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ON THE JORDAN STRUCTURE OF COMPLEX BANACH *ALGEBRAS

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All algebras considered are complex Banach algebras with identity and continuous involution. The principal results of § 1 are that for a Jordan *homomorphism T of $\mathfrak{A}_1$ into $\mathfrak{A}_2$ where $\mathfrak{A}_2$ is *semisimple, continuity is automatic, the kernel is a closed *ideal, and if $\mathfrak{A}_2$ is commutative then the factor algebra $\mathfrak{A}_1$/kernel T is also commutative. In § 2 a cone different from the usual cone is introduced and its relation to the usual cone is studied. The principal result is that if this cone coincides with the usual cone, then any Jordan *representation is the sum of a *representation and a *antirepresentation. § 3 is devoted to proving that for a *semisimple algebra, the axiom $\|xy\| \leqq \|x\| \|y\|$ follows from the weaker axiom $\|xy + yx\| \leqq 2\|x\| \|y\|$.

Our notation will be as follows: $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}'$ etc., are algebras; e, e_1, e' etc. are the identities of $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}'$ etc.; $\| \quad \|_{sp}$ is the spectral radius; H, H_1, H' etc. are the real subspace of hermitian elements of $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}'$ etc. We generally abbreviate "Jordan" as simply "J".

1. DEFINITION. A linear transformation $T: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ is called a *J-homomorphism* if $T(xy + yx) = T(x)T(y) + T(y)T(x)$ and $T(e_1) = e_2$. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have involutions and $T(x^*) = T(x)^*$ then T is called a *J-*homomorphism.*

The assumption $T(e_1) = e_2$ will usually be used in studying the spectrum. Since the adjunction of an identity merely adjoins 0 to any spectrum which does not already include 0, this assumption is removable in many situations.

LEMMA 1.1 *If $x \in \mathfrak{A}_1$ then the spectrum of Tx is contained in the spectrum of x.*

Proof. It suffices to show that if x has a two-sided inverse in $\mathfrak{A}_1$ then Tx has a two-sided inverse in $\mathfrak{A}_2$. So let $yx = xy = e_1, u = Tx$ and $v = Ty$. Then $uv + vu = T(xy + yx) = T(2e_1) = 2e_2$. Multiply on each side by v separately to get $vuv + v^2u = 2v, uv^2 + vuv = 2v$ so that $v^2u = uv^2$. Therefore u commutes with v^2 and hence so does u^2. Thus $2u^2v^2 = 2v^2u^2 = u^2v^2 + v^2u^2 = T(x^2y^2 + y^2x^2) = T(2e_1) = 2e_2$. Therefore $u^2v^2 = v^2u^2 = e_2$ and hence $u = Tx$ has a two-sided inverse in $\mathfrak{A}_2$.

An immediate consequence of Lemma 1.1 is the following lemma which has been stated in [1].

LEMMA 1.2. *If* $x \in \mathfrak{A}_1$, *then* $\| Tx \|_{sp} \leqq \| x \|_{sp}$.

THEOREM 1.3. *Suppose* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *have continuous involutions such that* $\mathfrak{A}_2$ *is *semisimple* ([6], p. 210) *and* $T: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ *is a J-*homomorphism. Then* T *is continuous.*

Proof. Since $\mathfrak{A}_2$ is *semisimple, it has an auxiliary norm $| \ |$ such that $|x| \leqq \|x\|_{sp}$ for $x \in H_2$. Now let $x_n \rightarrow x$ in $\mathfrak{A}_1$ and let each $x_n \in H_1$. Then $x \in H_1$, $Tx_n \in H_2$ and $Tx \in H_2$. Also suppose that $Tx_n \rightarrow y$ in $\mathfrak{A}_2$. Then $y \in H_2$. Therefore $\| x_n - x \|_{sp} \rightarrow 0$ so that $\| Tx_n - Tx \|_{sp} \rightarrow 0$, by Lemma 1.2. Also since $Tx_n - y \in H_2$, therefore $| T_n - y | \rightarrow 0$. It follows that $Tx = y$. By the Closed Graph Theorem T is continuous on H_1 and therefore on all of $\mathfrak{A}_1$.

