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PROVING THAT WILD CELLS EXIST

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In their famous paper Fox and Artin constructed several examples of wild cells in 3-space. The present authors construct a wild disk D in the 4-sphere S^4 with the property that the proof of nontameness is perhaps the most elementary possible. We require only the knowledge that if K is the trefoil knot in the 3-sphere S^3, then the fundamental group $\pi_1(S^3 - K)$ is not abelian. Parenthetically, the wild disk D constructed here has the property that every arc on D is tame, a fact which follows immediately from the construction.

In S^3 let $\{K_i\}$ be a sequence of polygonal trefoil knots that converge to a point q while each K_j lies interior to a 3-simplex that meets no other K_i. We consider S^3 as being the equator of S^4 while H is the upper hemisphere of S^4. In $H - S^3$ let $\{p_i\}$ be a sequence of points converging to q. If p_iK_i is the cone over K_i with vertex p_i, let $\{p_i\}$ be so chosen that the disks $\{p_iK_i\}$ are disjoint in pairs. Now in S^3 join p_1K_1 and p_2K_2 by a polyhedral disk D_1 so that $p_1K_1 \cup D_1 \cup p_2K_2$ is a disk disjoint from $(\bigcup_3^\infty p_iK_i) \cup q$. We next join p_2K_2 and p_3K_3 by a polyhedral disk, D_2, in S^3 so that $p_1K_1 \cup D_1 \cup p_2K_2 \cup D_2 \cup p_3K_3$ is a disk disjoint from $(\bigcup_4^\infty p_iK_i) \cup q$. This process is continued so that as $i \to \infty$ the diameter of D_i tends to 0 and the disk D is

$$\left(\bigcup_1^\infty (p_iK_i \cup D_i)\right) \cup q\ .$$

As a subset of S^4, D is locally tame [1] except perhaps at q.

THEOREM. *D is wild in S^4.*

The proof is given in two lemmas.

LEMMA 1. *If there is a homeomorphism h of S^4 onto S^4 such that $h(D)$ is the union of a finite number of triangles, then for some point p_j in D there is a neighborhood U_j of p_j in D and for each open set V'_j in S^4 containing p_j there is a neighborhood $V_j \subset V'_j$ of p_j such that $\pi_1(V_j - U_j)$ is abelian.*

Proof. If h exists then $\{h(p_i)\}$ contains a point that lies in the interior of a disk formed by the union of two triangles. Call this point $h(p_j)$. Then p_j has a neighborhood meeting the condition in the lemma while $\pi_1(V_j - U_j)$ is the infinite cyclic group.

LEMMA 2. *No point p_j in the disk D meets the conclusion of Lemma* 1.

Proof. If such a point were to exist we note that if K is a polygonal trefoil in S^3, S^4 the suspension of S^3, then the suspension of K in S^4 is a 2-sphere S^2. But $\pi_1(S^3 - K) \cong \pi_1(S^4 - S^2)$. All generators of $\pi_1(S^4 - S^2)$ may be selected in $S^3 - K$. So if Lemma 1 holds at a suspension vertex of K, $\pi_1(S^4 - S^2)$ is abelian.

Essentially the same construction yields a wild $(n - 2)$-disk in S^n for $n \geqq 4$.

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