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THE STONE-WEIERSTRASS THEOREM FOR VALUABLE FIELDS

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Let F be a topological field, which means in particular Hausdorff and commutative. The "Stone-Weierstrass" theorem for F would state that if X is a compact space and $\mathscr{A}$ an algebra of continuous functions from X to F which contains the constant functions and separates points, then $\mathscr{A}$ is uniformly dense in the algebra of all continuous functions from X to F. This is true for the reals but false for the complexes, and is commonly regarded as a special property of the reals. In fact, however, it is the complex field which is exceptional: the Stone-Weierstrass theorem and its characteristic corollaries hold for all valuable fields other than the complex numbers.

A subset B of a topological field F is called *bounded* if for every neighborhood U of 0 there is an $a \neq 0$ in F with $a \cdot B \subseteqq U$. The field F is *valuable* if for every neighborhood V of 0, the set $\{a^{-1}: a \notin V\}$ is bounded. Kaplansky originally called this condition "Type V"; Bourbaki [1] uses the term "local retroboundedness." All the topological fields usually encountered are valuable.

Lacking a unified proof of the Stone-Weierstrass theorem for valuable fields, we have to rely on a classification theorem: the valuable fields are precisely those whose topologies are induced either by a (Krull) valuation or an Archimedean absolute value [1, p. 181]. We treat these cases separately.

Case I. Suppose the topology of F is defined by a valuation v with value group Γ. We denote by A the valuation ring of F:

$$A = \{a \in F: v(a) \geqq 0\} .$$

If S is a subset of F and $\gamma \in \Gamma$, we set

$$S_\gamma = \{s \in S: v(s) > \gamma\} .$$

We recall that the ideals F_γ for $\gamma \geqq 0$ form a basis of neighborhoods of 0 in F.

We first prove two technical lemmas.

LEMMA 1. *Let $K \subseteqq F$ be compact with $0 \notin K$. Let $\alpha \in \Gamma$ be such that $F_\alpha \cap K = \varnothing$ and let $\beta > 0$. Then there is a polynomial $f(T) \in F[T]$ such that*

(1) $f(0) = 1$

(2) $f(F_\alpha) \subseteqq A$
(3) $f(K) \subseteqq F_\beta$.

Proof. K can be covered by a finite number of sets of the form $y + F_\alpha$ where $y \in K$. Since $v(y) \leqq \alpha$ for every $y \in K$, $v(y + F) = \{v(y)\}$. Hence v assumes only a finite number of values on K, say

$$\alpha_1 < \alpha_2 < \cdots < \alpha_m \leqq \alpha .$$

We write $K = K_1 \cup K_2 \cup \cdots \cup K_m$, where $K_i = \{x \in K : v(x) = \alpha_i\}$. The proof will proceed by induction on m. Note that we may assume $\beta \geqq \alpha$; this just decreases the size of F_β.

First suppose that $m = 1$. Cover $K = K_1$ by open sets $z_j + F_{2\beta}$, $j = 1, \cdots, n$, with $z_j \in K$. Everything necessary is done by the polynomial

$$f(T) = \prod_{j=1}^{n} (1 - T/z_j) .$$

Certainly (1) holds, while (2) follows since if $v(t) > \alpha$ then $t/z_j \in A$ because $v(z_j) = \alpha_1 \leqq \alpha$. To verify (3), write any given $x \in K$ as $z_i - b$ for some index i and $b \in F_{2\beta}$. Then

$$f(x) = (b/z_i) \cdot \prod_{j \neq i} (1 - x/z_j) ;$$

this is in F_β because each $v(x/z_j) = v(x) - v(z_j) = 0$ (v being constant on K), so that $1 - x/z_j \in A$.

