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HARMONIC ANALYSIS ON GROUPOIDS

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This paper generalizes harmonic analysis on groups to obtain a theory of harmonic analysis on groupoids. A system of measures is obtained for a locally compact locally trivial groupoid, Z, analogous to left Haar measure for a locally compact group. Then a convolution and involution are defined on $C_c(Z)$ = the continuous complex valued functions on Z with compact support. Strongly continuous unitary representations of Z on certain fiber bundles, called representation bundles, are lifted to $C_c(Z)$, yielding * representations of $C_c(Z)$. A norm, $\| \|_{12}$, is defined on $C_c(Z)$, and the convolution, involution, and representations all extend to $\mathscr{L}_{12}(Z)$ = the $\| \|_{12}$ completion of $C_c(Z)$. The main example given is that of the groupoid $Z = Z(G, H)$ that arises naturally from a Lie group G and a closed subgroup H. In this example, the representations of Z are related to induced representations of G. Finally, if Z_{ee} (=the group of elements in Z with left unit=right unit $= e$) is compact then we canonically represent $\mathscr{L}_2(Z)$ as a direct sum of certain simple H^*-algebras.

We use extensively the notation and results of [8], except that [8] assumes a C^r manifold structure on the groupoid Z, and we want to consider groupoids with just topological structure. There is no essential difficulty in developing the main results of [8] for locally trivial topological groupoids. In particular, a C^r coordinate (resp. C^r fiber) bundle in [8] becomes a coordinate (resp. fiber) bundle as defined in [7].

Reviewing [8, § 1], the algebraic structure of a (transitive) groupoid, Z (over M), consists of a subset M of Z (called the units of Z), a projection $l \times r$ of Z onto $M \times M$ sending $\Phi_{qp} \in Z$ into (left unit Φ_{qp}, right unit of Φ_{qp}) $= (q, p)$, and a law of composition defined for pairs Φ_{qp}, Ψ_{rs} such that $p = r$. For $B \subseteqq M \times M$, Z_B is defined as $(l \times r)^{-1}(B)$, and $Z_{qp} = (l \times r)^{-1}(q, p)$. The composition $\Phi_{qp} \cdot \Psi_{ps} \in Z_{qs}$, and $(\Phi_{qp} \cdot \Psi_{ps}) \cdot \Gamma_{st} = \Phi_{qp} \cdot (\Psi_{ps} \cdot \Gamma_{st})$. The unit $q \in M$ may be written 1_{qq}, and $1_{qq} \cdot \Phi_{qp} = \Phi_{qp} \cdot 1_{pp} = \Phi_{qp}$. Also, Φ_{qp} has an inverse, Φ_{qp}^{-1}, such that $\Phi_{qp}^{-1} \cdot \Phi_{qp} = 1_{pp}$ and $\Phi_{qp} \cdot \Phi_{qp}^{-1} = 1_{qq}$.

A coordinate groupoid (Z, Σ_e) over M consists of the following:

(1.1) An (algebraic, transitive) groupoid Z over M and a Hausdorff topological structure for M.

(1.2) A distinguished point $e \in M$ and a Hausdorff topological group structure for the group Z_{ee}.

(1.3) A set of functions $\Sigma_e = \{\alpha : U_\alpha \to Z_{U_\alpha \times e}\}$ such that U_α is open in M and $l \cdot \alpha =$ identity map, satisfying

(1.3.1) $\bigcup_{\alpha \in \Sigma_e} U_\alpha = M$.

(1.3.2) For α and $\beta \in \Sigma_e$, the map $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to Z_{ee}$; $g_{\alpha\beta}(q) = \alpha(q)^{-1} \circ \beta(q)$, is continuous.

Then the constructions of [8] lead to a topological structure for Z, making Z a locally trivial topological groupoid as defined by Ehresmann in [3]. Conversely, any such groupoid arises from a coordinate groupoid.

Finally, we stipulate that the letter "Z" will always represent a locally compact locally trivial groupoid. Note Z is locally compact if and only if both Z_{ee} and M are locally compact.

2. We first consider systems of measures on a groupoid, Z over M.

DEFINTION 2.1. A (continuous) *system of measures* on Z is an indexed set $\lambda = \{\lambda_{qp}\colon (q, p) \in M \times M\}$, where λ_{qp} is a regular Borel measure on Z_{qp}. We will write $\lambda_{qp}(f) = \int_Z f(\Phi_{qp}) d\lambda\Phi_{qp}$, where f is an integrable function on Z_{qp}, and will require that the function $\lambda(h)$: $M \times M \to C$; $\lambda(h)(q, p) = \lambda_{qp}(h \mid_{Z_{qp}})$ be in $C_c(M \times M)$ whenever $h \in C_c(Z)$.

The concepts of "left and right invariance" are easily applied to systems of measures.

