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A COUNTER-EXAMPLE TO A FIXED POINT CONJECTURE

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Let A be a finite-dimentional commutative Jordan algebra over a field F of characteristic zero. Then we may write $A = S + N$, S a semisimple subalgebra (Wedderburn factor), N the radical of A, [5], [6]. If G is a completely reducible group of automorphisms of A, then we may choose S to be invariant under G, [4]. If G is finite, then we showed in [10] that any two such G-invariant S were conjugate via an automorphism σ of A which centralizes G and which is a product of exponentials of nilpotent inner derivations of A of the form $\sum [R_{a_i}, R_{x_i}]$, x_i in N, a_i in A, where R_a is multiplication by a in A. It was conjectured in [10] that the various elements x_i and a_i which occur in the formulation of σ could be chosen as fixed points of G. This conjecture was based on analogous fixed point results proved for associative and Lie algebras, [7], [8], [9]. However, this conjecture is false, and we present in this note a simple counter-example.

We consider three-by-three matrices over F. Denoting by e_{ij} the usual matrix units, set $e = e_{11} + e_{22}$, $f = e_{33}$ and $x = e_{31}$. Consider the Jordan algebra A with basis e, f, x and multiplication table

	e	f	x
e	$2e$	0	x
f	0	$2f$	x
x	x	x	0

.

Clearly A has a one-dimensional radical $N = Fx$, and $S(0) = Fe + Ff$ is a Wedderburn factor of A. By [2], all Wedderburn factors are isomorphic, so are spanned by two orthogonal idempotents. The only idempotents (nonzero) of A are $(e/2) + \alpha x$, $(f/2) + \beta x$, α, β in F. The only pairs of orthogonal idempotents are $(e/2) + \alpha x$, $(f/2) - \alpha x$, α in F. Hence the Wedderburn factors of A are of the form $S(\alpha) = F(e + \alpha x) + F(f - \alpha x)$, and clearly $\alpha \rightarrow S(\alpha)$ is one-to-one.

A has two types of automorphisms, as can be seen by a direct check. The first type $A(\delta, \pi)$, δ, π in F, $\pi \neq 0$, is given by:

$$A(\delta, \pi)\begin{cases} e \to f + \delta x \\ f \to e - \delta x \\ x \to \pi x \end{cases}.$$

The second type $B(\delta, \pi)$, δ, π in F, $\pi \neq 0$, is given by:

$$B(\delta, \pi)\begin{cases} e \to e + \delta x \\ f \to f - \delta x \\ x \to \pi x \end{cases}.$$

A calculation shows that $S(\alpha)\, B(\delta, \pi) = S(\alpha\pi + \delta)$, so that if $\pi \neq 1$, $S((1-\pi)^{-1}\delta)$ is the only $B(\delta, \pi)$-invariant Wedderburn factor of A. If $\delta \neq 0$, then $B(\delta, 1)$ fixes no Wedderburn factor, and $B(0, 1) = I$, the identity mapping of A.

Turning to $A(\delta, \pi)$, we have that $S(\alpha) A(\delta, \pi) = S(-\alpha\pi - \delta)$. Hence if $\pi \neq -1$, $S(-\delta(1+\pi)^{-1})$ is the only $A(\delta, \pi)$-invariant Wedderburn factor of A. If $\delta \neq 0$, then $A(\delta, -1)$ fixes no Wedderburn factor, but $A(0, -1)$ fixes all Wedderburn factors $S(\alpha)$. Let G be the group of order two generated by $A(0, -1)$:

$$A(0, -1)\begin{cases} e \to f \\ f \to e \\ x \to -x \end{cases}.$$

Note that $e - f$ and x are eigenvectors for the eigenvalue -1 of $A(0, -1)$, so that $F(e + f)$ is the fixed point space of G. $R_{e+f} = 2I$, and N has no nonzero fixed points under G, which disproves the conjecture.

In checking the result of [10] in this example, let $D = [R_{e-f}, R_x] = R_{e-f}R_x - R_xR_{e-f}$. Then one can check that

$$\sigma = \exp\left(\left(\frac{\beta - \alpha}{2}\right)D\right) = I + \frac{\beta - \alpha}{2}D$$

will map $S(\alpha)$ onto $S(\beta)$ for any α, β in F. Since $e - f$ and x are in the -1 – eigenspace of $A(0, -1)$, the rule $g^{-1}R_a g = R_{ag}$ for a in A, g an automorphism of A, shows that D commutes with $A(0, -1)$, so that σ centralizes G. This leads to the more complicated conjecture that one can formulate σ in terms of inner derivations $[R_a, R_x]$, a in A, x in N, such that for any g in G, a and x are eigenvectors of g corresponding to eigenvalues $\alpha(g)$ and $\beta(g)$ respectively, such that $\alpha(g)\beta(g) = 1$. Such a σ will centralize G. We also note that this conjecture and the fixed point conjecture are still open for alternative algebras (see [10] for a precise formulation), although the fixed point conjecture now seems unlikely for alternative algebras, in view of the above counter-example for Jordan algebras, due to the close relation between alternative and Jordan algebras, [3]. We also remark that for completely reducible G, the existence of a σ centralizing G is still an open question. If $N^2 = 0$, this is trivial (see [10], § 5), and the difficulty lies in the case $N^2 \neq 0$. We also note that if F is any field of characteristic not two, then our example has A/N separable and $N^2 = 0$, in which case the Wedderburn-Malcev properties hold, [1], [2], [6], and any finite group G of order not divisible by the characteristic of F will fix a Wedderburn factor, [6]. So our example also shows that the fixed point conjecture is false for the case $N^2 = 0$, R/N separable.

