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NORMAL EXPECTATIONS IN VON NEUMANN ALGEBRAS

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Let h and k be two Hilbert spaces, $h \otimes k$ will denote the tensor product of h and k. Let $\mathscr{A}$ be a von Neumann algebra acting on h. Let ψ be an ampliation of $\mathscr{A}$ in $h \otimes k$, i.e., ψ is a map of $\mathscr{A}$ into bounded linear operators of $h \otimes k$ and $\psi(\mathscr{A}) = \mathscr{A} \otimes I_k$ (I_k is the identity map on k). Let $\tilde{\mathscr{A}}$ be the image of $\mathscr{A}$ by ψ.

The purpose of this paper is to prove the following result: If $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ and if $\mathscr{B}$ is the range of a normal expectation φ defined on $\mathscr{A}$, then there exists an ampliation of $\mathscr{A}$ in $h \otimes k$, independent of $\mathscr{B}$ and of φ, such that $\varphi \otimes I_k$ is a spatial isomorphism of $\tilde{\mathscr{A}}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two C^* algebras with identity. Suppose $\mathscr{B} \subset \mathscr{A}$. Let φ be a positive linear map of $\mathscr{A}$ on $\mathscr{B}$ such that φ preserves the identity and such that $\varphi(BX) = B\varphi(X)$ for all B in $\mathscr{B}$ and all X in $\mathscr{A}$. φ is then defined to be an expectation of $\mathscr{A}$ on $\mathscr{B}$. The extension of the notion of an expectation in the probability theory sense, to expectations on finite von Neumann algebra is largely due to J. Dixmier and H. Umegaki [1]. In [4] Tomiyama considers an expectation on von Neumann algebras to be a projection of norm one. If φ is an expectation in the sense $\varphi(BX) = B\varphi(X)$, φ positive and φ preserves identities, then $\varphi(XB) = \varphi(X)B$ for all X in $\mathscr{A}$, B in $\mathscr{B}$. $\mathscr{B}$ is the set of fixed points of φ. By writing $\varphi[(X - \varphi(X))^* (X - \varphi(x))] \geqq 0$ we have $\varphi(X^*X) \geqq \varphi(X)^*\varphi(X)$. In particular φ is a bounded map. The result stated in the previous paragraph extends a result by Nakamura, Takesaki, and Umegaki [2], who consider the case when $\mathscr{A}$ is a finite von Neumann algebra.

2. Preliminaries.

Basic definitions and some essentially known results will now be given for ready reference. Let M and N be C^* algebras and φ a positive linear map of M on N. Let M_n be the set of all $n \times n$ matrices whose entries are elements of M, call those entries $A_{i,j}$. Define for each n, $\varphi^{(n)}(A_{i,j}) = (\varphi(A_{i,j}))$; φ^n is then a map of M_n on N_n. φ is called *completely positive* if each φ^n is.

Let $\mathscr{A}$ and $\mathscr{B}$ be two von Neumann algebras, with $\mathscr{B} \subset \mathscr{A}$. Let φ be an expectation of $\mathscr{A}$ on $\mathscr{B}$. φ is called *faithful* if for any T in $\mathscr{A}$, $\varphi(TT^*) = 0$ implies $T = 0$. Let A_α be a net of uniformly bounded self adjoint operators in $\mathscr{A}$. φ is called *normal* if

$$\sup_a \varphi(A_\alpha) = \varphi(\sup_\alpha A_\alpha) .$$

The *ultra-weak topology* on α will be the weakest which will make all $\sum w_{x_i,y_i}(A) = \sum(Ax_i, y_i)$ continuous where

$$\sum \| x_i \|^2 < \infty \quad \text{and} \quad \sum \| y_i \|^2 < \infty .$$

In what follows if N is arbitrary von Neumann algebra, N' will denote the commutant of N. If h is any Hilbert space, dim h will denote the cardinality of the dimension of h.

LEMMA 1. *Let M and N be two von Neumann algebras acting on h_M and h_N. Let φ be a* isomorphism of M on N. Let k be a Hilbert space such that* $\dim k \geqq \operatorname{Max}(\chi_1, \dim h_M, \dim h_N)$, *then* $\varphi \otimes I_k$ *is a spatial isomorphism. This theorem says that there exists an isometry V of $h_M \otimes k$ on $h_N \otimes k$ such that*

$$\varphi \otimes I_k(A \otimes I_k) = \varphi(A) \otimes I_k = V(A \otimes I_k)V^*(= V\tilde{A}V^*) .$$

Tomiyama has shown this result in [5].

LEMMA 2. *Let M and N be two C^* algebras with identities. Let φ be an expectation of M on N, then φ is completely positive.* This result was shown by Nakamura, Takesaki, and Umegaki in [2].

One of the tools for the proof of the theorem will be the Stinespring construction which is given in [3] and which will be sketched here for completeness sake.