DEFINITION 2.2. A system of measures, λ, is said to be *left invariant* if and only if

$$\int_{Z_{rp}} f(\Psi_{qr} \cdot \Phi_{rp}) d\lambda\Phi_{rp} = \int_{Z_{qp}} f(\Gamma_{qp}) d\lambda\Gamma_{qp}\,, \tag{2.2.1}$$

for all $\Psi_{qr} \in Z$ and $p \in M$ and $f \in C_c(Z_{qp})$. Similarly, for *right invariance* the condition is (with $f \in C_c(Z_{pr})$):

$$\int_{Z_{pq}} f(\Phi_{pq} \cdot \Psi_{qr}) d\lambda\Phi_{pq} = \int_{Z_{pr}} f(\Gamma_{pr}) d\lambda\Gamma_{pr}\,. \tag{2.2.2}$$

If Z_{ee} is unimodular, it is easy to obtain a left and right invariant system of measures for Z from a Haar measure on Z_{ee} (use (2.6.1) with $\Delta \equiv 1$). In the general case, we extend the modular function for Z_{ee} to Z, and then obtain a left invariant system of measures for Z (depending on the extension).

DEFINITION 2.3. A function $\Delta: Z \to R^+$ is called a *modular* function for Z if and only if:

(2.3.1) Δ is a continuous homomorphism (multiplicative structure for R^+ = real numbers > 0.)

(2.3.2) $\Delta|_{Z_{ee}}$ is *the* modular function for Z_{ee}.

THEOERM 2.4. *If M is paracompact, then there exists a modular function for Z. Given two modular functions, Δ and Δ', on Z, we have $\Delta'(\Phi_{qp}) = h(q, p)\Delta(\Phi_{qp})$, $h: M \times M \to R^+$ is a continuous homomorphism* (with the trivial groupoid structure on $M \times M$, see (3.5b)).

Proof. Let Σ_e be a set of local sections in $Z_{M\times e}$ such that $\{U_\alpha = \text{dom}\,\alpha: \alpha \in \Sigma_e\}$ is a locally finite cover of M (using the paracompactness of M) and let $\{f_\alpha\}$ be a partition of 1 such that support $(f_\alpha) \subseteqq U_\alpha \cdot \Delta_{ee}$ is the modular function for Z_{ee}. We define $\Delta = e^\delta$, where

$$\delta(\Phi_{qp}) = \sum_{f\alpha, f\beta} f_\alpha(q) f_\beta(p) \log\left(\Delta_{ee}(\alpha(q)^{-1}\cdot\Phi_{qp}\cdot\beta(p))\right) .$$

Then Δ is a modular function for Z. Given a continuous homomorphism $h: M \times M \to R^+$, then Δ' defined by $\Delta'(\Phi_{qp}) = h(q, p)\Delta(\Phi_{qp})$ is a modular function for Z. Conversely, given two modular functions Δ and Δ' on Z, we find that $h(q, p) = \Delta'(\Phi_{qp})/\Delta(\Phi_{qp})$ is independent of Φ_{qp} for the given units, and that $h: M \times M \to R^+$ is a continuous homomorphism.

THEOREM 2.5. *If λ is a left (resp. right) invariant system of measures on Z, then λ_{qq} is a left (resp. right) Haar measure on Z_{qq} for each $q \in M$.*

From here on we assume λ_{ee} is a *fixed* left Haar measure on Z_{ee}, and will write $\lambda_{ee}(f) = \int_{Z_{ee}} f(\Phi_{ee}) d\Phi_{ee}$.

THEOREM 2.6. *There is a natural one-to-one correspondence between the left invariant systems of measures on Z and the modular functions on Z.*

Proof. Given a modular function, Δ, on Z, we define the system of measures, λ, by

$$\begin{aligned} \lambda_{qp}(f) &= \int_{Z_{qp}} f(\Phi_{qp}) d\Phi_{qp} \\ &= \int_{Z_{ee}} \Delta(\Gamma_{ep}) f(\Psi_{qe}\Lambda_{ee}\Gamma_{ep}) d\Lambda_{ee} . \end{aligned} \tag{2.6.1}$$

λ_{qp} is independent of the choice of Ψ_{qe} and Γ_{ep} with the indicated units, and λ is left invariant. Conversely if λ is a left invariant system of measures the above equation defines Δ on $Z_{e\times M}$. Then Δ may be extended to a continuous homomorphism of Z into R^+, and $\Delta \mid_{Z_{ee}}$ is the modular function of Z_{ee}.

THEOREM 2.7. *If Z_{ee} is unimodular, then there is a unique left and right system of measures on Z (recall λ_{ee} is a fixed left Haar measure).*

Proof. Just choose $\Delta \equiv 1$.