THEOREM 1.4. *With the hypothesis of Theorem* 1.3, *the kernel of* T *is a closed *ideal.*

Proof. The only nontrivial part is to show that the kernel is an ideal. Let $K =$ kernel of $T, a \in K$ and $b \in \mathfrak{A}_1$. Since $K^* = K$, the hermitian and the skew parts of any element of K are again in K. It therefore suffices to consider hermitian a and b. We present the proof in three brief steps.

Step 1. If $a^* = a \in K$ and $b^* = b \in \mathfrak{A}_1$ then $T(ba)$ and $T(ab)$ are both skew: $0 = T(a)T(b) + T(b)T(a) = T(ab + ba) = T((ab) + (ab)^*) = T(ab) + T(ab)^* = T(ba) + T(ba)^*$. Therefore $T(ab)^* = -T(ab)$ and $T(ba)^* = -T(ba)$.

Step 2. If $a^* = a \in K$ and $b^* = b \in \mathfrak{A}_1$ then aba, a^2b, ba^2 are in K: Since aba is hermitian, so is $T(aba)$. But we can prove $T(aba)$ skew because $aba + a^2b = (ab)a + a(ab) \in K$ so that $T(aba) = -T(a^2b)$ which is skew by Step 1. Therefore $T(aba) = 0 = T(a^2b)$ and similarly $T(ba^2) = 0$.

Step 3. If $a^* = a \in K$ and $b^* = b \in \mathfrak{A}_1$ then $ab \in K$: From Step 2, we know that $a^2b \in K$. Therefore

$$b(a^2b) + (a^2b)b \in K \quad \text{and} \quad a(ab^2) + (ab^2)a \in K .$$

Combining we get $ba^2b - ab^2a \in K$. But by Step 2, $ab^2a \in K$, and therefore $ba^2b \in K$. Therefore $0 = T(ba^2b + ab^2a) = T((ba)(ab) + (ab)(ba)) = T((ab)^*(ab)) + T((ab)(ab)^*)$. Since $\mathfrak{A}_2$ is *semisimple, it follows that $T(ab) = 0$.

THEOREM 1.5. *With the hypotheses of Theorem* 1.3, *let* $K =$ *kernel of* T. *Then if* $\mathfrak{A}_2$ *is commutative,* $\mathfrak{A}_1/K$ *is also commutative.*

Proof. Since K is an ideal, we can form the algebra $\mathfrak{A}_1/K$. But this algebra is J-isomorphic to $T(\mathfrak{A}_1)$, which is commutative and therefore has an associative Jordan product $a \circ b = ab + ba$. Therefore $\mathfrak{A}_1/K$ also has an associative Jordan product. This means that $\mathfrak{A}_1/K$ satisfies a polynomial identity of degree 3. By Lemma 1.1 it is semisimple. It follows that $\mathfrak{A}_1/K$ is commutative from a result of Kaplansky ([3], p. 580).

2. In studying the Jordan structure of *algebras it is natural to introduce the cone generated by the Jordan products $x^*x + xx^*$. This gives a cone in the subspace of hermitian elements even if the involution preserves the multiplication instead of reversing it. This cone turned out to be useful in the study of another problem not connected with Jordan homomorphisms. (See [7] where it is called the "quasicone"). For a commutative algebra, this cone is the same as the usual cone. For a symmetric algebra also this is quite obvious. Which other algebras have this property is an open question, which becomes all the more interesting because of Theorem 2.1 below. Since noncommutative nonsymmetric algebras are not easy to come by, it is hard to construct an example where this J-cone differs from the usual cone. Positivity with respect to this cone has also been used by Rickart in [6].

DEFINITION. The *J-cone* Q of a Banach *algebra $\mathfrak{A}$ is the closed cone in H generated by the elements of the form $x^*x + xx^*$.