Let M be any von Neumann algebra acting on h. Let $M \odot h$ denote the tensor product of M and h as linear spaces. Let N be von Neumann algebra of M which is the range of a normal expectation φ. On $M \odot h$ define an inner product by:

$$\left\langle \sum_{i=1}^{n} a_i \otimes x_i, \sum_{j=1}^{l} b_j \otimes y_j \right\rangle = \sum_{i,j} (\varphi(b_j^* \cdot a_i)x_i, y_j)$$

where a_i, b_j are in M, x_i, y_j are in h and where (,) denotes the inner product in h. Now:

$$\sum_{i,j} (a_j^* a_i x_i, x_j) = \left(\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} a_i x_i \right) \geqq 0 .$$

Let A be in M_n with $A_{ij} = a_j^* a_i$ then if $x = (x_1, x_2, \cdots, x_n)$

$$(Ax, x) = \sum_{i,j} (a_j^* a_i x_i, x_j) \geqq 0 .$$

By Proposition 2,

$$\sum_{i,j} (\varphi(a_j^* a_i)x_i, x_j) \geqq 0 .$$

Hence the inner product defined on $M \odot h$ is bilinear and positive. However, it is possible to have $\langle \zeta, \zeta \rangle = 0$ with $\zeta \neq 0$. Divide out the space $M \odot h$ by all vectors of norm zero. Then taking the completion of that space, one obtains a Hilbert space which will be denoted $M \otimes h$.

LEMMA 3. *h is embedded as a Hilbert space in $M \otimes h$.*

Proof. In fact we shall show that h is isomorphic to $N \otimes h$. Let $a_i i = 1, 2, \cdots, n$ be operators in N, consider the map

$$S(\sum_{i=1}^{n} a_i \otimes x_i) = \sum_{i=1}^{n} a_i x_i$$

then

$$\begin{aligned} \langle \sum_{i=1}^{n} a_i \otimes x_i \,, \quad \sum_{i=1}^{n} a_i \otimes x_i \rangle \\ &= \sum_{i,j} (\varphi(a_j^* a_i) x_i, x_j) \\ &= \sum_{i,j} (a_j^* a_i x_i, x_j) \\ &= (\sum_{i=1}^{n} a_i x_i \,, \quad \sum_{i=1}^{n} a_i x_i) \,. \end{aligned}$$

Hence S is an isometry of $N \otimes h$ on h. In particular then, one can view h as a subspace of $M \otimes h$.

LEMMA 4. *φ defines a self adjoint projection E of $M \otimes h$ on $N \otimes h$.*

Proof. Let a_i, $i = 1, 2, \cdots, n$ be operators of M. Define

$$E(\sum_{i=1}^{n} a_i \otimes x_i) = \sum_{i=1}^{n} \varphi(a_i) \otimes x_i$$

the proof in [2] shows that E is a well-defined self adjoint projection of $M \otimes h$ on $N \otimes h$. Recall for example how self adjointness is checked out.

$$\begin{aligned} &\langle E(\sum_i a_i \otimes x_i), \sum_j b_j \otimes y_i \rangle \\ &= \langle \sum_i \varphi(a_i) \otimes x_i, \sum_j b_j \otimes y_j \rangle = \sum_{i,j} (\varphi(b_j^* \varphi(a_i)) x_i, y_j \\ &= \sum_{i,j} (\varphi(\varphi(b_j^*) a_i) x_i, y_j) \\ &= \langle \sum a_i \otimes x_i, \sum_j \varphi(b_j) \otimes y_j \rangle \\ &= \langle \sum_i a_i \otimes x_i, E(\sum_j b_j \otimes y_j) \rangle \,. \end{aligned}$$

LEMMA 5. *There exists an ultra-weakly continuous representation l of M in $L(M \otimes h)$ such that $l(b)E = El(b)$ for all b in N. Moreover if h and $N \otimes h$ are identified by the isometry S of Lemma 3, then $\varphi(A) = El(a)E$ for all a in M.*

Proof. For each a in M define

$$l(a)(\sum a_i \otimes x_i) = \sum a\, a_i \otimes x_i$$

l is then a representation of M in $L(M \otimes h)$. Let b_i, $i = 1, 2, \cdots, n$ be operators in N then:

$$\begin{aligned} El(a)(E\, b_j \otimes x_j) &= E(\sum ab_j \otimes x_j) \\ &= \sum \varphi(a)b_j \otimes x_j = \varphi(a)(\sum b_j \otimes x_j) \end{aligned}$$

identifying $\sum b_j \otimes x_j$ with $\sum b_j x_j$ this shows that $El(a)E = \varphi(a)$. Let b be in N then

$$\begin{aligned} l(b)E(\sum a_i \otimes x_i) &= l(b)(\sum \varphi(a_i) \otimes x_i) \\ &= \sum b\varphi(a_i) \otimes x_i = El(b)(\sum a_i \otimes x_i) \; . \end{aligned}$$