From here on we will assume that a fixed modular function Δ has been given for Z, and the corresponding left invariant system of measures is λ as defined in (2.6.1). A fixed regular Borel measure, μ, is specified for M, and $\mu(f)$ will be written $\int_M f(q)dq$, for any integrable function f on M. We require support of $\mu = M$.

3. DEFINITION 3.1. Given f and $g \in C_c(Z)$ we define the *convolution* of f and g, f^*g, by $f^*g(\Phi_{qp}) = \int_M \int_{Z_{qr}} f(\Psi_{qr})g(\Psi_{qr}^{-1}\cdot\Phi_{qp})d\Psi_{qr}dr$.

THEOREM 3.2. *$C_c(Z)$ forms an algebra over C with convolution as the law of multiplication, and the usual addition and scalar multiplication.*

Proof. The main points to verify are:

(a) $f^*g \in C_c(Z)$ and

(b) $(f^*g)^*h = f^*(g^*h)$.

In regard to (a), if support $(f) \subseteqq A$ and support $(g) \subseteqq B$ then it is easy to show that support $(f^*g) \subseteqq A\cdot B$. $A\cdot B$ is the image of $(A \times B) \cap D \subseteqq Z \times Z$ under composition, where D is the (closed) subset of $Z \times Z$ where composition is defined. Hence $A\cdot B$ is compact if A and B are compact.

In regard to (b), we compute $(f^*g)^*h(\Phi_{qp})$

$$= \int_M \int_{Z_{qs}} \left(\int_M \int_{Z_{qr}} f(\Psi_{qr})g(\Psi_{qr}^{-1}\cdot\Gamma_{qs})d\Psi_{qr}dr \right) h(\Gamma_{qs}^{-1}\cdot\Phi_{qp})d\Gamma_{qs}ds \; .$$

Substitute $\Lambda_{rs} = \Psi_{qr}^{-1}\cdot\Gamma_{qs}$, and interchange the order of integration to obtain

$$= \int_M \int_{Z_{qr}} f(\Psi_{qr}) \left(\int_M \int_{Z_{rs}} g(\Lambda_{rs})h(\Lambda_{rs}^{-1}\cdot\Psi_{qr}^{-1}\cdot\Phi_{qp})d\Lambda_{rs}ds \right) d\Psi_{qr}dr$$
$$= f^*(g^*h)(\Phi_{qp}) \; .$$

Next, we define an involution for $C_c(Z)$.

DEFINITION 3.3. Given $f \in C_c(Z)$, we define f^* by

$$f^*(\Phi_{qp}) = \bar{f}(\Phi_{qp}{}^{-1})\Delta(\Phi_{qp}{}^{-1})$$

(where $\bar{f}$ is the complex conjugate of f).

THEOREM 3.4. *The map $f \to f^*: C_c(Z) \to C_c(Z)$ is an involution* (see [6]).

Proof. The only difficult part is to show $(f * g)^* = g^* * f^*$. We compute

$$(f * g)^*(\Phi_{qp}) = \int_M \int_{Z_{pr}} \bar{f}(\Psi_{pr})\bar{g}(\Psi_{pr}{}^{-1} \cdot \Phi_{qp}{}^{-1})\Delta(\Phi_{qp}{}^{-1})d\Psi_{pr}dr$$

$$= (\text{substituting } \Gamma_{qr} = \Phi_{qp} \cdot \Psi_{pr})$$

$$\int_M \int_{Z_{qr}} \bar{g}(\Gamma_{qr}{}^{-1})\bar{f}(\Phi_{qp}{}^{-1} \cdot \Gamma_{qr})\Delta(\Phi_{qp}{}^{-1})d\Gamma_{qr}dr = (g^* * f^*)(\Phi_{qp}) .$$

EXAMPLES 3.5. (a) Suppose $M = \{e\}$ and $\mu(1) = 1$. Then $Z = Z_{ee}$ is a locally compact group, f^*g is the ordinary convolution, and $f \to f^*$ is the usual involution.

(b) Suppose $Z = M' \times M'$ and $M =$ diagonal of $M' \times M'$. We define the *trivial groupoid* structure for Z over M as follows:

$$l(q, p) = (q, q) \quad \text{and} \quad r(q, p) = (p, p) ,$$

composition is given by $(q, p) \cdot (p, r) = (q, r)$, and $(q, q) \to (q, e)$ gives a global section of $l: Z_{M \times e} \to M$.

If M' is discrete, then f and $g \in C_c(Z)$ are matrices indexed by M', with a finite number of nonzero entries. If $\mu(\{q\}) = 1$ for all $q \in M$, and $\lambda_{ee}(1) = 1$, then f^*g is the matrix composition of f and g.