The following nine assertions are either obvious or can be proved as in the case of the usual cone:

1°. Q is the closed cone generated by $\{x^2 : x \in H\}$;

2°. Q is contained in the usual cone;

3°. If $\mathfrak{A}$ is *semisimple, then $Q \cap (-Q) = \{0\}$;

4°. Q includes the open ball in H of radius 1 about e;

5°. If the linear functional $f: \mathfrak{A} \to C$ is J-positive (i.e. $f(Q) \geqq 0$) then $f(x^*) = \overline{f(x)}$ and the norm of the restriction of f to H is $f(e)$;

6°. $|x|_1 = \sup \{|f(x)| : f$ J-positive, $f(e) = 1\}$ defines a pseudonorm on H;

7°. If $|\ \ |$ is the auxiliary pseudonorm on $\mathfrak{A}$, then for $x \in H$, $|x| = \sup\{|f(x)| : f$ positive, $f(e) = 1\}$ (see [4]);

8°. If $\mathfrak{A}$ is *semisimple, $|\ \ |_1$ is a norm;

9°. $|x| \leqq |x|_1 \leqq \|x\|_{sp} \leqq \|x\|$ for $x \in H$.

The following theorem says that Størmer's Theorem [8] extends to any algebra whose J-cone equals the usual cone.

THEOREM 2.1. *Any J-*representation T of a Banach *algebra*

*$\mathfrak{A}_1$ with identity e_1 and whose J-cone Q_1 is the usual cone, is the sum of a *representation and a *antirepresentation.*

Proof. Let $\mathfrak{A}_2$ be a B^*-algebra such that $T: \mathfrak{A}_1 \to \mathfrak{A}_2$ is a J-*homomorphism. Since $\mathfrak{A}_2$ is symmetric its J-cone is the usual cone, and for $\mathfrak{A}_1$ this is assumed to be the case. So we can freely interchange positivity and J-positivity in both algebras.

The relevance of the J-cone is that a J-*homomorphism preserves it. Consequently if f is a J-positive functional on $\mathfrak{A}_2$ then $f \circ T$ is a J-positive functional on $\mathfrak{A}_1$. Thus for any $x \in H_1$ we have

$$\begin{aligned} \| Tx \| = | Tx | = | Tx |_1 &= \sup\{| f(Tx) | : f \ J\text{-positive}, \ f(e_2) = 1\} \\ &= \sup\{| (f \circ T)(x) | : f \ J\text{-positive}, \ (f \circ T)(e_1) = 1\} \\ \text{(A)} \qquad &\leqq \sup\{| \phi(x) | : \phi \ J\text{-positive}, \ \phi(e_1) = 1\} \\ &\leqq | x |_1 = | x | \,. \end{aligned}$$

Therefore if $\mathfrak{A}_1$ is *semisimple, T is continuous with respect to the norm $| \ \ |$, and therefore extends to the completion. But the completion is a B^*-algebra and Størmer's Theorem ([8], p. 445) yields the desired conclusion. However if $\mathfrak{A}_1$ is not *semisimple, then (A) shows that the *radical of $\mathfrak{A}_1$ is contained in the kernel of T. Consequently, we may pass to the factor algebra which is *semisimple. Then the *representation and the *antirepresentation of the factor algebra can be lifted to $\mathfrak{A}_1$ via the natural map.

It is proved in [7] Lemma 3, that if $\mathfrak{A}$ is "J-symmetric" (i.e., every element of Q has a nonnegative spectrum-called "quasisymmetric" in [7]), then $\mathfrak{A}$ is symmetric. As a consequence, we have

THEOREM 2.2. *Let T: $\mathfrak{A}_1 \to \mathfrak{A}_2$ be a J-*isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. If $\mathfrak{A}_1$ is symmetric, then so is $\mathfrak{A}_2$.*

Proof. Consider any $Tx \in \mathfrak{A}_2$. Then

$$T(x)^*T(x) + T(x)T(x)^* = T(x^*x + xx^*)$$

so that by Lemma 1.1, $(Tx)^*(Tx) + (Tx)(Tx)^*$ has a nonnegative spectrum. Thus $\mathfrak{A}_2$ is J-symmetric and therefore symmetric.

Next we shall give two results regarding the relation between the J-cone and the usual cone. The first result is quite trivial, though interesting. $\mathfrak{A}$ is *semisimple in the rest of this section.

LEMMA 2.3. *Any $| \ \ |$-continuous J-positive functional on $\mathfrak{A}$ is positive.*

Proof. Let f be a $| \ \ |$-continuous J-positive functional. Since f

is $|\ \ |$-continuous, it extends to the completion, where it continues to be J-positive. But the completion is a B^*-algebra and therefore f is positive.