Next, the induction step. Set $K' = K_2 \cup \cdots \cup K_m$. By induction there is a polynomial g such that

$$g(0) = 1$$
$$g(F_\alpha) \subseteqq A$$
$$g(K') \subseteqq F_\beta .$$

Now choose $\gamma \in \Gamma$ so that $g(K_1) \cdot F_\gamma \subseteqq F_\beta$. Using the case $m = 1$ choose a polynomial h such that

$$h(0) = 1$$
$$h(F_{\alpha_1}) \subseteqq A$$
$$h(K_1) \subseteqq F_\gamma .$$

Then $f(T) = g(T)h(T)$ does what is wanted. Since $h(F_\alpha) \subseteqq A$, (2) is obvious, and only (3) needs checking. Observe that $K' \subseteqq F_{\alpha_1}$, so that $K \subseteqq K_1 \cup F_{\alpha_1}$; then $f(K_1) \subseteqq g(K_1)h(K_1) \subseteqq g(K_1) \cdot F_\gamma \subseteqq F_\beta$, while

$$f(K') \subseteqq g(K') \cdot h(F_{\alpha_1}) \subseteqq F_\beta \cdot A = F_\beta .$$

LEMMA 2. *Let $K \subseteqq F$ be compact and let a be a nonzero element of F. Then there is a polynomial $f(T)$ such that*
(1) $f(0) = 0$
(2) $f(a) = 1$
(3) $f(K) \subseteqq A$.

Proof. Choose $\beta \geqq 0$ such that $a \notin F_\beta$ and $(1 - a^{-1} \cdot K) \cdot F_\beta \subseteqq A$. By Lemma 1 there is a polynomial $g(T)$ such that

$$g(0) = 1$$
$$g(F_\beta) \subseteqq A$$
$$g(K \backslash K_\beta) \subseteqq F_\beta .$$

Set $f(T) = 1 - (1 - a^{-1}T)g(T)$. Then obviously $f(0) = 0$ and $f(a) = 1$. To verify (3) we observe that if $x \in K_\beta$ then $(1 - a^{-1}x) \in A$ and $g(x) \in A$ so that $f(x) \in A$. On the other hand, if $x \in K \backslash K_\beta$ then $g(x) \in F_\beta$ so that $(1 - a^{-1}x)g(x) \in A$ by the choice of β.

The next lemma may be regarded as the "Weierstrass" part of our generalized Stone-Weierstrass theorem.

LEMMA 3. *Let K be a compact subset of F. Then the polynomials are uniformly dense in the continuous F-valued functions on K.*

Proof. The compact set K is totally disconnected, and therefore its topology has a basis of clopen (i.e., simultaneously closed and open) sets. Let $g: K \rightarrow F$ be continuous and let U be a neighborhood of 0 in F. K may be covered by finitely many clopen subsets V_i on each of which any two values of g differ only by an element of U. Moreover, because the intersection of finitely many clopen sets is clopen, we may take the V_i to be disjoint. Choose $x_i \in V_i$ and set $a_i = g(x_i)$. Define $g_U = \Sigma_i a_i \chi_{V_i}$. Then g_U is continuous on K because the V_i are clopen. Moreover it is obvious that $g_U(x)$ and $g(x)$ differ only by an element of U for every $x \in K$; thus g_U approximates g uniformly within U on K. To approximate g by polynomials, it will therefore suffice to approximate all the characteristic functions χ_V for V a clopen subset of K.

Let $\gamma \in \Gamma$ be given. For any $x \in V$ and $y \in K \backslash V$ there is a polynomial g_{xy} such that $g_{xy}(x) = 1$, $g_{xy}(y) = 0$, and $g_{xy}(K) \subseteqq A$; this follows from Lemma 2 applied to $x - y$, $K - y$. By continuity, g_{xy} maps some neighborhood of x into $1 + F_\gamma$. Cover V by finitely many such neighborhoods and take the product of the corresponding polynomials; this is a polynomial h_y with $h_y(y) = 0$, $h_y(K) \subseteqq A$, and $h_y(V) \subseteqq 1 + F_\gamma$. Now do it again: h_y lies in F_γ on some neighborhood of y, so covering $K \backslash V$

with finitely many of these neighborhoods and multiplying we get a polynomial which maps $K \backslash V$ into F_γ and V into $1 + F_\gamma$. That is, it is within F_γ of χ_V on K.

PROPOSITION 1. Let F be any totally disconnected field. Suppose that the polynomials are uniformly dense in the continuous F-valued functions on every compact subset of F. Then the Stone-Weierstrass theorem holds for F.