(c) Suppose G is a Lie group and H is a closed subgroup of G. We define the *homogeneous space groupoid* for G and H, $Z(G, H) = Z = \{(q, \Phi, p): \Phi \in G, p \text{ and } q \in G/H, \text{ and } \Phi p = q\}$. The groupoid structure for Z is given as follows: $M = \{(q, 1, q): q \in G/H\}$ is the set of units, and $q \to (q, 1, q)$ identifies M with G/H to give M the required topology; $l(q, \Phi, p) = (q, 1, q)$ and $r(q, \Phi, p) = (p, 1, p)$. Composition is defined by $(q, \Phi, p) \cdot (p, \Psi, r) = (q, \Phi \cdot \Psi, r)$; the local sections of $l: Z_{M \times e} \to M$ come from local sections of $G \to G/H$ (identifying G/H with M as above, and taking $e = 1H$.): $(e, \Phi, e) \to \Phi$ is a group isomorphism sending Z_{ee} onto H, giving Z_{ee} the required topology.

We note that $Z_{M \times e}$ is essentially the usual principal bundle obtained from G and H.

For simplicity we only consider in this paper the case where Δ_H (the modular function for H) $= \Delta_G$ (the modular function for G), restricted to H. Then, by a theorem in [5, Chapter 10], there is a G

invariant measure on M, which we take for μ. There is a canonical (continuous) homomorphism $\zeta: Z \to G$, defined by $\zeta(q, \Phi, p) = \Phi$. Note that ζ maps Z onto G, and that $\zeta \mid_{Z_{ee}}$ is an isomorphism mapping Z_{ee} onto H. The above consideration leads to the following:

THEOREM 3.5.1. *$\Delta_G \cdot \zeta$ is a modular function for Z. Unless otherwise mentioned we will always use $\Delta = \Delta_G \cdot \zeta$ for $Z(G, H)$.*

If M is compact and $\mu(1) = 1$, then $\zeta^*(f) = f \cdot \zeta \in C_c(Z)$ for $f \in C_c(Z)$, and we obtain the

THEOREM 3.5.2. *$\zeta^*: C_c(G) \to C_c(Z)$ is a one-to-one* homomorphism (with the usual convolution and involution on $C_c(G)$, using a suitable left Haar measure on G).*

Proof. The first point is that $f \to \int_M \int_{Z_{qp}} \zeta^*(f)(\Phi_{qp}) d\Phi_{qp} dp$ (writing $(q, \Phi, p) = \Phi_{qp}$) defines a left invariant measure on G which we take as the desired left Haar measure on G. Note, this measure on G is independent of the choice of $q \in M$. Next, we compute

$$\begin{aligned} \zeta^*(f) * \zeta^*(g)(\Phi_{qp}) &= \int_{Z_q \times M} \zeta^*(f)(\Psi_{qr}) \zeta^*(g)(\Psi_{qr}^{-1} \cdot \Phi_{qp}) d\Psi_{qr} dr \\ &= \int_G f(\Psi) g(\Psi^{-1} \cdot \Phi) d\Psi \\ &= (f * g)(\Phi) = \zeta^*(f * g)(\Phi_{qp}), \text{ as required.} \end{aligned}$$

Finally, for $f \in C_c(G)$,

$$(\zeta^*(f))^*(\Phi_{qp}) = (\zeta^*(f))(\Phi_{qp}^{-1}) \Delta(\Phi_{qp}^{-1}) = f(\Phi^{-1}) \Delta_G(\Phi^{-1}) = \zeta^*(f^*)(\Phi_{qp}) ,$$

as required.

4. DEFINITION 4.1. A (unitary) *representation bundle, E,* is a fiber bundle with a Hilbert space structure for the fiber Y, and group $U(Y)$ = the unitary operators on Y with the strong operator topology.

We note that there is a natural *inner product field*, $\langle \, , \rangle$, on E. For $q \in M$, $\langle \, , \rangle_q$ is an inner product on E_q defined via any admissable map from the fiber Y. Then $\langle \, , \rangle_q$ makes E_q a Hilbert space and the unitary maps from Y to E_q are the admissable maps from Y to E_q.

Using the given regular Borel measure, μ, on M, we obtain an inner product on $\Gamma_c(E)$, the continuous sections in E with compact support. For γ and $\delta \in \Gamma_c(E)$,

$$\langle \gamma, \delta \rangle = \int_M \langle \gamma_q, \delta_q \rangle_q \, dq \, .$$

The completion of $\Gamma_c(E)$ with respect to this inner product is then a Hilbert space, to be called $\Gamma_2(E)$.

DEFINITION 4.2. A (strongly continuous) *unitary representation* ρ of Z on a representation bundle E is a continuous homomorphism $\rho: Z \to A(E)$ = the (locally trivial) groupoid of admissable maps between the fibers of E, such that ρ is the identity map on the units of Z (see [8]).

The main results listed below are obtained essentially as in [8, § 4].