Denote the set of all positive (respectively J-positive) functionals f for which $f(e) = 1$ by $\mathscr{F}$ (respectively $\mathscr{G}$). Then clearly both are weak*-compact convex sets in the dual space of H.

Our next proof was simplified by the referee, to whom the author's thanks are due.

THEOREM 2.4. *If $\mathfrak{A}$ is *semisimple then $\mathscr{F}$ is a face of $\mathscr{G}$. In other words, if $f = tf_1 + (1 - t)f_2$ where $f \in \mathscr{F}, f_1 \in \mathscr{G}, f_2 \in \mathscr{G}$, $0 < t < 1$, then $f_1 \in \mathscr{F}, f_2 \in \mathscr{F}$.*

Proof. Let $f \in \mathscr{F}$ and $f = tf_1 + (1 - t)f_2$ where $f_1 \in \mathscr{G}, f_2 \in \mathscr{G}$ and $0 < t < 1$. Then $f - tf_1 = (1 - t)f_2$ is J-positive, so that $f(v^2) \geqq tf_1(v^2)$ for all $v \in H$. Using the Cauchy-Schwarz inequality for J-positive functionals,

$$f_1(v)^2 \leqq f_1(v^2) \leqq t^{-1}f(v^2) \leqq t^{-1}\,|\,v^2\,| \leqq t^{-1}\,|\,v\,|^2 \qquad \text{for all } v \in H\,.$$

Therefore f is continuous on H. Since the involution is continuous in the norm $|\ \ |, f_1$ is continuous on $\mathfrak{A}$. So by Lemma 2.3, $f_1 \in \mathscr{F}$. Similarly $f_2 \in \mathscr{F}$.

For the rest of the section, the reader is assumed to be familiar with the material on GM-spaces and Takeda's Theorem given in [4]. We note that H becomes a GM-space with the usual cone as well as the J-cone Q (see 3° and 4° above). The norm obtained from the GM-structure given by the usual cone is $|\ \ |$ (see 7° above). The norm obtained from the GM-structure given by Q is $|\ \ |_1$. Therefore Takeda's Theorem ([4], p. 221) applies to both norms: Any linear functional f: $H \to \boldsymbol{R}$ which is $|\ \ |$-continuous (respectively $|\ \ |_1$-continuous) can be written as $\phi - \psi$ where ϕ and ψ are positive (respectively J-positive) and $|\,\phi\,| + |\,\psi\,| = |\,f\,|$ (respectively $|\,\phi\,|_1 + |\,\psi\,|_1 = |\,f\,|_1$). In the former case however one can pass to the completion with respect to the norm $|\ \ |$, which gives a B^*-algebra; and then one can also assert that ϕ and ψ are unique, because of Grothendieck's Theorem [2]. This procedure is not available for the $|\ \ |_1$ case, because $|\ \ |_1$ seems to have no extension to all of $\mathfrak{A}$ which would make multiplication continuous. For our problem, the extension must also make the involution continuous, and it can be shown that if there is a norm on $\mathfrak{A}$ which makes multiplication and involution both continuous, and which agrees with $|\ \ |_1$ on H, then Q is the usual cone, $|\ \ |_1 = |\ \ |$. It would be interesting to know whether the uniqueness holds for the

$|\ \ |_1$-case also. However, if it is shown that Q is always the usual cone, then this problem is trivialised. (It can also be shown that in the linear space case, as in Takeda's Theorem, uniqueness fails. The counterexample is three-dimensional).

3. The notion of a *radical does not depend upon the norm. Suppose $\mathfrak{A}$ is any (complex) *algebra. One can define the *radical as the set of all $x \in \mathfrak{A}$ such that $f(x) = 0$ for all positive f. One may use the condition that $f(x^*x) = 0$ in place of $f(x) = 0$. The *radical is a *ideal. All this is worked out in [5]; the norm is nowhere used. Thus one can speak of a *semisimple algebra. Our main result of this section is:

THEOREM 3.1. *Let $\mathfrak{A}$ be a *semisimple complex algebra with an identity e. Let $\|\ \ \|$ be a Banach space norm on $\mathfrak{A}$ such that $\|x^*\| = \|x\|$, $\|e\| = 1$ and $\|xy + yx\| \leqq 2\|x\|\,\|y\|$. Then there is an equivalent norm on $\mathfrak{A}$ such that $\mathfrak{A}$ is a Banach *algebra.*

We prove this theorem in the form of three lemmas.

