PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 2, 1968

TOTALLY GEODESIC HYPERSURFACES OF KAEHLER MANIFOLDS

SAMUEL I. GOLDBERG

It is known that a C^∞ orientable totally umbilical hypersurface P with nonzero mean curvature of a Kaehler manifold M is a normal contact manifold. Moreover, if $M = C_n$ with the flat Kaehler metric, P can be realized as a normal contact metric manifold of positive constant curvature. It is the main purpose of this paper to obtain corresponding results for cosymplectic manifolds.

The direct product of two normal almost contact manifolds can be endowed with a complex structure. For cosymplectic manifolds more is obtained. Indeed, the direct product of two cosymplectic manifolds can be given a Kaehlerian structure. This is particularly true of orientable totally geodesic hypersurfaces of a Kaehler manifold.

Our notion of a cosymplectic manifold differs from the one given by P. Libermann in [3] and was given by D. Blair [1].

THEOREM 1. *A necessary and sufficient condition that a C^∞ orientable hypersurface P of a Kaehler manifold M be cosymplectic with almost contact form η is that its second fundamental form H be proportional to $\eta \otimes \eta$, that is*

$$H = h\eta \otimes \eta \ ,$$

where $h = H(\xi, \xi)$, the vector field ξ being the contravariant form of η with respect to the almost contact metric.

COROLLARY 1. *A C^∞ orientable totally geodesic hypersurface of a Kaehler manifold is a cosymplectic manifold.*

A corresponding result was obtained by Y. Tashiro [6] for totally umbilical hypersurfaces.

For complete simply connected cosymplectic manifolds an application of the de Rham decomposition theorem (see [2]) yields

COROLLARY 2. *A C^∞ complete simply connected orientable totally geodesic hypersurface of a Kaehler manifold is a product with one factor Kaehlerian.*

THEOREM 2. *A cosymplectic hypersurface of C_n with the flat Kaehler metric is locally flat.*

For the corresponding statement concerning normal contact hypersurfaces the reader is referred to [6].

A. Morimoto [4] has shown that the direct product of two normal almost contact manifolds can be given a complex structure. For cosymplectic manifolds we obtain more.

THEOREM 3. *The direct product of two cosymplectic manifolds can be given a Kaehlerian structure.*

Applying Corollary 1 we obtain

COROLLARY 3. *The direct product of two C^∞ orientable totally geodesic hypersurfaces of a Kaehler manifold is a Kaehler manifold.*

2. Almost contact manifolds. An *almost contact structure* (ϕ, ξ, η) on a $(2n+1)$-dimensional C^∞ manifold P is given by a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η on P called the *contact form* such that

$$\eta(\xi) = 1 \,, \tag{2.1}$$

$$\phi(\xi) = 0 \,, \qquad \eta \circ \phi = 0 \,, \tag{2.2}$$

$$\phi^2 = -I + \eta(\,\cdot\,)\xi \,, \tag{2.3}$$

where I is the identity transformation field. If P has a (ϕ, ξ, η)-structure then we can find a Riemann metric $(\ ,\ )$ such that

$$\begin{gathered} \eta = (\xi, \cdot) \,, \\ (\phi X, \phi Y) = (X, Y) - \eta(X)\eta(Y) \,, \end{gathered} \tag{2.4}$$

so that ϕ is skew-symmetric with respect to $(\ ,\ )$. P is then said to have a $(\phi, \xi, \eta, (\ ,\ ))$-structure.

The almost contact structure is called *normal* if for any vector fields X, Y on P

$$[X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] = d\eta(X, Y)\xi \,. \tag{2.5}$$

A $(2n+1)$-dimensional C^∞ manifold is said to have a *contact structure*, and is then called a *contact manifold*, if it carries a global 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0 \,.$$

It can be shown that there exists a $(\phi, \xi, \eta, (\ ,\ ))$-structure on a contact manifold P such that

$$d\eta = (\phi X, Y) .$$

P is then called a *contact metric manifold*.