THEOREMS 4.3. (a) *If* ρ *is given as in* (4.2) *then* $\rho\,|_{Z_{ee}} = \rho_e$ *defines a unitary representation of* Z_{ee} *on* E_e.

(b) *Given a unitary representation* ρ_e *of* Z_{ee} *on a Hilbert space* E_e, *there is a representation bundle* E' *and representation* ρ' *of* Z *on* E' *such that* $\rho'\,|_{Z_{ee}} \cong \rho_e$ *(a unitary equivalence).*

(c) *Two representations* ρ *and* ρ' *of* Z *on* E *and* E' *respectively are equivalent* (as in [8]) *if and only if* $\rho\,|_{Z_{ee}} \cong \rho'\,|_{Z_{ee}}$.

A groupoid representation, ρ, of Z on E^ρ defines a representation of the algebra $C_c(Z)$; $\rho: C_c(Z) \to \mathscr{L}(\Gamma_2(E^\rho))$ = the bounded linear maps of $\Gamma_2(E^\rho)$ into itself.

DEFINITION 4.4. Given $f \in C_c(Z)$ and $\gamma \in \Gamma_c(E^\rho)$, we define $\rho(f)\gamma$ by $(\rho(f)\gamma)_q = \int_M \int_{Z_{qp}} f(\Phi_{qp})\rho(\Phi_{qp})\gamma_p d\Phi_{qp} dp$. Alternatively,

$$\langle \rho(f)\gamma, \delta \rangle = \int_M \int_M \int_{Z_{qp}} f(\Phi_{qp}) \langle \rho(\Phi_{qp})\gamma_p, \delta_q \rangle \, d\Phi_{qp} dq dp \; .$$

THEOREM 4.5. $\| \rho(f)\gamma \|_2 \leqq \| f \|_{12} \| \gamma \|_2$, *where*

$$\| f \|_{12}^2 = \int_{M \times M} \left(\int_{Z_{qp}} | f(\Phi_{qp}) | \, d\Phi_{qp} \right)^2 dq dp \; .$$

Proof. See (5.4). Accordingly $\rho(f)$ extends to a bounded operator on $\Gamma_2(E^\rho)$ of norm $\leqq \| f \|_{12}$. $\mathscr{L}(\Gamma_2(E^\rho))$ has a natural Banach* algebra structure.

THEOREM 4.6. *The representation* $\rho: C_c(Z) \to \mathscr{L}(\Gamma_2(E^\rho))$ *is a *homomorphism.*

Proof. For f and $g \in C_c(Z)$, we compute

$$(\rho(f*g)\gamma)_q = \int_M \int_{Z_{qp}} \left(\int_M \int_{Z_{qr}} f(\Psi_{qr}) g(\Psi_{qr}^{-1} \cdot \Phi_{qp}) d\Psi_{qr} dr \right) \rho(\Phi_{qp}) \gamma_p d\Phi_{qp} dp$$

$=$ (substituting $\Gamma_{rp} = \Psi_{qr}^{-1} \cdot \Phi_{qp}$ and interchanging the order of integration)

$$\int_M \int_{Z_{qr}} f(\Psi_{qr}) \rho(\Psi_{qr}) \left(\int_M \int_{Z_{rp}} g(\Gamma_{rp}) \rho(\Gamma_{rp}) \gamma_p d\Gamma_{rp} dp \right) d\Psi_{qr} dr$$

$= (\rho(f)(\rho(g)\gamma))_q$ as desired.

Finally, we compute

$$\begin{aligned} \langle \rho(f^*)\gamma, \delta \rangle &= \int_{M \times M} \int_{Z_{qp}} f^*(\Phi_{qp}) \langle \rho(\Phi_{qp}) \gamma_p, \delta_q \rangle_q d\Phi_{qp} dp\, dq \\ &= \int_{M \times M} \int_{Z_{qp}} \bar{f}(\Phi_{qp}^{-1}) \Delta(\Phi_{qp}^{-1}) \langle \rho(\Phi_{qp}^{-1}) \delta_q, \gamma_p \rangle_p^- d\Phi_{qp} dp\, dq \\ \text{(see (5.2.1))} &= \int_{M \times M} \int_{Z_{pq}} \bar{f}(\Psi_{pq}) \langle \rho(\Psi_{pq}) \delta_q, \gamma_p \rangle_p^- d\Psi_{pq} dp \\ &= \langle \gamma, \rho(f)\delta \rangle, \text{ so } \rho(f^*) = \rho(f)^* . \end{aligned}$$

The following example provides a representation analogous to the left regular representation for groups.