LEMMA 3.2. *Let $\mathfrak{A}$ be a complex *algebra with identity e and let $\mathscr{F}$ be the set of all positive functionals f on $\mathfrak{A}$ such that $f(e) = 1$. Assume $\mathscr{F}$ nonempty and define $|x| = \sup\{f(x^*x)^{1/2}: f \in \mathscr{F}\}$. Then $|\ \ |$ has the following properties*:

(a) $|x + y| \leqq |x| + |y|$,

(b) $|\lambda x| = |\lambda|\,|x|$, $\quad \lambda \in C$,

(c) $|xy| \leqq |x|\,|y|$ *and*

(d) $|x^*| = |x|$.

Proof. Since the Cauchy-Schwarz inequality holds, all is trivial. In (c) use the fact that if $f \in \mathscr{F}$ and $f(y^*y) \neq 0$, then

$$\phi(z) = f(y^*zy)/f(y^*y)$$

defines an element of $\mathscr{F}$.

The lemma says that $|\ \ |$ has all the properties of a norm except that it may take the value ∞, and that $|x| = 0$ may not imply that $x = 0$. The second difficulty is taken care of by the *semisimplicity and the first by an argument analogous to the case of Banach algebras.

LEMMA 3.3. *Let $\mathfrak{A}$ be as in Theorem 3.1. Then every positive functional f restricted to H is continuous with norm f(e); i.e. $|f(h)| \leqq f(e)\,\|h\|$ if $h \in H$. Consequently the function $|\ \ |$ of Lemma 3.2 is* *a norm and* $|h| \leqq \|h\|$ *if* h *is either hermitian or skew.*

Proof. The proof for Banach *algebras on pp. 189–190 of [5] goes over word for word.

Now let $\mathfrak{A}'$ denote the algebra $\mathfrak{A}$ with norm $|\ \ |$. Then for any $x \in \mathfrak{A}$ the map $y \to xy$ is continuous from $\mathfrak{A}$ to $\mathfrak{A}'$ because of Lemma 3.2 (c) and Lemma 3.3. We exploit the Closed Graph Theorem to show that it is continuous from $\mathfrak{A}$ to $\mathfrak{A}$.

LEMMA 3.4. *Let* $\mathfrak{A}$ *be as in Theorem* 3.1. *For any* $x \in \mathfrak{A}$ *let* $T_x\colon \mathfrak{A} \to \mathfrak{A}$ *map* y *into* xy. *Then the linear operator* T_x *is continuous.*

Proof. Let $\|h_n - h\| \to 0$ and let each $h_n \in H$. Then h is also hermitian so that by Lemma 3.3, $|h_n - h| \to 0$. Then by Lemma 3.2 (c) and (d) we have

$$\text{(A)} \qquad |xh_n - xh| \longrightarrow 0, \qquad |h_n x^* - hx^*| \longrightarrow 0 .$$

Assume that $\|xh_n - z\| \to 0$. If we can prove $z = xh$ then by the Closed Graph Theorem T_x is continuous on H and therefore on all of $\mathfrak{A}$. Now

$$\|xh_n - z\| \longrightarrow 0, \qquad \|h_n x^* - z^*\| \longrightarrow 0 .$$

Therefore

$$\text{(B)} \qquad \|(xh_n + h_n x^*) - (z + z^*)\| \longrightarrow 0 ,$$

and

$$\text{(C)} \qquad \|(xh_n - h_n x^*) - (z - z^*)\| \longrightarrow 0 .$$

But since these elements are hermitian and skew respectively, therefore by Lemma 3.3, (B) and (C) hold if the norm $|\ \ |$ is used. Combining this with (A), $xh + hx^* = z + z^*, xh - hx^* = z - z^*$, so that $z = xh$.

We have so far shown that multiplication is a continuous operation in $\mathfrak{A}$. Our theorem now follows from a well-known result (Prop VIII, subsection 5, § 9 of [5]).

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