We conclude with an example of an infinite group G which illustrates the conjecture for completely reducible G that σ can be chosen to centralize G, in a case where $N^2 \neq 0$. Again considering three-by-three matrices over F, let $e = e_{11} + e_{33}$, $x = e_{12}$, $y = e_{23}$, $z = e_{13}$. Let A be the Jordan algebra with basis e, x, y, z and multiplication table

	e	x	y	z
e	$2e$	x	y	$2z$
x	x	0	z	0
y	y	z	0	0
z	$2z$	0	0	0

Clearly the radical N of A is $N = Fx + Fy + Fz$, $N^2 = Kz$ and $N^3 = 0$. Clearly $S(0, 0) = Ke$ is a Wedderburn factor, and if we calculate the elements f for which $f^2 = 2f$, we find

$$f = e + \alpha x + \beta y - \alpha\beta z, \ \alpha, \beta \in F .$$

Since all Wedderburn factors are isomorphic (we are assuming characteristic zero), the Wedderburn factors are of the form

$$S(\alpha, \beta) = F(e + \alpha x + \beta y - \alpha\beta z) ,$$

and the correspondence $(\alpha, \beta) \to S(\alpha, \beta)$ is one-to-one on $F \times F$.

Let $\delta \in F$, $\phi \in F$, $\phi \neq 0, 1$. Let $A(\delta, \phi)$ be the automorphism of A given by:

$$A(\delta, \phi) \begin{cases} e \to e + \delta y \\ x \to x - \delta z \\ y \to \phi y \\ z \to \phi z \end{cases} .$$

$A(\delta, \phi)$ is completely reducible, since A has a basis of eigenvectors y, z, $(1-\phi)e+\delta y$, $(1-\phi)x-\delta z$, the latter two being fixed points of $A(\delta, \phi)$. One can check that $S(\alpha, \beta)A(\delta, \phi) = S(\alpha, \delta+\beta\phi)$, so that $S(\alpha, \delta(1-\phi)^{-1})$ is fixed by G, the group generated by $A(\delta, \phi)$, for any α in F. For α, α' in F, set

$$D = (\alpha'-\alpha)(1-\phi)^{-2}[R_{(1-\phi)e+\delta y}, R_{(1-\phi)x-\delta z}] .$$

Then one can calculate that $\sigma = \exp D = I + D + (D^2/2)$ carries $S(\alpha, \delta(1-\phi)^{-1})$ onto $S(\alpha', \delta(1-\phi)^{-1})$, and centralizes G since the elements $(1-\phi)e+\delta y$, $(1-\phi)x-\delta z$ are fixed points of $A(\delta, \phi)$. Note that if ϕ is not a root of unity, then G is an infinite group.

Another automorphism $B(\delta, \tau)$ of A, for δ, τ in F, $\tau \neq 0$, is given by:

$$B(\delta, \tau)\begin{cases} e \longrightarrow e - \delta\tau x + \delta y + \delta^2\tau z \\ x \longrightarrow \tau^{-1}y + \delta z \\ y \longrightarrow \tau x - \delta\tau z \\ z \longrightarrow z \end{cases} .$$

$B(\delta, \tau)$ has a three-dimensional fixed point space spanned by $e+\delta y$, z and $\tau x + y$, and an eigenvector $\tau x - y - \delta\tau z$ for the eigenvalue -1, so that $B(\delta, \tau)$ is completely reducible. Actually $B(\delta, \tau)^2 = I$, so G here is a group of order two. One calculates that $S(\alpha, \beta)B(\delta, \tau) = S(-\delta\tau+\beta\tau, \delta+\alpha\tau^{-1})$. Hence $S(\alpha, \delta+\alpha\tau^{-1})$ is G-invariant for any $\alpha \in F$. Set $D' = \tau^{-1}(\alpha'-\alpha)[R_{e+\delta y}, R_{\tau x+y}]$ for $\alpha, \alpha' \in F$. Then

$$\sigma = \exp D' = I + D' + \frac{(D')^2}{2}$$

carries $S(\alpha, \delta+\alpha\tau^{-1})$ onto $S(\alpha', \delta+\alpha'\tau^{-1})$, and centralizes G since $e+\delta y$ and $\tau x + y$ are fixed points of $B(\delta, \tau)$. Hence, in this case, the fixed point property holds, although, as we have seen in our first example, it does not hold for every finite group G.

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