EXAMPLE 4.7. Let ρ_e be the strongly continuous unitary representation of Z_{ee} on $\mathscr{L}_2(Z_{e\times M})$ given by $(\rho_e(\Phi_{ee})f_e)(\Psi_{ep}) = f_e(\Phi_{ee}^{-1} \cdot \Psi_{ep})$. The representation bundle F arising from ρ_e and Z may be regarded as $= \bigcup_{q \in M} \mathscr{L}_2(Z_{q\times M})$. The map $f \to f'$; $C_c(Z) \to \Gamma_c(F)$, defined by $f'(q) = f|_{q\times M}$ is bijective, and $\|f\|_2 = \|f'\|_2$. Accordingly, we can identify $\mathscr{L}_2(Z)$ and $\Gamma_2(F)$. Given f and $g \in C_c(Z)$, then $\rho(f)g' = (f*g)'$.

5. DEFINITION 5.1. For $f \in C_c(Z)$, we define

$$\|f\|_{12} = \left(\int_M \int_M \left(\int_{Z_{qp}} |f(\Phi_{qp})| d\Phi_{qp} \right)^2 dq dp \right)^{\frac{1}{2}} .$$

$\| \ \|_{12}$ defines a norm on $C_c(Z)$; we complete $C_c(Z)$ with respect to $\| \ \|_{12}$ to form $\mathscr{L}_{12}(Z)$.

To simplify matters, we recall the map: $\lambda: C_c(Z) \to C_c(M \times M)$, where $\lambda(f)(q, p) = \int_{Z_{qp}} f(\Phi_{qp}) d\Phi_{qp}$.

THEOREM 5.2. *$\lambda(f*g) = \lambda(f)*\lambda(g)$ and $\lambda(f^*) = \lambda(f)^*$, using the trivial groupoid structure on $M \times M$ over the diagonal of $M \times M$. (on $(M \times M)_{ee} = \{(e, e)\}$ the Haar measure is taken as 1).*

Proof. We write f_{qp} for $\lambda(f)(q, p)$. Then

$$\begin{aligned}\lambda(f*g)(q, p) &= \int_{Z_{qp}}\int_M\int_{Z_{qr}} f(\Psi_{qr})g(\Psi_{qr}{}^{-1}\cdot\Phi_{qp})d\Psi_{qr}drd\Phi_{qp}\\ &= \int_M\int_{Z_{qr}} f(\Psi_{qr})g_{rp}d\Psi_{qr}dr\\ &= \int_M f_{qr}g_{rp}dr = (\lambda(f)*\lambda(g))(\Phi_{qp})\ .\end{aligned}$$

Next, to show $\lambda(f^*) = \lambda(f)^*$ we should show

$$(5.2.1)\qquad \int_{Z_{qp}} f(\Phi_{qp}{}^{-1})\Delta(\Phi_{qp}{}^{-1})d\Phi_{qp} = \int_{Z_{qp}} f(\Phi_{pq})d\Phi_{pq}\ .$$

If $p = q = e$ this is a standard theorem. The extension to the general case is routine, using (2.6.1).

Accordingly, $f \to \lambda(f)$ defines a $*$homomorphism. Also, $\|f\|_{12} = \|\lambda(|f|)\|_2$, where $\|\ \|_2$ is the $\mathscr{L}_2$ norm on $C_c(M \times M)$. For f and $g \in C_c(M \times M)$ it is easy to show that $\|f*g\|_2 \leqq \|f\|_2\|g\|_2$. Finally, we obtain the

THEOREM 5.3. *Given f and $g \in C_c(Z)$ then $\|f*g\|_{12} \leqq \|f\|_{12}\|g\|_{12}$ and $\|f\| = \|f^*\|$.*

Proof.

$$\|\lambda(|f*g|)\|_2 \leqq \|\lambda(|f|*|g|)|_2 = \|\lambda(|f|)*\lambda(|g|)\|_2 \leqq \|f\|_{12}\|g\|_{12}$$

settles the first part, and $\|\lambda(|f^*|)\|_2 = \|\lambda(|f|)^*\|_2 = \|\lambda(|f|\|_2$ settles the second part.

Accordingly, the convolution and (*) involution extend to $\mathscr{L}_{12}(Z)$, making $\mathscr{L}_{12}(Z)$ a Banach algebra with a natural involution. Representations also extend to $\mathscr{L}_{12}(Z)$ as shown below.

