* if there exists a sequence $(y_i) \subset \mathscr{H}$ such that $(x_i, y_j) = \delta_{ij}$. Every basis is minimal, and the sequence (a_i) in the expansion of h (above) is given by $a_i = (h, y_i)$. In this paper, we restrict our attention to *real* Hilbert space.

We derive, from classical characterizations of bases in B-spaces, criterea for (x_i) to be a basis for $\mathscr{H}$, as well as for (x_i) to be minimal in $\mathscr{H}$. We show that the sequence is minimal if and only if there are sequences $(g_i) \subset \mathscr{H}$ whose Gram matrices have a prescribed form. Similar conditions are obtained for (x_i) to be a basis for $\mathscr{H}$.

Let (x_i) be a linearly independent sequence of elements of $\mathscr{H}$. Using the Gram-Schmidt process, one finds an orthonormal basis, (w_i), for the closed span, $[x_i]$ of the sequence (x_i). We assume throughout that $[x_i] = \mathscr{H}$. Then, we may write

$$x_i = \sum_{j=0}^{i} p_{ij} w_j \ ,$$

and

$$w_i = \sum_{j=0}^{i} q_{ij} x_j \ .$$

If we let P and Q denote the matrices (p_{ij}) and (q_{ij}), respectively, then each is lower triangular, and $PQ = QP = I = (\delta_{ij})$. It is a classical result that Q is the unique inverse of P.

For (x_i) to be minimal, we need a sequence (y_i) such that $(x_i, y_j) = \delta_{ij}$. It is easy to see that, formally, $y_i = \sum_{y=i}^{\infty} q_{ji} w_j$. Further, the sequence is minimal if and only if the distance from x_k to $[x_j], j \neq k$ is positive. Using these facts, we get the following theorem. The second part is similar to the characterization of minimality due to Foias and Singer [2].

THEOREM 1. *Let $H = (h_{ij})$ denote the Gram matrix of (x_i), i.e., $h_{ij} = (x_i, x_j)$. Then the sequence is minimal if and only if any of the following conditions holds:*

(a) *The matrix $R = Q^T Q$ exists.*

(b) *There exists a sequence, (δ_i), with $\delta_i > 0$ for all i, such* *that for all real vectors* $A = (a_0, a_1, \cdots, a_n, 0, \cdots)$, $AHA^T \geqq \sum \delta_i a_i^2$.

(c) *There exists a sequence* (ε_i) *with* $\varepsilon_i > 0$ *for all* i *such that* $ARA^T \geqq \sum \varepsilon_i a_i^2$, *with* A *as in* (b).

Proof. (a) Follows from the formal relation $y_i = \sum q_{ji} w_j$. For (b), notice that $AHA^T \geqq \| \sum a_i x_i \|^2$. If (x_i) is minimal, then $AHA^T \geqq \lambda_i \| x_i \|^2 a_i^2$, where $\lambda_i^{1/2}$ is the distance from $x_i / \| x_i \|$ to $[x_j]$, $j \neq i$. Therefore, for each permutation (n_i) of the nonnegative integers,

$$AHA^T \geqq \sum a^{-(n_i+1)} \lambda_i a_i^2 \| x_i \|^2 .$$

So $\delta_i = 2^{-(n_i+1)} \lambda_i \| x_i \|^2$ works. On the other hand, if $AHA^T \geqq \sum \delta_i a_i^2$, then $AHA^T \geqq \delta_i a_i^2 = \lambda_i \| x_i \|^2 a_i^2$ for each i. Part (c) follows since (y_i) is minimal if and only if (x_i) is minimal.

2. Here we derive further criteria, for minimal and basic sequences, which depend upon the existence of certain Gram matrices. First, we recall that a fundamental sequence (x_i) in a B-space is minimal if and only if, for each n, there exists a constant $K_n \geqq 1$ such that, for all m and all sequences (a_j),

$$\left\| \sum_{j=0}^{n} a_j x_j \right\| \leqq K_N \left\| \sum_{j=0}^{n+m} a_j x_j \right\| .$$

Further, such a sequence is basic if and only if (K_n) is bounded (that is, if and only if a bounded sequence (K_n) can be chosen) [1]. In either case, K_n is to be chosen in such a way that

$$\left\{ K_n^2 \left\| \sum_{j=0}^{n+m} a_j x_j \right\|^2 - \left\| \sum_{j=0}^{n} a_j x_j \right\|^2 \right\}$$

defines a positive definite form on the collection of all finite real sequences. Associated with this form is the matrix $S = S(n, K_n)$, defined as follows:

$$S_{ij} = \begin{cases} (K_n^2 - 1)(x_i, x_j);\ 1 \leqq i, j \leqq n \\ K_n^2 (x_i, x_j);\ \text{otherwise} . \end{cases}$$

The positive definiteness of the form ASA^T will be achieved over the finite vectors $A = (a_1, a_2, \cdots, a_n, 0, \cdots)$ if and only if each principal $k \times k$ submatrix, $S^{(k)}$ of S is positive definite. Each $S^{(k)}$ is positive definite if and only if there exists a real, nonsingular, lower triangular matrix T such that $S^{(k)} = T^{(k)} T^{(k)^T}$. A routine calculation shows that

$$T_{ij} = \begin{cases} \sqrt{K_n^2 - 1}\, p_{ij};\ 1 \leqq i, j \leqq n \\ \dfrac{K_n^2}{\sqrt{K_n^2 - 1}}\, p_{ij};\ i > n, 1 \leqq j \leqq n . \end{cases}$$