So $l(b)E = El(b)$ for all b in N. To show now that l is u. w. continuous, let

$$\zeta_k = \sum_{i=1}^{n_k} a_i^{(k)} \otimes x_i^{(k)},\ \eta_h = \sum_{j=1}^{n_h} b_j^{(h)} \otimes y_j^{(h)}$$

with $\sum \| \zeta_k \|^2 < \infty$ and $\sum \| \eta_h \|^2 < \infty$. Let a_α be a net converging u. w. to a in M. Then it is sufficient to show that A tends to zero where

$$A = \sum_{k,h} \langle l(a - a_\alpha)\zeta_k, \eta_h \rangle \; .$$

we have

$$A = \sum_{k,h} \sum_{i,j} (\varphi(b_j^{(h)*}(a - a_\alpha)a_i^{(k)})x_i^{(k)}, y_j^{(k)}) \; .$$

Now $b_j^{(h)*}(a - a_\alpha)a_i^{(k)}$ tends to zero u.w. As φ is normal, A tends to zero. Let $N \subset M$ be two von Neumann algebras acting on h. Let φ be a faithful, normal expectation of M on N.

3. Main results.

First the following result will be established.

PROPOSITION 6. There exists a Hilbert space k such that:

(1) h can be embedded in k.

(2) There exists an u.w. continuous representation l of M in $L(k)$ such that $\varphi(A) = p_{hl}(A)p_h$ where p_h is the projection of k on h.

(3) l is a^* isomorphism.

(4) p_h commutes with all $l(b)$ with b in N.

Proof. Let $k = M \otimes h$, if $l(a) = 0$ then $l(a^*a) = 0$ so $\varphi(a^*a) = 0$. By faithfulness of φ, this implies $a = 0$. Hence l is a * isomorphism of M in $L(k)$. The rest of Proposition 6 is a restatement of Lemma 5. The main result of this paper can now be given.

THEOREM 7. *There exists an ampliation of M in $h \otimes k$ such that if N is any von Neumann subalgebra of M which is the range of a normal expectation φ, then there exists an isometry V in $(N \otimes I_k)'$ such that $\varphi \otimes I_k(\tilde{A}) = V\tilde{A}V^*$, $VV^* = I$, on putting $V^*V=P$, then P is in $(N \otimes I_k)'$, $\varphi \otimes I_k(\tilde{A})P = P\tilde{A}P$. If φ is faithful then $\tilde{A}P = 0$ $(A \geqq 0)$ implies $\tilde{A} = 0$.*

Proof. Let s be a Hilbert space with cardinality greater or equal to the maximum of ψ_1 and cardinality of a Hammel basis of $M \otimes h$. Define $\tilde{l}(\tilde{A}) = l(A) \otimes I_s$, $\tilde{\varphi} = \varphi \otimes I_s$. Then $\tilde{\varphi}(\tilde{A}) = (P_h \otimes I_s)\tilde{l}(\tilde{A})(P_h \otimes I_s)$. By Lemma 1, l is spatial. There exists an isometry U of $h \otimes s$ onto $k \otimes s$ such that $\tilde{\varphi}(\tilde{A}) = U(\tilde{A})U^*$. Hence

$$\tilde{\varphi}(\tilde{A}) = P_{h\otimes s}U(A \otimes I_s)U^*P_{h\otimes s}$$

where $P_{h\otimes s}$ denotes the projection of $k \otimes s$ on $h \otimes s$. Moreover $P_{h\otimes s}$ commutes with all $U\tilde{B}U^*$ as B ranges over N (Proposition 6). So $UP_{h\otimes s}U$ commutes with all $\tilde{B}$ for B in N.

Let $V = P_{h\otimes s}U$, then $VV^* = P_{h\otimes s}$ $(=I_{h\otimes s})$. Define $V^*V = P = U^*P_{h\otimes s}U$. Then P is in $(N \otimes I_s)'$. So $\tilde{\varphi}(\tilde{A}) = V\tilde{A}V^*$ for all A in M. Claim: V is in $(N \otimes I_s)'$. Let B be in N, $\tilde{B} = \tilde{\varphi}(\tilde{B}) = V\tilde{B}V^*$ so $V^*\tilde{B} = P\tilde{B}V^* = \tilde{B}PV^* = (\tilde{B})V^*$ so V is in $\tilde{N}'$. Now

$$\begin{aligned} P\tilde{A}P &= V^*V\tilde{A}V^*V \\ &= V^*\tilde{\varphi}(\tilde{A})V \\ &= V^*V\tilde{\varphi}(\tilde{A}) = P\tilde{\varphi}(\tilde{A}) = \tilde{\varphi}(\tilde{A})P \text{ (as } \tilde{\varphi}(\tilde{A}) \in N \otimes I_s) . \end{aligned}$$

Now let $\tilde{A}P = O(A \geqq 0)$ then $\tilde{A}V^*V = 0$ so $V\tilde{A}V^*V = 0 = \tilde{\varphi}(\tilde{A})V$ so $\tilde{\varphi}(\tilde{A})P_{h\otimes s}U = 0$ and $\tilde{\varphi}(\tilde{A})P_{h\otimes s} = 0$ so $(\varphi(A) \otimes I_s)(x \otimes u) = 0$ for all x in h and u in s implies $\varphi(A) = 0$ so $A = 0$, by faithfulness of φ.

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