THEOREM 5.4. *For $f \in C_c(Z)$ and $\gamma \in \Gamma_c(E)$, $\|\rho(f)\gamma\|_2 \leqq \|f\|_{12}\|\gamma\|_2$.*

Proof. $\langle\rho(f)\gamma, \rho(f)\gamma\rangle$

$$\begin{aligned}&= \int_M\int_M\int_M\int_{Z_{qr}}\int_{Z_{qp}} f(\Phi_{qp})\bar{f}(\Psi_{qr})\langle\rho(\Phi_{qp})\gamma_p, \rho(\Psi_{qr})\gamma_r\rangle d\Psi_{qp}d\Psi_{qr}drdpdq\\ &\leqq \int_{M\times M\times M} |f_{qp}|\,\|\gamma_p\|\,|f_{qr}|\,\|\gamma_r\|\,drdpdq\\ &= \int_M\left(\int_M |f_{qp}|\,\|\gamma_p\|\,dp\right)\left(\int_M |f_{qr}|\,\|\gamma_r\|\,dr\right)dq\\ &\leqq \int_M\left(\int_M |f_{qp}|\,\|\gamma_p\|\,dp\right)^2 dq\end{aligned}$$

$$\leq \int_M \left(\int_M |f_{qp}|^2 dp \int_M \|\gamma_p\|^2 \, dp \right) dq$$
$$= \|f\|_{12}^2 \|\gamma\|_2^2 .$$

Accordingly, ρ of Z on E lifts to a *representation of $\mathscr{L}_{12}(Z)$ on $\Gamma_2(E)$.

EXAMPLE 5.5. Suppose $Z = Z(G, H)$ as in (3.5 c), and that G/H is compact and $\mu(1) = 1$. Then $\zeta^*: C_c(G) \to C_c(Z)$ (see (3.5.2) is a norm increasing *homomorphism.

Furthermore, a representation ρ of Z on E defines a representation ρ' of G on $\Gamma_2(E)$, by $(\rho'(\Phi)\gamma)_q = \rho(\Phi_{qp})\gamma_p$, where $p = \Phi^{-1}(q)$ and $\Phi_{qp} = (q, \Phi, p)$. ρ' is a unitary representation since μ is invariant under G. Then ρ' is the induced representation (well known in group theory) from the representation ρ_e of $Z_{ee}(\cong H)$ on E_e. The diagram below, relating Z and G, commutes.

$$\begin{array}{ccc} C_c(Z) & \xrightarrow{\rho} & \mathscr{L}(\Gamma_2(E)) \\ \zeta^* \uparrow & & \| \\ C_c(G) & \xrightarrow{\rho'} & \mathscr{L}(\Gamma_2(E)). \end{array}$$

Note that the case $H = G$, $\mu(1) = \lambda_{ee}(1) = 1$, is the same as the Example 3.5a, where $Z = Z_{ee}$.

6. Suppose Z_{ee} is compact, $\Delta \equiv 1$, and $\lambda_{ee}(1) = 1$ (the vertically compact case). Then the completion of $C_c(Z)$ with respect to the $\| \|_2$ norm forms the Hilbert space $\mathscr{L}_2(Z)$. We will extend the "orthogonality relations" for compact groups to the above case, and represent $\mathscr{L}_2(Z)$ as a direct sum of simple H^* algebras.

DEFINITION 6.1. Given γ and $\delta \in \Gamma_c(E^\rho)$, where ρ is a representation of Z on E^ρ, we define $T_{\rho\gamma\delta}: Z \to \boldsymbol{C}$, by

$$T_{\rho\gamma\delta}(\Phi_{qp}) = \langle \gamma_q, \rho(\Phi_{qp})\delta_p \rangle_q .$$

THEOREM 6.2. *If ρ_e and ρ'_e are irreducible, then*

$$\langle T_{\rho\gamma\delta}, T_{\rho'\gamma'\delta'} \rangle = \begin{cases} \dfrac{\langle \gamma, \gamma' \rangle \langle \delta', \delta \rangle}{\dim \rho_e} \; \textit{if } \rho = \rho' \\ 0 \; \textit{if } \rho \textit{ is not equivalent to } \rho'. \end{cases}$$

Proof. Integrating both sides of (6.2.1) over $M \times M$ yields the desired result.

$$(6.2.1)\qquad \int_{Z_{qp}} \langle \gamma_q, \rho(\Phi_q)\delta_p\rangle_q \langle \gamma'_q, \rho'(\Phi_{qp})\delta'_p\rangle d\Phi_{qp}$$

$$= \begin{cases} \dfrac{\langle \gamma_q, \gamma'_q\rangle\langle \delta'_p, \delta_p\rangle}{\dim \rho_e} \text{ if } \rho = \rho' \\ 0 \text{ if } \rho \text{ is not equivalent to } \rho'. \end{cases}$$

For $q = p = e$, (6.2.1) is just the orthogonality relations for compact groups. The proof of (6.2.1) for general p and q is similar to the usual derivation of the orthogonality relations, for example see [1].