Proof. Let X be a compact space and $\mathscr{F} = \{f_i\}$ a separating family of continuous F-valued functions on X. Let $f_i(X) = K_i$, a compact, totally disconnected subset of F. The map $\Pi_i f_i \colon X \to Y = \Pi_i K_i$ is a homeomorphic embedding, since X is compact and $\mathscr{F}$ separates the points of X. We may therefore regard X as a subset of Y; what we must then prove is that every continuous F-valued function on X can be uniformly approximated by polynomials in the basic coordinate functions of Y.

By a *basic covering* of Y we mean a partition of Y into clopen "rectangular" sets obtained by partitioning finitely many of the K_i into clopen sets and taking products. The Lebesgue covering theorem then implies that any open covering of X is refined by the restriction to X of some basic covering of Y. Hence, just as in Lemma 3, to approximate any continuous function on X it is enough to be able to approximate every χ_V, where the clopen set V is an element of a basic covering. By definition, χ_V is a product of finitely many characteristic functions depending on only one coordinate; and by hypothesis, each of these factors can be uniformly approximated by polynomials in the corresponding coordinate. The product of these polynomials is a polynomial in the coordinate functions which uniformly approximates χ_V.

Proposition 1 and Lemma 3 together prove the Stone-Weierstrass theorem in Case I. This case was proved for the p-adics by Dieudonné [3], and later independently for the p-adics by Mahler [5]. It was proved for fields with rank one valuations by Kaplansky [4]; the deduction of Lemma 3 from Lemma 2 follows his treatment.

We might also remark that every *discrete* field trivially satisfies the hypothesis of Proposition 1, the compact subsets being finite. Hence the Stone-Weierstrass theorem holds for these fields.

Case II. Suppose the topology of F is defined by an Archimedean absolute value. These fields, as is well known, have been completely classified.

First suppose that F is a dense subfield of $\boldsymbol{R}$. Then we can regard F-valued functions as $\boldsymbol{R}$-valued. The Stone-Weierstrass theorem for $\boldsymbol{R}$ then implies that if $\mathscr{F}$ is a separating family of F-valued continuous functions on the compact space X, then any $\boldsymbol{R}$-valued continuous function on X may be uniformly approximated by real-coefficient polynomials in the elements of $\mathscr{F}$. But the real coefficients of these polynomials may be approximated by elements of the dense subfield F. (This argument shows that, more generally, any field with the Stone-Weierstrass property bequeaths it to its dense subfields.)

We are left finally with the case of a proper dense subfield F of $\boldsymbol{C}$. We first need

LEMMA 4. *Let G be a proper subgroup of the additive group of $\boldsymbol{C}$. Then no compact subset of G disconnects the plane.*

Proof. Suppose, on the contrary, that $K \subseteqq G$ is a compact set disconnecting the plane. The same is then true of some one of its components [6, p. 123], so we may assume that K is connected. Let U be a bounded component of its complement, and let V be the union of all the other components. Choose an integer n large enough that the diameter of $n \cdot K$ is larger than the diameter of U. We assert that $n \cdot K + U \subseteqq n \cdot K + K$.

Indeed, take any $x = a + u$ with $a \in n \cdot K, u \in U$. We shall show that $x - K$ meets $n \cdot K$. For if it does not, then $n \cdot K$ is contained in the union of the two disjoint open sets $x - U, x - V$; being homeomorphic to K, it is connected, and hence contained in some one of the two. But $a \in n \cdot K \cap (x - U)$, so we must have $n \cdot K \subseteqq x - U$; this, however, is impossible because $n \cdot K$ has a larger diameter.

Thus we have shown that G contains the open set $n \cdot K + U$, which implies that $G = \boldsymbol{C}$, contrary to hypothesis.

Returning to our proper dense subfield F of $\boldsymbol{C}$, we now know that if $K \subseteqq F$ is compact then K has no interior and does not separate the plane. Therefore we can invoke Mergelyan's theorem [7, p. 386] to show that every continuous complex function on K can be approximated by polynomials over $\boldsymbol{C}$, and therefore by polynomials over F. In particular, $z \mapsto \bar{z}$ can be so approximated.

