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ON A CHARACTERIZATION OF INFINITE COMPLEX MATRICES MAPPING THE SPACE OF ANALYTIC SEQUENCES INTO ITSELF

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Let S be the space of all complex sequences. An element $u = \{u_n\}_{n=0}^{\infty}$ of S is called analytic if for some constant $M > 0$, $|u_n| \leqq M^{n+1}$ for $n = 0, 1, 2, \cdots$. By A denote the space of all analytic sequences. Clearly A is the space of all complex functions analytic at zero. I. Heller has proved

Theorem 1. The transformation $y_n = \sum_{m=0}^{\infty} c_{nm} u_m$ maps A into A if and only if for every $p > 0$ there exists a $q > 0$ and a constant $M > 0$ such that $|c_{nm}| \leqq Mp^m/q^n$ for $m, n = 0, 1, 2, \cdots$; and also if and only if the function G of two complex variables (i.e., in $E \times E$, where E is the complex plane) respresented by the double power series $G(z, y) = \sum_{m,n=0}^{\infty} c_{nm} z^m y^n$ be regular on $E \times 0$.

The present paper provides an alternative proof for the theorem in order to give insight into the structure of A as a countable union of BK spaces, that is, Banach spaces with coutinuous coordinates.

Let $q > 0$ be fixed and $A_q = \{u \in S \mid \sup_n |q^n u_n| = \|u\|_q < \infty, n = 0, 1, 2, \cdots\}$.

THEOREM 2. (1) $A = \bigcup_{n=0}^{\infty} A_{q_n}$ *where* $q_n \downarrow 0$, *and*
(2) *for any* $q > 0, (A_q, \|u\|_q)$ *is a BK space.*

Proof. (1) A complex sequence $u = \{u_n\}_{n=0}^{\infty}$ is analytic if and only if the $\sup_n |q^n u_n| \leqq M$ for some $q > 0$, some constant $M > 0$ and $n = 0, 1, 2, \cdots$. It now follows that $A = \bigcup_{0<q<\infty} A_q$. The proof is completed by a set theoretic argument showing that $\bigcup_{0<q<\infty} A_q = \bigcup_{n=0}^{\infty} A_{q_n}$ after observing that if $0 < r < s$, then $A_s \subset A_r$.

(2) It suffices to observe that $(A_q, \|u\|_q)$ is isometrically isomorphic with the Banach space of all bounded complex sequences

$$(m) = \{u \in S \mid \|u\|_{(m)} = \sup_n |u_n|\} .$$

The operator E_q from A_q into (m) establishing this isomorphism is defined by $E_q: \{u_n\}_{n=0}^{\infty} \rightarrow \{q^n u_n\}_{n=0}^{\infty}$. Finally for each $n, |u_n| \leqq \|u\|_q/q^n$. Thus the coordinate functional $P_n(u) = u_n$ is continuous, being a linear operator on A_q. This proves that the space $(A_q, \|u\|_q)$ is a BK space.

By a mapping C of a sequence space X into a sequence space Y generated by an infinite complex matrix $(c_{nm})\, m, n = 0, 1, 2, \cdots$ is meant $(y = C(u), u \in X)$ if and only if $(y_n = \sum_{m=0}^{\infty} c_{nm} u_m, y = \{y_n\}_{n=0}^{\infty} \in Y)$.

THEOREM 3. *Let C be the transformation from A into A generated by an infinite complex matrix (c_{nm}) $n, m = 0, 1, 2, \cdots$. For each $p > 0$ and $q > 0$ fixed let $A_{pq} = \{u \in A_p \mid C(u) \in A_q\}$. Then*

(1) *$A_p = \bigcup_{n=0}^{\infty} A_{pq_n}$ where $q_n \downarrow 0$, and*

(2) *for each $p > 0$ and $q > 0$ fixed,*

$$(A_{pq}, \| u \|_{pq} = \| u \|_p + \| C(u) \|_q)$$

is a BK space.

Proof. (1)

$$A_p = \left\{u \in A_p \mid C(u) \in A = \bigcup_{n=0}^{\infty} A_{q_n}, q_n \downarrow 0\right\}$$
$$= \bigcup_{n=0}^{\infty} \{u \in A_p \mid C(u) \in A_{q_n}\} = \bigcup_{n=0}^{\infty} A_{pq_n} .$$

(2) For each $u = \{u_n\}_{n=0}^{\infty}$ belonging to the BK space A_p, $(C(u))_k = C_k(u) = \sum_{n=0}^{\infty} c_{kn} u_n$ on A_p is the limit of the sequence of continuous linear operators $\sum_{n=0}^{j} c_{kn} u_n$ $j = 0, 1, 2, \cdots$ on A_p. So C_k is a continuous linear operator on A_p for each $k = 0, 1, 2, \cdots$ by [2, Th. 17, p. 54]. This shows that C is a continuous linear operator from A_p into A.