An almost contact metric structure $(\phi, \xi, \eta, (\ ,\ ))$ is called *quasi-Sasakian* if it is normal and its fundamental form Φ, where $\Phi(X, Y) = (\phi X, Y)$, is closed. The quasi-Sasakian manifolds may be classified according to the rank of η. The 1-form η has rank $2p$ if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$, and has rank $2p + 1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$. There are no quasi-Sasakian structures of even rank [1]. If the rank is maximal, the almost contact manifold is a *Sasakian manifold*, and if the rank is 1, it is a *cosymplectic manifold*.

3. Almost contact hypersurfaces.

Let M be an almost hermitian manifold of real dimension $2n$ with almost complex structure tensor J. Then, in terms of the hermitian metric $\langle\ ,\ \rangle$ of M

$$\langle Jx, Jy \rangle = \langle x, y \rangle \tag{3.1}$$

for every pair of tangent vectors $x, y \in M_m$—the tangent space at $m \in M$. If P is a smooth orientable hypersurface imbedded in M with imbedding $i: P \to M$, the induced metric on P is defined in terms of the metric on M by

$$(x, y) = \langle i_* x, i_* y \rangle \tag{3.2}$$

for each pair of tangent vectors $x, y \in P_m$.

A *Riemannian connexion* D on a Riemannian manifold with metric $\langle\ ,\ \rangle$ is characterized by the properties:

(a) $$D_X Y - D_Y X = [X, Y] \quad \text{and}$$

(b) $$Z\langle X, Y \rangle = \langle D_Z X, Y \rangle + \langle X, D_Z Y \rangle .$$

Let N_m be the unit normal of P at m with orientation determined by that of P. Let ∇ be the Riemannian connexion on M. The *Weingarten map* $W: P_m \to P_m$ is given by

$$W(x) = \nabla_x N_m , \qquad x \in P_m . \tag{3.3}$$

(We write x for $i_* x$ in the sequel, with no resulting confusion, in order to simplify our notation.) The second fundamental form $H: P_m \times P_m \to R$ of P is the symmetric bilinear form

$$H(x, y) = (Wx, y) . \tag{3.4}$$

If D is the induced Riemannian connexion on P, then

$$D_x Y = \nabla_x Y + H(x, y) N_m , \tag{3.5}$$

where $x \in P_m$ and Y is a vector field on P extending y. Set

$$\eta(x) = \langle Jx, N\rangle\ , \qquad x \in P_m \tag{3.6}$$

and

$$\Phi(x, y) = \langle Jx, y\rangle\ , \qquad x, y \in P_m\ . \tag{3.7}$$

Formula (3.1) says that Φ is skew-symmetric with respect to $\langle\ ,\ \rangle$. If $\xi = (\eta,\ )$ is the contravariant form of η with respect to $(\ ,\ )$, then ξ is a vector field on P with the property

$$J\xi = N\ . \tag{3.8}$$

Thus,

$$\begin{aligned}(\iota(\xi)\Phi)(X) &= \Phi(\xi, X)\\ &= \langle J\xi, X\rangle \text{ by } (3.7)\\ &= \langle N, X\rangle \text{ by } (3.8)\\ &= 0\ ,\end{aligned} \tag{3.9}$$

where $X \in E(P)$, the module of vector fields on P.

Now, in terms of Φ and $(\ ,\ )$ an endomorphism ϕ of $E(P)$ is defined by the equation

$$(\phi X, Y) = \Phi(X, Y)\ , \qquad X, Y \in E(P)\ . \tag{3.10}$$

Since Φ is skew-symmetric

$$(\phi X, Y) = -(X, \phi Y)\ . \tag{3.11}$$

Moreover,

$$\phi X = JX - \eta(X)N\ , \qquad X \in E(P)\ .$$

It follows that

$$(\phi X, \phi Y) = (X, Y) - \eta(X)\eta(Y)\ . \tag{3.13}$$

For, by (3.10), (3.7), (3.12) and (3.6), $(\phi X, \phi Y) = \Phi(X, \phi Y) = \langle JX, \phi Y\rangle = \langle JX, JY - \eta(Y)N\rangle = \langle JX, JY\rangle - \eta(Y)\langle JX, N\rangle = \langle X, Y\rangle - \eta(Y)\eta(X)$. In addition, since $(\phi^2 X, Y) = -(\phi X, \phi Y) = \eta(X)\eta(Y) - (X, Y)$

$$\phi^2 X = -X + \eta(X)\xi\ . \tag{3.14}$$

Applying (3.9) and (3.10), we obtain

$$\phi\xi = 0\ . \tag{3.15}$$

Thus, from (3.11) and (3.13)

$$\eta \circ \phi = 0\ . \tag{3.16}$$

We conclude that, P has a $(\phi, \xi, \eta, (\ ,\ ))$-structure.