Now let X be a compact space and $\mathscr{A}$ a separating algebra of continuous F-valued functions containing the constants; we regard them as complex-valued functions. If $f \in \mathscr{A}, f(X) = K \subseteqq F$ is a compact subset. Since $\bar{z}$ can be uniformly approximated on $f(X)$ by polynomials over $F, \bar{f}$ is in the uniform closure of $\mathscr{A}$. Therefore the ordinary Stone-Weierstrass theorem implies that every complex-valued continuous function, and in particular every F-valued one, is in the uniform closure of $\mathscr{A}$. Thus, F has the Stone-Weierstrass property. This completes the proof of the

THEOREM. *Let F be any valuable topological field except $\boldsymbol{C}$. Then the Stone-Weierstrass theorem holds for F.*

REMARK. An analysis of our argument shows that it can be generalized to the following situation. Assume that the topological field F has a basis $\{N\}$ of neighborhoods of 0 satisfying the following conditions: for every compact $K \subseteqq F$ and every open covering of K there is a family of continuous functions $f_i: K \to F$ subordinate to the covering and satisfying (1) $\Sigma_i f_i = 1$ and (2) for arbitrary ε_i in N, the values of $\Sigma_i \varepsilon_i f_i$ are in N. (This is a generalized *partition of unity*; the condition (2) replaces the usual positivity condition for real-valued partitions of unity.) Then if the f_i are approximable by polynomials, an argument along the lines of Proposition 1 shows that the Stone-Weierstrass theorem holds for F.

Moreover, this argument is valid even if $\{N\}$, the values of the f_i, and the coefficients of the approximating polynomials are in the completion of F (a topological ring, though not always a field). For it yields approximation by polynomials over the completion, which can in turn be uniformly approximated with polynomials over F by approximating all their coefficients.

Corollaries and further results. Our theorem can be significantly strengthened by eliminating the requirement that the algebra $\mathscr{A}$ contain the constant functions. To this end we prove

PROPOSITION 2. Let F be any topological field except $\boldsymbol{C}$. Let $\mathscr{A}$ be an algebra of F-valued continuous functions on a compact space X having no common zeros. Then $\mathscr{A}$ contains a function vanishing nowhere on X.

Proof. The usual compactness argument gives us finitely many functions $f_i \in \mathscr{A}$ and compact sets X_i such that $\bigcup X_i = X$ and f_i is nowhere zero on X_i. We can proceed inductively if we can show that there is a function in the algebra generated by f_1 and f_2 vanishing nowhere on $X_1 \cup X_2$; thus we may assume $X_1 \cup X_2 = X$.

Since f_1 and f_2 are never both zero, f_1/f_2 defines a continuous function from X to the projective line over F, $\boldsymbol{P}_1(F)$. If the image of X omits at most one point, $\boldsymbol{P}_1(F)$ must be compact, whence F is locally compact. So if F is *not* locally compact, there is an $a \in F \subset \boldsymbol{P}_1(F)$ not in the image of X, which means that $f_1 - af_2$ is nowhere zero on X.

Suppose, then, that F is locally compact. If $F = \boldsymbol{R}$, $f_1^2 + f_2^2$ never vanishes. If F is discrete, then $\langle f_1, f_2 \rangle$ maps X onto a finite subset S of $F \times F$ which does not contain $\langle 0, 0 \rangle$. It is easy to find a polynomial P in two variables over F which has no constant term and is identically 1 on S. Then $P(f_1, f_2)$ is identically 1 on X.

Finally, suppose that F is locally compact and has a non-Archimedean absolute value. We may multiply f_1 by a constant and so assume that $|f_1| < 1$ everywhere. Now let $C = \{x: f_2(x) = 0\}$, and let $U_n = \{x: |f_2(x)| \leqq 1/n\}$. If f_1 has a zero in each U_n, then it has one in $\bigcap_n U_n = C$, contrary to assumption. So say that f_1 never vanishes on U_k; since U_k is compact, there is an $\varepsilon > 0$ with $|f_1| \geqq \varepsilon$ on U_k. Because the valuation is discrete, we can find an $a \in F$ such that $U_k = \{x: |af_2(x)| < 1\}$. If r is a sufficiently large integer, $|af_2|^r$ is smaller than ε on U_k, and of course still $\geqq 1$ off U_k. Hence $f_1 + (af_2)^r$ can vanish neither on U_k nor off it, and the proof is complete.