Notation. The representations ρ and ρ' of Z on E^ρ and $E^{\rho'}$ respectively will be such that ρ_e and ρ'_e are irreducible. The map $\delta \to \delta^*$: $\Gamma_2(E) \to \Gamma_2(E)^* =$ dual of $\Gamma_2(E)$, is defined by $\delta^*(\gamma) = \langle \gamma, \delta\rangle$. $\Gamma_c(E)^*$ is the image of $\Gamma_c(E)$ under $\delta \to \delta^*$. The (algebraic) tensor product $\Gamma_c(E^\rho) \otimes \Gamma_c(E^\rho)^*$ many be regarded as a (dense) subalgebra of $C_\rho =$ the Schmidt operators on $\Gamma_2(E^\rho)$. In particular $(\gamma \otimes \delta^*)(\beta) = \langle \beta, \delta\rangle\gamma$. Conversely, α and $\beta \in C_\rho$ can be regarded as elements of the (Hilbert space) tensor product $\Gamma_2(E^\rho) \otimes \Gamma_2(E^\rho)^*$. The inner product on C_ρ is defined by $\langle \alpha, \beta\rangle' = \langle \alpha, \beta\rangle \dim \rho_e$ where $\langle\ ,\ \rangle$ is the inner product on $\Gamma_2(E^\rho) \otimes \Gamma_2(E^\rho)^*$, making C_ρ a simple H^* algebra.

THEOREM 6.4. *The canonical map* $T_\rho: \Gamma_c(E^\rho) \otimes \Gamma_c(E^\rho) \to C_c(Z)$ *defined by* $T_\rho(\gamma \otimes \delta^*) = T_{\rho\gamma\delta} \dim \rho_e$ *extends to a *homomorphism and isometry of* C_ρ *into* $\mathscr{L}_2(Z)$.

Proof. To show T_ρ defines an isometry from C_ρ we compute $\langle T_{\rho\gamma\delta} \dim \rho_e, T_{\rho\gamma'\beta} \dim \rho_e\rangle = \langle \gamma \otimes \delta^*, \gamma' \otimes \beta^*\rangle \dim \rho_e$ (by the orthogonality relations,) $= \langle \gamma \otimes \delta^*, \gamma' \otimes \beta^*\rangle'$ in C_ρ. In C_ρ, $(\gamma \otimes \delta^*) \circ (\gamma' \otimes \beta^*)(\alpha) = \langle \alpha, \beta\rangle\langle \gamma', \delta\rangle\gamma$. To show T_ρ is a homomorphism we need $T_{\rho\gamma\delta} * T_{\rho\gamma'\beta} = (\langle \gamma', \delta\rangle T_{\rho\gamma\beta})/\dim \rho_e$. We compute

$$T_{\rho\gamma\delta} * T_{\rho\gamma'\beta}(\Phi_{qp}) = \int_M \int_{Z_{qr}} \langle \gamma_q, \rho(\Psi_{qr})\delta_r\rangle\langle \gamma'_r, \rho(\Psi_{qr}{}^{-1} \cdot \Phi_{qp})\beta_p\rangle d\Psi_{qr} dr$$
$$= \int_M \langle \gamma_q, \rho(\Phi_{qp})\beta_p\rangle\langle \gamma'_r, \delta_r\rangle dr/\dim \rho_e = T_{\rho\gamma\beta}(\langle \gamma', \gamma\rangle/\dim \rho_e)$$

as desired. Finally, it is easy to show that

$$T_\rho((\gamma \otimes \delta^*)^*) = (T_\rho(\gamma \otimes \delta^*))^*.$$

THEOREM 6.5. *Let* $\mathscr{E}$ *be a set of irreducible representations of* Z *containing exactly one member from each equivalence class. Then* $\sum_{\rho \in \mathscr{E}} T_\rho$ *is a *isomorphism and isometry of* $\sum_{\rho \in \mathscr{E}} C_\rho$ *onto* $\mathscr{L}_2(Z)$.

Proof. The main point is that the functions $T_{\rho\gamma\delta}$ for $\rho \in \mathscr{E}$, γ and $\delta \in \Gamma_c(E^\rho)$, separate the points of Z, and $T_{\rho\gamma\delta}$ is orthogonal to $T_{\rho'\gamma'\delta'}$ if $\rho \neq \rho'$ and ρ and $\rho' \in \mathscr{E}$.

7. REMARKS. 7.0. The algebra $C_c(Z)$ forms a quasi-unitary algebra as defined by Dixmier in [2] if we use the inner product

$$\langle f \cdot g \rangle = \int_M \int_M \int_{Z_{qr}} \sqrt{\Delta(\Phi_{qp})} f(\Psi_{qr}) \bar{g}(\Psi_{qr}) d\Psi_{qr} dq dr, \; f^* = f^s, \text{ and}$$

$$f^j(\Phi_{qp}) = f(\Phi_{qp}) / \sqrt{\Delta(\Phi_{qp})} \; .$$

Then $C_c(Z)$ is essentially the same as the algebra Dixmier defines on page 310, [2] in the special case that Z is the example of (3.5c). Also, in this special case, the representation defined in (4.4) is substantially the same as that defined by Glimm in Theorem 1.5, [4].

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