PROPOSITION 1. *A C^∞ orientable hypersurface P of an almost hermitian manifold M has a naturally induced almost contact structure.*

If the fundamental 2-form Ω of M where $\Omega(X, Y) = \langle JX, Y\rangle$ is closed (that is, if M is an almost Kaehler manifold) then Φ is closed. Indeed, $\Phi = i^*\Omega$ directly from the definitions of Φ and Ω.

We compute $D_X\Phi$:

$$\begin{aligned}(D_X\Phi)(Y, Z) &= X\Phi(Y, Z) - \Phi(D_XY, Z) - \Phi(Y, D_XZ)\\ &= X(\phi Y, Z) + (\phi Z, D_XY) - (\phi Y, D_XZ)\\ &= X(\phi Y, Z) + \langle\phi Z, \nabla_XY\rangle - \langle\phi Y, \nabla_XZ)\\ &= X\langle JY, Z\rangle + \langle JZ, \nabla_XY\rangle - \langle JY, \nabla_XZ\rangle\\ &\quad - \langle\eta(Z)N, \nabla_XY\rangle + \langle\eta(Y)N, \nabla_XZ\rangle\\ &= (\nabla_X\Omega)(Y, Z) - \eta(Z)\langle N, \nabla_XY\rangle + \eta(Y)\langle N, \nabla_XZ\rangle\\ &= (\nabla_X\Omega)(Y, Z) + \eta(Z)H(X, Y) - \eta(Y)H(X, Z)\end{aligned}$$

for all $X, Y, Z \in E(P)$.

If M is Kaehlerian, $\nabla_X\Omega = 0$, so

$$(D_X\Phi)(Y, Z) = \eta(Z)H(X, Y) - \eta(Y)H(X, Z)\ . \tag{3.17}$$

Moreover,

$$(D_X\eta)(Y) = H(X, \phi Y)\ . \tag{3.18}$$

For,

$$\begin{aligned}(D_X\eta)(Y) &= (Y, D_X\xi)\\ &= \langle Y, \nabla_X\xi\rangle \text{ by } (3.5)\\ &= \langle JY, J\nabla_X\xi\rangle \text{ by } (3.1)\\ &= \langle JY, \nabla_XJ\xi\rangle \text{ since } \nabla_X\Omega \text{ vanishes}\\ &= \langle JY, \nabla_XN\rangle \text{ by } (3.8)\\ &= (\phi Y, WX) \text{ by } (3.3)\\ &= (W\phi Y, X) \text{ since } W \text{ is self-adjoint}\\ &= H(X, \phi Y) \text{ by } (3.4).\end{aligned}$$

4. **Proof of Theorem 1.** If $H = h\eta\otimes\eta$, then by (3.17), $D_X\Phi = 0$, and by (3.18), $(D_X\eta)(Y) = h\eta(X)\eta(\phi Y) = 0$ since P is almost contact. Since ϕ has vanishing covariant derivative it is easily seen that P is normal. Indeed, it is easily checked that (2.5) is satisfied for a basis of coordinate vector fields compatible with ϕ. Hence, P is cosymplectic. Observe that $h = H(\xi, \xi)$.

Conversely, if the almost contact structure on P is cosymplectic,

$D\eta$ vanishes (see [1]). Consequently, by (3.18), $H(X, \phi Y) = 0$, $X, Y \in E(P)$. Hence, $H(X, \phi^2 Y) = 0$ implies $H(X, Y) = \eta(Y)H(X, \xi)$. But $H(X, \xi) = \eta(X)H(\xi, \xi)$, so

$$H(X, Y) = H(\xi, \xi)\eta(X)\eta(Y) .$$

REMARKS. (a) A simply connected totally geodesic orientable hypersurface of a Kaehler manifold cannot be compact, since otherwise its first betti number is not zero by virtue of the fact that η is harmonic.