Thus, we must solve, in the reals, the equations

$$\sum_{j=n+1}^{i} T_{ij}T_{kj} = K_n^2(x_i, x_k) - \frac{K_n^4}{K_n^2 - 1}(\pi_n x_i, x_k) ,$$

where $\pi_n x_i = \sum_{j=1}^{n} p_{ij} w_j$. If these equations are solvable, then S is positive definite (over finite A), if and only if $T_{ii} \neq 0$. Now let $(f_i)_{i=n+1}^{\infty}$ be any linearly independent sequence in $\mathscr{H}$ for which

$$(f_i, f_j) = K_n^2(x_i, x_j) - \left(\frac{K_n^4}{K_n^2 - 1}\right)(\pi_n x_i, x_j) ,$$

if it exists. If we orthonormalize (f_i), we get a sequence $(g_i)_{i=n+1}^{\infty}$ and

$$f_i = \sum_{j=n+1}^{i} T_{ij} g_j .$$

Linear independence of (f_i) gives $T_{ii} \neq 0$. On the other hand, if the equations above are solvable, for (T_{ij}), we may set $f_i = \sum_{j=n+1}^{i} T_{ij} w_j$.

We have the following theorem:

THEOREM 2. *The sequence* (x_i) *is*

(a) *minimal if and only if, for each* n, *there exists* $K_n \geqq 1$ *and a linearly independent sequence* $(f_i)_{i=n+1}^{\infty}$ *such that*

$$(f_i, f_j) = K_n^2(x_i, x_j) - \frac{K_n^4}{K_n^2 - 1}(\pi_n x_i, x_j) .$$

(b) *a basis if and only if it is minimal, and the sequence* (K_n) *may be chosen so that it is bounded.*

The sequence (x_j) is minimal if and only if, for each n, there exists $C_n \geqq 1$ such that, for all m and sequences (a_i),

$$\left\|\sum_{i=n+1}^{n+m} a_i x_i\right\| \leqq C_n \left\|\sum_{i=1}^{n+m} a_i x_i\right\| .$$

It is basic if and only if (C_n) may be chosen as a bounded sequence (see, e.g., [4]). Using these facts, and arguments similar to those for Theorem 2, we obtain,

THEOREM 3. *The sequence* (x_i) *is*

(a) *minimal if and only if, for each* n, *there exists* $C_n \geqq 1$ *and a linearly independent sequence* $(g_i)_{i=n+1}^{\infty}$ *such that, for* $i, j > n$,

$$(g_i, g_j) = (C_n^2 - 1)(x_i, x_j) - C_n^2(\pi_n x_i, x_j) ,$$

and

(b) *basic if and only if it is minimal and* (C_n) *may be chosen as a bounded sequence.*

In deriving Theorem 3, one must determine the positive definiteness of the matrices S defined by

$$S_{ij} = \begin{cases} C_n^2(x_i, x_j);\ 1 \leqq i \leqq n \text{ or } 1 \leqq j \leqq n \\ (C_n^2 - 1)(x_i, x_j);\ i, j > n . \end{cases}$$

An interesting characterization of minimal sequences and bases is the following.

PROPOSITION. *The sequence* (x_i) *is*

(a) *minimal if its Gram matrix,* H, *is strictly diagonally dominant, and*

(b) *a basis if its Gram matrix is uniformly diagonally dominant.*[1]

Proof. If H is strictly diagonally dominant, for each n there exists $\gamma_n \in (0, 1)$ such that $\gamma_n |(x_n, x_n)| < \sum_{j \neq n} |(x_n, x_j)|$. Then, for $C_n^2 = 1/\gamma_n$, the matrix S is strictly diagonally dominant, and hence positive definite over finite A[5]. Part (b) follows in the same manner.

Using the same method of proof, Theorems 3 and 4, and the fact that the positive definite $n \times n$ matrices define a cone in the linear space of all $n \times n$ matrices, we obtain the most general form of our characterization of minimal sequences and bases in $\mathscr{H}$.

THEOREM 4. *The sequence* (x_i) *is*

(a) *minimal if and only if, some (and hence all)* $\alpha, \beta > 0$ *and all* n, *there exist* $K_n, C_n \geqq 1$ *and* $(g_i)_{i=n+1}^{\infty}$ *such that, for* $i, j > n$,

$$(g_i, g_j) = (\alpha K_n^2 + \beta C_n^2 - \beta)(x_i, x_j) - \left(\frac{(\alpha K_n^2 + BC_n^2)^2}{\alpha K_n^2 + \beta C_n^2 - \alpha}\right)(\pi_n x_i, x_j) ,$$

and

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[1] A symmetric matrix A is *strictly diagonally dominant* [5] if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i, and is *uniformly diagonally dominant* if there exists $\gamma \in (0, 1)$ such that $\gamma\, |a_{ii}| \geqq \sum_{j \neq i} |a_{ij}|$ for each i.

(b) *a basis if and only if* (K_n) *and* (C_n) *may be chosen as bounded sequences.*

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