The BK spaces $(A_p, \| u \|_p)$, $(A_q, \| u \|_q)$ and the continuous linear map $C : A_p \to A$ satisfy the conditions of [4, Th. 1, p. 226]. This together with [4, Th. 3, p. 205] prove that $A_p \cap C^{-1}(A_q) = A_{pq}$ is a FK space (Frechet space with continuous coordinates) with the norm $\| u \|_p + \| C(u) \|_q$ (as the sup of two normed topologies is given by the sum of the norms). That $(A_{pq}, \| u \|_{pq})$ is a BK space is now immediate.

THEOREM 4. *Let C be the transformation from A into A generated by an infinite complex matrix (c_{nm}) $n, m = 0, 1, 2, \cdots$. Then*

(1) *for every $p > 0$ there exists a $q > 0$ such that C maps A_p into A_q.*

The transformation C from A_p into A_q generated by (c_{nm}) for fixed $p > 0$ and $q > 0$

(2) *is linear and continuous, and*

(3) *its norm, $\| C \| = \sup_n \sum_{m=0}^{\infty} q^n \mid c_{nm} \mid p^{-m}$, $n = 0, 1, 2, \cdots$.*

Proof. (1) For any $p > 0, C : A_p \to A = \bigcup_{n=0}^{\infty} A_{q_n}, q_n \downarrow 0$. Moreover $A_p = \bigcup_{n=0}^{\infty} A_{pq_n}$. And by defintion of the Banach norm $\| u \|_{pq} = \| u \|_p + \| C(u) \|_q$ on A_{pq}, the injective maps from A_{pq_n} into A_p are continuous for any $p > 0$. Thus by [4, Corollary 6, p. 205] or [5, Satz 4.6, p. 472], there exists an index k such that $A_p = A_{pq_k}$. This q_k is the desired q.

(2) The lineary of C is clear. Continuity follows from [4, Corollary 5, p. 204].

(3) Map A_p into (m) by the operator $E_p: u = \{u_m\}_{m=0}^{\infty} \rightarrow \{p^m u_m\}_{m=0}^{\infty}$. Define the operator B to be $E_q C E_p^{-1}$. Clearly B is an operator from (m) into (m) which is generated by the infinite matrix

$$(b_{nm}) = (q^n c_{nm} p^{-m}) .$$

And so B is linear and continuous from (m) into (m) and $\|B\| = \sup_n \sum_{m=0}^{\infty} q^n |c_{nm}| p^{-m}$ $n = 0, 1, 2, \cdots$. But $\|C\| = \|B\|$.

Proof of Theorem 1. By Theorem 4 (1) and (3) for every $p > 0$ there exists a $q > 0$ such that C maps A_p into A_q and

$$C\| = \sup_n \sum_{m=0}^{\infty} q^n |c_{nm}| p^{-m} \leqq M, n = 0, 1, 2, \cdots$$

respectively. Thus $|c_{nm}| \leqq Mp^m/q^n, m, n = 0, 1, 2, \cdots$. This proves necessity.

Since $A = \bigcup_{0<p<\infty} A_p$, it suffices to show that the operator C is well defined on A_p. Let $0 < r < 1$. For the number pr there exists a number $q > 0$ such that $|c_{nm}| \leqq M(pr)^m q^{-n}$ for all m and n and some M, and so $|c_{nm}u_m| \leqq Mr^m q^{-n} \|u\|_p$ for all m and n. This implies that the series $\sum_{m=0}^{\infty} c_{nm}u_m$ is convergent and

$$\left| \sum_{m=0}^{\infty} c_{mn}u_m \right| \leqq M(1 - r)^{-1} q^{-n} \|u\|_p .$$

Thus the sequence $y = C(u)$ belongs to the space A_q and therefore also to the space A. This proves the sufficiency of the condition.

The functional analysis method employed herein has implications beyond the proof of Theorem 1. It enables us to extend Heller's result to the space of Borel measurable functions bounded with respect to a weight function. This will be the subject of a forthcoming paper.

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