(b) Observe that the vector field ξ of Theorem 1 is a characteristic vector of H.

PROPOSITION 2. *A sufficient condition that a C^∞ orientable hypersurface P of a Kaehler manifold M be a contact metric manifold is that*

$$H = \lambda(\ ,\ ) + \mu\eta \otimes \eta , \qquad \lambda \neq 0$$

where $(\ ,\ )$ *is the almost contact metric and* η *the almost contact form of* P.

Proof. If $H = \lambda(\ ,\ ) + \mu\eta \otimes \eta$, then by (3.18),

$$\begin{aligned} 2d\eta(X, Y) &= (D_Y\eta)(X) - (D_X\eta)(Y) = H(Y, \phi X) - H(X, \phi Y) \\ &= \lambda[(Y, \phi X) - (X, \phi Y)] = 2\lambda\Phi(X, Y) . \end{aligned}$$

Since M is a Kaehler manifold Φ is closed. Thus, λ is a constant and $\Phi = d\eta'$ where $\lambda\eta' = \eta$.

Observe that $\lambda + \mu = H(\xi, \xi)$.

Theorem 2 is an immediate consequence of the well-known formula

$$K_S(X, Y) = K_R(X, Y) - [H(X, X)H(Y, Y) - (H(X, Y))^2]$$

where $X, Y \in P_m$ is an orthonormal pair. For, $K_S = 0$ and $H(X, Y) = h\eta(X)\eta(Y)$.

Proof of Theorem 3. Let P_1 and P_2 be almost contact manifolds with almost contact structures (ϕ_i, ξ_i, η_i), $i = 1, 2$, respectively. Then, an almost complex structure J is induced on the product manifold $P_1 \times P_2$ (see [4]). In fact, for $x_i \in P_{im_i}$, $i = 1, 2$ we set

$$J_{(m_1, m_2)}(x_1, x_2) = (\phi_1 x_1 - \eta_2(x_2)\xi_1, \phi_2 x_2 + \eta_1(x_1)\xi_2) . \tag{4.1}$$

Now, for $i = 1, 2$, let P_i be given a cosympletic structure with underlying almost contact structure (ϕ_i, ξ_i, η_i). Since these structures on P_1 and P_2 are normal the almost complex structure J defined by (4.1) on the product manifold $M = P_1 \times P_2$ comes from a complex structure. Define a metric g on M by $g_1 + g_2$ where $g_i (i = 1, 2)$ is the almost contact metric of P_i. Defining a 2-form Ω on M by

$$\Omega = \Phi_1 + \Phi_2 + \eta_1 \wedge \eta_2 ,$$

we see that Ω has maximal rank since Φ_i has this property on P_i, $i = 1, 2$. Moreover, since Φ_1, Φ_2, η_1 and η_2 are closed, so is Ω. It is not difficult to check that

$$g(X, Y) = \Omega(X, JY) ,$$

so Ω is the Kaehler form of the Kaehler manifold (M, J, g).

REFERENCES

1. D. E. Blair, *The theory of quasi-Sasakian structures*, J. of Diff. Geom. **1** (1967), 331-345.
2. D. E. Blair and S. I. Goldberg, *Topology of almost contact manifolds* J. of Diff. Geom. **1** (1967), 347-354.
3. P. Libermann, *Sur les automorphismes infinitésimaux des structures symplectiques et des structures de contact*, Coll. de géom. diff. globale, CBRM (1959), 37-59.
4. A. Morimoto, *On normal almost contact structures*, J. Math. Soc. Japan **15** (1963), 420-435.
5. M. Okumura, *Cosymplectic hypersurfaces in Kaehlerian manifold of constant holomorphic sectional curvature*, Kōdai Math. Sem. Rep. **17** (1965), 63-73.
6. Y. Tashiro, *On contact structure of hypersurfaces in complex manifolds*, I, Tôhoku Math. J. **15** (1963), 62-78.

Received December 20, 1967. Research supported by NSF Grant GP-5477.

UNIVERSITY OF ILLINOIS, URBANA