COROLLARY. *Let F be any field for which the Stone-Weierstrass theorem holds. Let $\mathscr{A}$ be an algebra of F-valued continuous functions on the compact space X, not necessarily containing the constants. If $\mathscr{A}$ separates points and does not vanish identically at any point, it is uniformly dense in the ring of all continuous F-valued functions.*

Proof. By Proposition 2, $\mathscr{A}$ contains a function f vanishing nowhere. The function $1/f$ is then continuous; by the Stone-Weierstrass theorem for F we can approximate it by $b_\alpha + g_\alpha$, with $b_\alpha \in F$ and $g_\alpha \in \mathscr{A}$. Hence $b_\alpha f + g_\alpha f \to 1$, so that 1 is in the closure of $\mathscr{A}$, and the Stone-Weierstrass theorem applies to it.

REMARK. Proposition 2 is definitely false for $\boldsymbol{C}$. For example, let X be the unit sphere $|z_1|^2 + |z_2|^2 = 1$ in $\boldsymbol{C}^2$. The polynomials in z_1 and z_2 with no constant term form an algebra on X having no common zero; yet since each vanishes at the origin it must vanish somewhere on the sphere, as follows, e.g., from a well-known theorem of Hartogs.

PROPOSITION 3. Let $\mathscr{A}$ be a closed subalgebra of the ring of continuous F-valued functions on a compact space X. Suppose the Stone-Weierstrass theorem holds for F. Then $\mathscr{A}$ is determined precisely by the set of relations $f(x_1) = f(x_2)$ and $f(x) = 0$ satisfied by all its members.

Proof. Set $x \sim y$ if, for all $f \in \mathscr{A}, f(x) = f(y)$. The quotient space under this equivalence relation is a compact space X', and $\mathscr{A}$ is canonically identified with a separating closed subalgebra of the ring of continuous functions on X'. If $\mathscr{A}$ has no common zeros it is all of that ring, i.e., consists of all functions on X satisfying the relations defining X'. Suppose, on the other hand, that $f(x) = 0$ for all $f \in \mathscr{A}$, and let g be any continuous function on X' vanishing at x. Adjoining the constant functions to $\mathscr{A}$ and using the Stone-Weierstrass theorem for F, we can find functions $f_\alpha \in \mathscr{A}$ and $b_\alpha \in F$ with $b_\alpha + f_\alpha \to g$. But since $g(x) = 0 = f_\alpha(x)$, $b_\alpha \to 0$. Thus $f_\alpha \to g$, $g \in \mathscr{A}$, and $\mathscr{A}$ contains all functions on X' vanishing at x.

From this result it follows that the closure of any subalgebra $\mathscr{A}$ is the set of all continuous F-valued functions satisfying the same relations $f(x_1) = f(x_2)$ and $f(x) = 0$ as $\mathscr{A}$ does. Also, suppose $\mathscr{A}$ is a closed ideal in the ring of continuous F-valued functions. Then $f(x_1) = f(x_2)$ for all $f \in \mathscr{A}$ if and only if either (1) $f(x_1) = f(x_2) = 0$ for all $f \in \mathscr{A}$ or (2) $g(x_1) = g(x_2)$ for all continuous $g\colon X \to F$; otherwise we could multiply an $f \in \mathscr{A}$ having $f(x_1) = f(x_2) \neq 0$ by a continuous g having $g(x_1) \neq g(x_2)$. In either case the relation $f(x_1) = f(x_2)$ is unnecessary for determining $\mathscr{A}$, and we get

COROLLARY. *The closed ideals are determined by their zero sets.*

We could go on to generalize such things as the extension of the Stone-Weierstrass theorem to locally compact spaces, or Dieudonné's theorem concerning the approximation of functions on product spaces. However here the proofs as well as the statements remain valid without change, so we simply refer the reader to Stone's article [8]. Similarly, all of the theorems of Cantor [2], stated for fields with rank one valuations on the basis of [4], extend immediately to fields with arbitrary valuations. Doubtless many other results can also be generalized.

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